Collatz Conjecture Proof

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Abstract. Collatz sequences are formed by applying the Collatz algorithm to any positive integer. If it is even repeatedly divide by two until it is odd, then multiply by three and add one to get an even number and vice versa. If the Collatz conjecture is true eventually you always get back to one. A connected Collatz Structure is created, which contains all positive integers exactly once. The terms of the Collatz Structure are joined together via the Collatz algorithm. Thus, every positive integer forms a Collatz sequence with unique terms terminating in the number one.

History. The Collatz conjecture was made in 1937 by Lothar Collatz. Through 2017 the conjecture has been checked for all starting values up to $(87)(2^{60})$, but very little progress has been made toward proving the conjecture. Paul Erödos said about the Collatz conjecture: "Mathematics may not be ready for such problems." https://en.wikipedia.org/wiki/Collatz_conjecture

Introduction.

The Collatz Structure (displayed in the diagram below) consists of horizontal branches and vertical towers. Vertical arrows ↓ represent descending Collatz towers, where each term is half the previous term. Horizontal arrows ← indicate the Collatz algorithm is applied to move from term to term in the branch.

We show how different integer types fit in the Collatz Structure (Section 1) exactly once (Section 2). Section 3 defines the binary series of branches, which are used in section 4 to show that all positive integers are in the branches and towers. To prove the Collatz Conjecture we show that the Collatz Structure contains all positive integers (Section 5). Appendix 1 proves there can be no more than two consecutive even integers in a branch. Appendix 2 gives vertical tower details. Appendix 3 provides details about the Collatz Structure.

Section 1

Defining and populating the Collatz Structure

Collatz Structure Branches and Towers ↓ indicates a descending Collatz tower

		↓		↓				↓				↓				↓		↓							↓
↓		84		160	←	53	←	106	←	35	←	70	←	23	←	46	←	15							36
128		42		80					↓				↓				↓						↓		18
64	←	21		40←	13	←	26	←	52	←	17	←	34	←	11	←	22	←	7	←	14	←	28	←	9
32				20	↓																				
16	←	5	←	10←	3																				
8																									
4																									
2																									
1																									

The **Trunk Tower** is the left-most tower, where each term is a power of two 2^s, $s=0,1,2,3\ldots$. A Collatz sequence can begin anywhere within the Collatz Structure and eventually by applying the Collatz algorithm a 2^s term in the **Trunk Tower** will be reached. From there we repeatedly divide by two until the base term 1 is reached. Every Collatz sequence terminates at the **Trunk Tower** base term 1.

Notice that every **red tower** base term is of the form $24m+4$, $24m+10$, or $24m+22$. The rest of the **red tower** terms alternate between $12k+8$ terms $20, 80$ in **blue** and $24k+16$ terms $40, 160$ in **brown.**

We trace a **red tower** from its ***n-th*** term $24k_n+16 \rightarrow 12k_n+8 \rightarrow 6k_n+4 = 24k_{n-1}+16$ $(k_n = 4k_{n-1}+2)\ldots$to its ***first*** (**base**) term. $24k_2+16 \rightarrow 12k_2+8 \rightarrow 6k_2+4 = 24k_1+16$ $(k_2 = 4k_1+2) \rightarrow 12k_1+8 \rightarrow \underline{6k_1+4}$.

If $k_1=4m$, $\underline{6k_1+4} = 24m+4$. **If** $k_1=4m+1$, $\underline{6k_1+4} = 24m+10$. **If** $k_1=4m+3$, $\underline{6k_1+4} = 24m+22$.

Every $24k+16$ term can be written as $4^j a$, $j = 1,2,3\ldots$ $a = 24m+4, 24m+10,$ **or** $24m+22, m=0,1,2,3\ldots$

The Collatz Structure starts with the **Trunk Tower.** Each ***$(4^j)(4)$, $j=1,2,3$***...Trunk Tower term is the last term in a branch. At every ***$a=24m+4$, $24m+10$***, and ***$24m+22$*** base term in the Trunk Tower branches is a ***$4^j a$, $j=1,2,3\ldots$*** secondary **red tower**. Each of these ***$4^j a$*** terms in the secondary **red towers** is the last term in a branch. At every ***$a=24m+4$, $24m+10$,*** and ***$24m+22$*** base term in these secondary branches is a ***$4^j a$*** secondary **red tower**. Each ***$4^j a$*** is the last term in a branch. This process is repeated indefinitely.

Note that ***$24k+16$*** terms, which are divisible by eight are the last term in a branch. All the other even terms that appear in the middle of a branch ***$24m+4 \rightarrow 12m+2$, $24m+10$,*** or ***$24m+22$,*** have even factors of at most four or two. In appendix 1 we show there can be no more than two consecutive even terms in a branch. Since they are divisible by eight, *$24k+16$* terms must appear at the end of a branch. We will show in section 4 that there are no unending branches.

Collatz Structure Branches and Towers ↓ indicates an descending Collatz tower

		↓		↓				↓				↓				↓		↓							↓
↓		84		160	←	53	←	106	←	35	←	70	←	23	←	46	←	15							36
128		42		80					↓				↓				↓						↓		18
64	←	21		40←	13	←	26	←	52	←	17	←	34	←	11	←	22	←	7	←	14	←	28	←	9
32				20	↓																				
16	←	5	←	10←	3																				
8																									
4																									
2																									
1																									

The successor of any odd term is an even term ***$2j+1 \rightarrow 6j+4$*** that leaves a reminder of one when divided by three. The **green** first terms in a branch are of the form ***$6j+3$.*** They all divisible by three, as are all other terms in a **green tower.** They are of the form ***$(2^s)(6j+3)$ $s=1,2,3\ldots$*** No odd term can appear above a ***$6j+3$*** term in a **green tower. *$6j+3$*** terms can only appear at the beginning of a branch. Each term above the ***$6j+3$*** base term in a **green tower** is also of the form ***$24k$, $24k+6$, $24k+12$,*** or ***$24k+18$.*** The exact relation between the two form types is shown below.

$$(2j+1)(24)(2^k) = (2^{k+3})(6j+3) \quad j=0,1,2,3\ldots \ k=0,1,2,3\ldots$$
$$24k+6 = (2)(6j+3), \ (j=2k) \quad j=0,2,4\ldots \ k=0,1,2,3\ldots$$
$$24k+12=(4)(6j+3), \ (j=k) \quad j=0,1,2,3\ldots \ k=0,1,2,3\ldots$$
$$24k+18=(2)(6j+3), \ (j=2k+1) \ j=1,3,5\ldots \ k=0,1,2,3\ldots$$

$2k+1$ **(within a branch) →** ***$6k+4$ ($24k+4$, $24k+10$, $24k+16$, $24k+22$)***

Terms of the form ***$24k+2s$, $0 \le s \le 11$,*** and ***$6j+t$, $t=1,3,5$*** fit within the Collatz Structure as follows:

$24k$ **green tower**
$24k+2$ **successor of *$24j+4$, $j=2k$***
$24k+4$ **red tower base middle of a branch**
$24k+6$ **green tower**
$24k+8$ **red tower successor of *$24j+16$, $j = 2k$***
$24k+10$ **red tower base middle of a branch**
$24k+12$ **green tower**
$24k+14$ **successor of *$24j+4$, $j = 2k+1$***
$24k+16$ **red tower end of a branch**
$24k+18$ **green tower**
$24k+20$ **red tower successor of *$24j+16$, $j = 2k+1$***
$24k+22$ **red tower base middle of a branch**
$6j + 1$ **middle of a branch**
$6j + 3$ **green tower and beginning of a branch**
$6j + 5$ **middle of a branch**

Section 2

No individual term appears more than once in the Collatz structure. There can be no duplicate terms in a branch. All the predecessors of a duplicate pair of terms would be duplicates. This would require ***24h+3, 24h+9,*** or ***24h+15*** to be a duplicate term, and those terms only appear at the beginning of a branch. ***24h+21* have a *24(3h+2)+16*** term as an immediate successor without duplicates. There can be no duplicates in a tower. They are strictly increasing sequences. Since they all start with a different base, no duplicates can appear in different towers. Finally, no duplicate terms can appear in different branches. From the second term forward until the last term is reached all terms in branches have unique predecessors and successors.

Section 3

We define the branch binary series, and provide examples. It will be used to prove all positive integers are in the Collatz Structure, and that there are no unending Collatz sequences.

The ***6n+3*** branch first terms are sub-divided into four types: ***24h+3, 24h+9, 24h+15* and *24h+21, h ≥ 0.*** A branch **binary series** counts the number of divisions by two on its **red tower base** terms: ***24m+4 (2), 24m+10 (1),*** and ***24m+22 (1)***. Only ***24h+3, 24h+9,* and *24h+15*** first terms appear in branches with binary series. These three groups of branches are characterized by their first term ***24h+3, 24h+9*** or ***24h+15*** and a binary series of ***1*'s** and ***2*'s** (see ***2,1,1,2*** below) counting the divisions by two on their **red tower base** terms ***24m+4 (2), 24m+10 (1),*** or ***24m+22 (1)*** and a last term ***24k+16.*** The **length *r*** of its binary series is the number of **red tower base** terms in a branch.

If the sum of ***r 1's*** and ***2's*** in the binary series is ***s,*** there are three different formulas for the first terms of branches that have the same binary series.

$$24h+3+(p-1)(24)(2^s),$$
$$24h+9+(p-1)(24)(2^s),$$
$$24h+15+(p-1)(24)(2^s),\ 2^s > h \geq 0,\ p=1,2,3\ldots$$

Each individual value of ***h*** is part of a different group of branches with the same binary series.

All branches end with $24k+16+(p-1)(24)(3^{r+1}),\ 3^{r+1} > k \geq 0,\ r \geq 0,\ p=1,2,3\ldots.$

We have 3 branches with the binary series ***(2,1,1,2)*** counting divisions by two on their **red tower** base terms.

The first branch is ***9, 28(2), 14, 7, 22(1), 11, 34(1), 17, 52(2), 26, 13, 40.***
The second branch is ***1545, 4636(2), 2318, 1159, 3478(1), 1739, 5218(1), 2609, 7828(2), 3914 1957, 5872.***
The third branch is ***3081, 9244(2), 4622, 2311, 6934(1), 3467, 10402(1), 5201, 15604(2), 7802, 3901, 11704.***

The sum of this binary series is six. These are a series of branches whose first terms differ by $(24)(2^6)=1536$. The first term sequence is $9+(p-1)(24)(2^6)$ ***9, 1545, 3081,…*** The length of this binary series is four. There are five applications of ***2j+1→6j+4*** to the odd terms in the branches. These are a series of branches whose last terms differ by $(24)(3^5)=5832$. The last term sequence is $40+(p-1)(24)(3^5)$ ***40, 5872, 11704,…***

Apply the Collatz algorithm to the first term ***24h+q, q=3,9 or 15*** of a branch with a binary series of **length *r.*** If ***s*** divisions by two on even terms and ***r+1*** applications of ***2j+1→6j+4*** to odd terms result in a last term of ***24k+16,*** then for $24h+q+(p)(24)(2^s)$, ***s*** divisions by two on even terms and ***r+1*** applications of ***2j+1→6j+4*** to odd terms will produce a branch last term of $24k+16+(p)(24)(3^{r+1})$.

Dividing by two ***s*** times eliminates the 2^s term from $(p)(24)(2^s)$. Applying ***2j+1→6j+4*** to $24h+q+(p)(24)(2^s)$ multiplies $(p)(24)(2^s)$ by three. $24h+q+(p)(24)(2^s)\rightarrow 72h+3q+1+(p)(24)(2^s)(3)$.

Starting with $(p)(24)(2^s)$ ***s*** divisions by two on even terms and ***r+1*** applications of ***2j+1→6j+4*** to odd terms gives $(p)(24)(3^{r+1})$. ***24h+21*** has an empty (**length *r = 0***) binary series. Branches that begin with ***24h+21*** are followed immediately by the branch last term ***(24)(3h+2)+16, h ≥ 0.***

Section 4

All positive integers appear branches and towers

A **branch segment** has a first term of the form ***24h+1, 24h+7, 24h+13, 24h+19, 24h+5, 24h+11, 24h+17,*** or ***24h+23*** and a ***24k+16*** last term. A **branch** has a first term of the form ***24h+3, 24h+9, 24h+15,*** or ***24h+21*** and a ***24k+16*** last term. ***h = 0,1,2,3,… k = 0,1,2,3,…***

For each first term form above there is a formula for the proportion of that term form belonging to a **branch** or **branch segment** with **binary series** of **length *r***. Each formula generates a geometric series that sums to one; the total proportion of each individual term form. Thus, all positive integers with these term forms are in branches, and there are no unending branches. We use induction arguments to prove each of these formulas.

Section 4.1 *24k+16* are the last terms of branches with binary series of every combination of *1's* and *2's* for every value of *r*.

Theorem 4.1: The proportion of ***24k+16*** terms in branches with a binary series of length $r \geq 0$ is $2^r/3^{r+1}$.

Lemma 4.1.1: The proportion of ***24k+16*** terms in branches with an empty (**length** $r = 0$) binary series is ***1/3.***

A branch with no binary series has the form: $24h+21 \rightarrow 72h+64 = 24(3h+2)+16.$
$24(3h+2)+16,\ h=0,1,2,\ldots\ 24(2)+16,\ 24(5)+16,\ 24(8)+16\ \ 2,5,8\ldots$ ***is 1/3*** of the terms.

Lemma 4.1.2: The proportion of all ***24k+16*** terms with the same binary series of **length *r*** is $1/3^{r+1}$.

The formula (section 3) for the last term in a group of branches with the same binary series of **length *r*** is $24k+16+(p-1)(24)(3^{r+1})\ p=1,2,3\ldots\ 0 \leq k < 3^{r+1}$. They comprise $1/3^{r+1}$ of the terms.

By lemma 4.1.1 the proportion for length ***0*** is $2^0/3^{0+1}=1/3.$

Assume the proportion of ***24k+16*** terms in branches with a binary series of **length *r*** is $2^r/3^{r+1}$.

Let ***a*** denote ratio. The total proportion of ***24k+16*** terms is $1 = (1/3)/(1-a).\ \ a = 2/3.$

The proportion of ***24k+16*** terms in branches with a binary series of **length *r+1*** is $(2/3)(2^r/3^{r+1})=2^{r+1}/3^{r+2}$.

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By lemma 4.1.2 the proportion of all ***24k+16*** terms with the same binary series of **length *r*** is $1/3^{r+1}$. The proportion of all binary series of **length *r*** is $2^r/3^{r+1}$. There are 2^r branch binary series of **length *r.*** There are branches with ***24k+16*** last terms with binary series of every combination of ***1's*** and ***2's*** for every value of ***r.***

Section 4.2 *24h+3, 24h+9,* and *24h+15* are the first terms of branches with binary series of every combination of *1's* and *2's* for every value of *r*.

There are first term formulas (from section 3) for three groups of branches whose binary series sum to ***s*:**

$$24h+3+(p-1)(24)(2^s),$$
$$24h+9+(p-1)(24)(2^s)$$
$$24h+15+(p-1)(24)(2^s),\ p=1,2,3\ldots\ 0 \leq h < 2^s.$$

If all ***24h+3, 24h+9,*** and ***24h+15 h=0,1,2,3,…*** terms are put in three separate ascending sequences, terms with the same binary series occur every 2^s terms: $1/2^s$ proportion of the sequence terms. We show by induction arguments that each of ***24h+3, 24h+9,*** and ***24h+15*** have formulas for the proportion of terms that are in branches with a binary series of **length *r.*** We show that collectively all ***24h+3, 24h+9,*** and ***24h+15*** terms are in branches with binary series of every combination of ***1's*** and ***2's*** for every value of ***r.***

Theorem 4.2.1: The proportion of $24h+3$ terms in branches with a binary series of **length** $r \geq 2$ is $3^{r-2}/2^{2r-1}$.

Lemma 4.2.1.1: The first two $24h+3$ binary series are (1) if $h=2,4,6,\ldots$ and $(1,2)$ if $h=3, 11, 19,\ldots$

$24h+3 \to 72h+10 \to 36h+5 \to 108h+16.$
For $h=2,\ 51 \to 154(1) \to 77 \to 232=24(9)+16.$
For $h=3,\ 75 \to 226(1) \to 113 \to 340(2) \to 85 \to 256=24(10)+16$

For $r=2,\ 3^{r-2}/2^{2r-1} = 1/2^3$. By Lemma 4.2.1.1 The binary series for $r=2$ is $(1,2) = 1/2^3$.

Assume the proportion of $24h+3$ terms in branches with a binary series of **length** $r \geq 2$ is $3^{r-2}/2^{2r-1}$.

By lemma 4.2.1.1 the first $24h+3$ binary series is (1) for $h=2,4,6,\ldots$ That is $1/2$ the terms in the $24h+3$ terms sequence. Let a denote ratio. The total proportion of $24h+3$ terms is $1 = 1/2 + (1/8)/(1-a)$. $a = 3/4$.

The $r+1$ position of every binary series of that length contains either (1) one or (2) two divisions by two. This increases the distance between $24h+3$ terms with the same binary series by a factor of 2 for (1) and 2^2 for (2), decreasing the proportion by a factor of $1/2$ for (1) and $1/2^2$ for (2).

The proportion of $24h+3$ terms of binary series **length** $r+1$ is

$(1/2)(3^{r-2}/2^{2r-1})+(1/2^2)(3^{r-2}/2^{2r-1}) = (3/4)(3^{r-2}/2^{2r-1}) = 3^{r-1}/2^{2(r+1)-1}$.

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The first two $24h+3$ binary series are (1) for h **even** and $(1,2)$ if $h=3,11,19,\ldots$
All other binary series with h **odd** begin with $(1,2,\ldots)$.

Theorem 4.2.2: The proportion of $24h+9$ terms in branches with a binary series of **length** $r \geq 1$ is $3^{r-1}/2^{2r}$.

Lemma 4.2.2.1: For $h=3$ the $24h+9$ branch binary series binary series is (2).

$24h+9 \to 72h+28 \to 18h+7 \to 54h+22.$
For $h=3,\ 81 \to 244(2) \to 61 \to 184=24(7)+16.$

For $r=1,\ 3^{r-1}/2^{2r} = 1/2^2$. By Lemma 4.2.2.1 The binary series for $r=1$ is $(2) = 1/2^2$.

Assume the proportion of $24h+9$ terms in branches with a binary series of **length** $r \geq 1$ is $3^{r-1}/2^{2r}$.

The total proportion of $24h+9$ terms is $1 = (1/4)/(1-a)$. $a = 3/4$.

The $r+1$ position of every binary series of that **length** contains either (1) one or (2) two divisions by two.

The proportion of $24h+9$ terms of binary series **length** $r+1$ is

$(1/2)(3^{r-1}/2^{2r})+(1/2^2)(3^{r-1}/2^{2r}) = (3/4)(3^{r-1}/2^{2r}) = 3^r/2^{2(r+1)}$.

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For $h=3,7,11,\ldots$ the $24h+9$ branch binary series binary series is (2).
All other binary series begin with $(2,\ldots)$.

Theorem 4.2.3: The proportion of $24h+15$ terms in branches with a binary series **length** $r \geq 2$ is $3^{r-2}/2^{2r-2}$.

Lemma 4.2.3.1: For $h=3$ the $24h+15$ branch binary series binary series is $(1,1)$.

$24h+15 \to 72h+46 \to 36h+23 \to 108h+70 \to 54h+35 \to 162h+106.$
$87 \to 262(1) \to 131 \to 394(1) \to 197 \to 592=24(24)=16.$

For $r=2,\ 3^{r-2}/2^{2r-2} = 1/2^2$. By Lemma 4.2.3.1 The binary series for $r=2$ is $(1,1) = 1/2^2$.

Assume the proportion of $24h+15$ terms in branches with a binary series of **length** $r \geq 2$ is $3^{r-2}/2^{2r-2}$.

The total proportion of all $24h+15$ terms is $1 = (1/4)/(1-a)$. $a = 3/4$.

The proportion of $24h+15$ terms of binary series **length** $r+1$ is

$(1/2)(3^{r-2}/2^{2r-2})+(1/2^2)(3^{r-2}/2^{2r-2}) = (3/4)(3^{r-2}/2^{2r-2}) = 3^{r-1}/2^{2(r+1)-2}$.

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For $h=3,7,11,\ldots$ the $24h+15$ branch binary series is $(1,1)$.
All other binary series begin with $(1,1,\ldots)$.

Collectively all $24h+3$, $24h+9$, and $24h+15$ are first terms in branches with binary series of all 2^r combinations of *1's* and *2's* for every value of r. All $24h+16$ are last terms in branches with binary series of all 2^r **combinations of** *1's* and *2's* for every value of r. **Thus, there are no unending branches.**

Section 4.3 $24h+1$, $24h+7$, and $24h+19$ are the first terms of branch segments with binary series of every combination of *1's* and *2's* for every value of r.

$24h+13 \rightarrow 72h+40 = 24(3h+1)+16$ is a branch segment with an empty (**length** $r = 0$) binary series.

In the same manner that first term formulas were developed for branches in section 3, there are first term formulas for three groups of branch segments whose binary series sums to s:

$$24h+1+(p-1)(24)(2^s),$$
$$24h+7+(p-1)(24)(2^s)$$
$$24h+19+(p-1)(24)(2^s),\ p=1,2,3\ldots\ 0 \le h < 2^s.$$

If all $24h+1$, $24h+7$, and $24h+19$ terms are put in three separate ascending sequences, terms with the same binary series occur every 2^s terms: $1/2^s$ proportion of the sequence terms. We show by induction arguments that each of $24h+1$, $24h+7$, and $24h+19$ have formulas using **length** r for the proportion of terms that are in branch segments with a binary series of **length** r. We show that collectively all $24h+1$, $24h+7$, and $24h+19$ terms are in branch segments with binary series of every combination of *1's* and *2's* for every value of r.

Theorem 4.3.1: The proportion of $24h+19$ in branch segments with a binary series **length** $r \ge 2$ is $3^{r-2}/2^{2r-1}$.

Lemma 4.3.1.1: The first two $24h+19$ binary series are (1) if $h=2$ and $(1,2)$ if $h=4$.

$24h+19 \rightarrow 72h+58 \rightarrow 36h+29 \rightarrow 108h+88.$
For $h=2$, $67 \rightarrow 202(1) \rightarrow 101 \rightarrow 304=24(12)+16$
For $h=5$, $139 \rightarrow 418(1) \rightarrow 209 \rightarrow 628(2) \rightarrow 157 \rightarrow 472=24(19)+16$

For $r=2$, $3^{r-2}/2^{2r-1} = 1/2^3$. By Lemma 4.3.1.1 The binary series for $r=2$ is $(1,2) = 1/2^3$.

Assume the proportion of $24h+19$ terms in branch segments with a binary series of **length** $r \ge 2$ is $3^{r-2}/2^{2r-1}$.

By lemma 4.3.1.1 the first $24h+19$ binary series is (1) for $h=2,4,6,\ldots$ That is $1/2$ the terms in the $24h+19$ terms sequence. Let a denote ratio. The total proportion of $24h+19$ terms is $1 = 1/2 + (1/8)/(1-a)$. $a = 3/4$.

The proportion of $24h+19$ terms with a binary series of **length** $r+1$ is

$$(1/2)(3^{r-2}/2^{2r-1})+(1/2^2)(3^{r-2}/2^{2r-1}) = (3/4)(3^{r-2}/2^{2r-1}) = 3^{r-1}/2^{2(r+1)-1}.$$

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The first two $24h+19$ binary series are (1) for h **even** and $(1,2)$ if $h=5,13,21,\ldots$
All other binary series with h **odd** begin with $(1,2,\ldots)$.

Theorem 4.3.2: The proportion of $24h+1$ in branch segments with a binary series of **length** $r \ge 1$ is $3^{r-1}/2^{2r}$.

Lemma 4.3.2.1: For $h=2$ the $24h+1$ branch segment binary series binary series is (2).

$24h+1 \rightarrow 72h+4 \rightarrow 18h+1 \rightarrow 54h+4.$
For $h=2$, $49 \rightarrow 148(2) \rightarrow 37 \rightarrow 112=24(4)+16.$

For $r=1$, $3^{r-1}/2^{2r} = 1/2^2$. By Lemma 4.3.2.1 The binary series for $r=1$ is $(2) = 1/2^2$.

Assume the proportion of $24h+1$ terms in branch segments with a binary series of **length** $r \ge 1$ is $3^{r-1}/2^{2r}$.

The total proportion of $24h+1$ terms is $1 = (1/4)/(1-a)$. $a = 3/4$.

The proportion of $24h+1$ terms with a binary series **length** $r+1$ is

$$(1/2)(3^{r-1}/2^{2r})+(1/2^2)(3^{r-1}/2^{2r}) = (3/4)(3^{r-1}/2^{2r}) = 3^r/2^{2(r+1)}.$$

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For $h=2,6,10,\ldots$ the $24h+1$ branch segment binary series binary series is (2).
All other binary series begin with $(2,\ldots)$.

Theorem 4.3.3: The proportion of $24h+7$ in branch segments with binary series of **length** $r \geq 2$ is $3^{r-2}/2^{2r-2}$.

Lemma 4.3.3.1: For $h=2$ the $24h+7$ branch segment binary series binary series is $(1,1)$.

$24h+7 \to 72h+22 \to 36h+11 \to 108h+34 \to 54h+17 \to 108h+52.$
$55 \to 166(1) \to 83 \to 250(1) \to 125 \to 376 = 24(15)+16.$

For $r=2$, $3^{r-2}/2^{2r-2} = 1/2^2$. By Lemma 4.3.3.1 The binary series for $r=2$ is $(1,1) = 1/2^2$.

Assume the proportion of $24h+7$ terms in branch segments with a binary series of **length** $r \geq 2$ is $3^{r-2}/2^{2r-2}$.

The total proportion of all $24h+7$ terms is $1 = (1/4)/(1-a)$. $a = 3/4$.

The proportion of $24h+7$ terms with a binary series **length** $r+1$ is

$$(1/2)(3^{r-2}/2^{2r-2})+(1/2^2)(3^{r-2}/2^{2r-2}) = (3/4)(3^{r-2}/2^{2r-2}) = 3^{r-1}/2^{2(r+1)-2}.$$

***

For $h=2,6,10,\ldots$ the $24h+7$ branch segment binary series is $(1,1)$.
All other binary series begin with $(1,1,\ldots)$.

Collectively all $24h+1$, $24h+7$, and $24h+19$ are first terms in branches with binary series of all 2^r combinations of *1's* and *2's* for every value of r. **There are no unending $24h+1$, $24h+7$, or $24h+19$ branch segments.**

Section 4.4 $24h+11$, $24h+17$, and $24h+23$ are the first terms of branch segments with binary series of every combination of *1's* and *2's* for every value of r.

A branch segment with an empty (**length** $r = 0$) binary series has the form: $24h+5 \to 72h+16 = 24(3h)+16$.

There are three groups of branch segments whose binary series sums to s:

$$24h+11+(p-1)(24)(2^s),$$
$$24h+17+(p-1)(24)(2^s)$$
$$24h+23+(p-1)(24)(2^s),\ p=1,2,3\ldots\ 0 \leq h < 2^s.$$

If all $24h+11$, $24h+17$, and $24h+23$ terms are put in three separate ascending sequences, terms with the same binary series occur every 2^s terms: $1/2^s$ proportion of the sequence terms. We show by induction arguments that each of $24h+11$, $24h+17$, and $24h+23$ have formulas using **length** r for the proportion of terms that are in branches with a binary series of **length** r. We show that collectively all $24h+11$, $24h+17$, and $24h+23$ terms are in branch segments with binary series of every combination of *1's* and *2's* for every value of r.

Theorem 4.4.1: The proportion of $24h+19$ in branch segments with a binary series **length** $r \geq 2$ is $3^{r-2}/2^{2r-1}$.

Lemma 4.4.1.1: The first two $24h+11$ binary series are (1) if $h=1$ and $(1,2)$ if $h=8$.

$24h+11 \to 72h+34 \to 36h+17 \to 108h+52.$
For $h=1$, $35 \to 106(1) \to 53 \to 160 = 24(6)+16$
For $h=8$, $203 \to 610(1) \to 305 \to 916(2) \to 229 \to 688 = 24(28)+16$

For $r=2$, $3^{r-2}/2^{2r-1} = 1/2^3$. By Lemma 4.4.1.1 The binary series for $r=2$ is $(1,2) = 1/2^3$.

Assume the proportion of $24h+11$ terms in branch segments with a binary series of **length** $r \geq 2$ is $3^{r-2}/2^{2r-1}$.

By lemma 4.3.2.1 the first $24h+11$ binary series is (1) for $h=1,3,5,\ldots$ That is $1/2$ the terms in the $24h+11$ terms sequence. Let a denote ratio. The total proportion of $24h+11$ terms is $1 = 1/2 + (1/8)/(1-a)$. $a = 3/4$.

The proportion of $24h+11$ terms of binary series **length** $r+1$ is

$$(1/2)(3^{r-2}/2^{2r-1})+(1/2^2)(3^{r-2}/2^{2r-1}) = (3/4)(3^{r-2}/2^{2r-1}) = 3^{r-1}/2^{2(r+1)-1}.$$

***

The first two $24h+11$ binary series are (1) for $h=1,3,5,\ldots$ and $(1,2)$ if $h=8,16,24,\ldots$
All other binary series with h **even** begin with $(1,2,\ldots)$.

Theorem 4.4.2: The proportion of $24h+17$ in branch segments with binary series of **length** $r \geq 1$ is $3^{r-1}/2^{2r}$.

Lemma 4.4.2.1: For $h=4$ the $24h+17$ branch segment binary series binary series is (2).

$24h+17 \to 72h+52 \to 18h+13 \to 54h+40.$
For $h=4,\ 113 \to 340(2) \to 85 \to 256 = 24(10)+16.$

For $r=1$, $3^{r-1}/2^{2r} = 1/2^2$. By Lemma 4.4.2.1 The binary series for $r=1$ is $(2) = 1/2^2$.

Assume the proportion of $24h+17$ terms in branch segments with a binary series of **length** $r \geq 1$ is $3^{r-1}/2^{2r}$.

The total proportion of all $24h+17$ terms is $1 = (1/4)/(1-a)$. $a = 3/4$.

The proportion of $24h+17$ terms of binary series **length** $r+1$ is

$$(1/2)(3^{r-1}/2^{2r})+(1/2^2)(3^{r-1}/2^{2r}) = (3/4)(3^{r-1}/2^{2r}) = 3^r/2^{2(r+1)}.$$

***

For $h=4,8,12,\ldots$ the $24h+17$ branch segment binary series is (2).
All other binary series begin with $(2,\ldots)$.

Theorem 4.4.3: The proportion of $24h+23$ in branch segments with a binary series **length** $r \geq 2$ is $3^{r-2}/2^{2r-2}$.

Lemma 4.4.3.1: For $h=4$ the $24h+23$ branch segment binary series binary series is $(1,1)$.

$24h+23 \to 72h+70 \to 36h+35 \to 108h+106 \to 54h+53 \to 108h+160.$
$119 \to 358(1) \to 179 \to 538(1) \to 269 \to 808 = 24(33)+16.$

For $r=2$, $3^{r-2}/2^{2r-2} = 1/2^2$. By Lemma 4.4.3.1 The binary series for $r=2$ is $(1,1) = 1/2^2$.

Assume the proportion of $24h+23$ terms in branch segments with a binary series of **length** $r \geq 2$ is $3^{r-2}/2^{2r-2}$.

The total proportion of all $24h+17$ terms is $1 = (1/4)/(1-a)$. $a = 3/4$.

The proportion of $24h+23$ terms of binary series **length** $r+1$ is

$$(1/2)(3^{r-2}/2^{2r-2})+(1/2^2)(3^{r-2}/2^{2r-2}) = (3/4)(3^{r-2}/2^{2r-2}) = 3^{r-1}/2^{2(r+1)-2}.$$

***

For $h=4,8,12,\ldots$ the $24h+23$ branch binary series is $(1,1)$.
All other binary series begin with $(1,1,\ldots)$.

Collectively all $24h+11$, $24h+17$, and $24h+23$ are first terms in branches with binary series of all 2^r combinations of ***1's*** and ***2's*** for every value of r. **There are no unending $24h+11$, $24h+17$, or $24h+23$ branch segments.**

Section 4 Summary

$24h+5 \to 24(3h)+16$. $24h+13 \to 24(3h+1)+16$. $24h+21 \to 24(3h+2)+16$. All $24k+16$ terms are in branches and so are all $24h+5$, $24h+13$ and $24h+21$.

Thus, all odd terms and all $(2n+1 \to 6n+4)$ $24m+4$ $(n = 4m) \to 12m+2$ $(24j+2,\ m=2j,\ 24j+14,\ m=2j+1)$, $24m+10$ $(n = 4m+1)$, $24m+16$ $(n = 4m+2)$, and $24m+22$ $(n = 4m+3)$ are in branches.

All $(2^s)(6j+3)$ $24k$, $24k+6$, $24k+12$, and $24k+18$ terms are in **green towers.**

All $24k+16 \to 12k+8$ $(24j+8,\ k=2j,\ 24j+20,\ k=2j+1)$ terms are in **red towers.**

The terms of the form $24h+1$, $24h+9$, and $24h+17$ have proportion formulas $3^{r-1}/2^{2r}$ and are the first terms in branch or branch segments with binary series of (2) and $(2,\ldots)$.

The terms of the form $24h+3$, $24h+11$, and $24h+19$ have proportion formulas $3^{r-2}/2^{2r-1}$ and are the first terms in branch or branch segments with binary series of (1), $(1,2)$ and $(1,2,\ldots)$.

The terms of the form $24h+5$, $24h+13$, and $24h+21$ are the first terms in branch or branch segments with an empty ***length*** $r = 0$ binary series.

The terms of the form $24h+7$, $24h+15$, and $24h+23$ have proportion formulas $3^{r-2}/2^{2r-2}$ and are the first terms in branch or branch segments with binary series of $(1,1)$ and $(1,1,\ldots)$.

A circular Collatz sequence could not contain any ***6j+3*** terms. The only predecessors of ***6j+3*** terms are of the form $(2^s)(6j+3)$ and they cannot be in a circular sequence. They have no predecessors but themselves. No ***6j+1*** or ***6j+5*** terms can be in a circular Collatz sequence. They are all in branches, which contain ***6j+3*** terms. Therefore, there are no circular Collatz sequences.

Section 5

There are no unending Collatz sequences.

To prove this we need to define a new item that is a part of all Collatz sequences. An L_8 begins with a ***24k+16 (280)*** term in a secondary tower. The Collatz algorithm is applied until the **red tower base** term ***(70)*** appears**.** The Collatz algorithm is applied to the branch segment until a ***24k+16*** term ***(160)*** appears in an adjoining tower. Thus, we have an L_8**.** It has an **L** shape and joins two ***24k+16*** terms both divisible by eight. The adjoining L_8 is between ***160*** and ***16.*** We have reached the Trunk Tower. The process stops.

$$\begin{array}{lllll}
 & & & & 280\\
 & & & & 140\\
 & & 160 \leftarrow 53 \leftarrow & 106 \leftarrow 35 \leftarrow & 70\\
 & & 80 & & \\
 & & 40 & & \\
 & & 20 & & \\
16 \leftarrow 5 \leftarrow & 10 & & &
\end{array}$$

Definition of an L_8 chain binary series and its usage factor. A chain of adjoining L_8 moves through Collatz Structure until reaching a ***24k+16*** Trunk Tower term. An L_8 chain binary series is made of the number of divisions by two in each **red tower base** term in the individual L_8 of the L_8 chain. The length of an L_8 chain binary series is the number of **red tower base** terms in the individual L_8 of the L_8 chain.

The above L_8 chain has a binary series of ***(1,1,1).*** The usage factor for each L_8 chain binary series is calculated by multiplying together the even factors of all the **red tower base** terms and inverting the product. We take the sum of the usage factors for each of the individual binary series of **length *r*** to get the usage factor for all binary series of **length *r*.**

Theorem 5.1 The usage factor for all L_8 chains with a binary series of **length *r* is** $3^r/4^r$**.**

We will prove theorem 5.1 by induction.

There are two binary series of length one ***(1), (2)*** for an L_8 chain with one **red tower base** term.
The usage factor is $1/2^1+1/2^2= 3/4$ verifying the formula for $r = 1$.

Assume the the sum of the 2^r individual usage factors for all binary series of **length *r*** is $3^r/4^r$.
The ***r+1*** position of every L_8 chain binary series of that length contains ***(1)*** one or ***(2)*** two divisions by two.
The **length *r+1*** usage factor is $(1/2)(3^r/4^r)+(1/4)(3^r/4^r) = (3/4)(3^r/4^r) = 3^{r+1}/4^{r+1}$**.**

***

The sum of the geometric series of the L_8 chain binary series usage factors is $(3/4)/(1 – 3/4)=3$**.** This equals the total proportion of ***24h+3, 24h+9,*** and ***24h+15*** terms in branches. This total proportion is the sum of three geometric series, each of which (like the geometric series sum of the L_8 chain usage factors) are calculated by multiplying together the even factors of all the **red tower base** terms in each binary series of every different length, inverting the products, and taking the sum.

The equality between the sum of the geometric series of the L_8 chain binary series usage factors and the total of the geometric series sums for the proportion of ***24h+3, 24h+9,*** and ***24h+15*** terms in branches indicates that the **red tower base** terms in all branches are also in L_8 chains. Each L_8 chain binary series is of finite length, but there is no longest L_8 chain binary series.

Theorem 5.2 The proportion of L_8 chains with a binary series of length $r \geq 0$ is $2^r/3^{r+1}$**.**

We prove theorem 5.2 by induction.
$4^3 \leftarrow 21,\ 4^6 \leftarrow (56)(24)+21,\ 4^9 \leftarrow (3640)(24)+21, \ldots$ (Appendix 2).

One third of the Trunk Tower $24k+16$ terms are last terms of branches with an empty binary series.
The proportion of L_8 chains with binary series of **length *0*** is $2^0/3^{0+1}=1/3$,

Assume the proportion of L_8 chains with binary series **length** r is $2^r/3^{r+1}$.
Let a denote ratio. The total proportion of L_8 chains is $1 = (1/3)/(1 - a)$. $a = 2/3$.
The proportion of L_8 chains with a binary series of **length** $r+1$ is $(2/3)(2^r/3^{r+1}) = 2^{r+1}/3^{r+2}$.

***

A $2^r/3^{r+1}$ proportion of all tower **branches** in the Collatz Structure have a binary series **length** r (appendix 2). These are the same branches whose $2^r/3^{r+1}$ proportion of all $24k+16$ last terms are in **branches** with a binary series **length** r (theorem 4.1). **Every last term of a branch is the first term of a finite L_8 chain. Note:** in most cases the length of the **branch** binary series is different from the length of the binary series of the L_8 **chain,** whose first term is the last term of the **branch**.

There are no unending L_8 chains. They would never reach a Trunk Tower term and could not be part of either the binary series usage factor geometric series sum or the binary series length geometric series sum for L_8 chains. The Collatz Structure generated from the Trunk Tower contains all positive integers exactly once. Thus, every positive integer forms a Collatz sequence with unique terms terminating in the number one.

Appendix 1. A branch cannot have more than two consecutive even terms.

$6n+1 \rightarrow 18n+4$
If $n = 4j$, $18n+4 = 72j+4$ $(24m+4,\ m=3j) \rightarrow 36j+2 \rightarrow 18j+1$.
If $n = 4j+1$, $18n+4 = 72j+22$ $(24m+22,\ m=3j) \rightarrow 36j+11$.
If $n = 4j+2$, $18n+4 = 72j+40$ $(24m+16,\ m=3j+1)$ Last term in the branch.
If $n = 4j+3$, $18n+4 = 72j+58$ $(24m+10,\ m=3j+2) \rightarrow 36j+29$

$6n+3 \rightarrow 18n+10$.
If $n = 4j$, $18n+10 = 72j+10$ $(24m+10,\ m=3j) \rightarrow 36j+4$.
If $n = 4j+1$, $18n+10 = 72j+28$ $(24m+4,\ m=3j+1) \rightarrow 36j+14 \rightarrow 18j+7$
If $n = 4j+2$, $18n+10 = 72j+46$ $(24m+22,\ m=3j+1) \rightarrow 36j+23$.
If $n = 4j+3$, $18n+10 = 72j+64$ $(24m+16,\ m=3j+2)$ Last term in the branch.

$6n+5 \rightarrow 18n+16$.
If $n = 4j$, $18n+16 = 72j+16$ $(24m+16,\ m=3j)$ Last term in the branch.
If $n = 4j+1$, $18n+16 = 72j+34$ $(24m+10,\ m=3j+1) \rightarrow 36j+17$.
If $n = 4j+2$, $18n+16 = 72j+52$ $(24m+4,\ m=3j+2) \rightarrow 36j+26 \rightarrow 18j+13$.
If $n = 4j+3$, $18n+16 = 72j+70$ $(24m+22,\ m=3j+2) \rightarrow 36j+34$.

Appendix 2. The repeating binary series structure of towers.

Within a tower if the sum of r *1's* and *2's* in the binary series of a branch is s, there are three groups of branches having the same binary series.

The first begins with $24h+3+(2^s)(24k+16)(4^{(x)(p-1)} - 1) / 3^{r+1}$, $k=0,1,2,3,\dots$, $h=0,1,2,3,\dots$, $x=3^{r+1}$, $p=1,2,3\dots$ and ends with $(24k+16)(4^{(x)(p-1)})$, $k=0,1,2,3,\dots$, $x=3^{r+1}$, $p=1,2,3\dots$

The other two groups that begin with $24h+9\dots$ and $24h+15\dots$ have the same form as $24h+3\dots$

$r+1$ applications of $2j+1 \rightarrow 6j+4$ applied to $24h+3$ and its odd successors
and applied to $(2^s)(24k+16)(4^{(x)(p-1)} - 1) / 3^{r+1}$

and s divisions by two applied to $72h+10$ and its even successors
and applied to $(2^s)(24k+16)(4^{(x)(p-1)} - 1) / 3^r$ gives $(24k+16)+(24k+16)(4^{(x)(p-1)} - 1) = (24k+16)(4^{(x)(p-1)})$.

A branch with no binary series starts with $24h+21+((24)(3h+2)+16)(4^{(3)(p-1)} - 1)/3$

and ends with $((24)(3h+2)+16)(4^{(3)(p-1)})$.

Link between the formulas for branch and tower first terms.

For some t, $24h+3+(t-1)(24)(2^s) = 24h+3+(2^s)(24k+16)(4^{(x)(p-1)} - 1) / 3^{r+1}$.

For $x=3^{r+1}$ every power of three in $4^{(x)(p-1)} - 1 = (3+1)^{(x)(p-1)} - 1$ has a coefficient divisible by 3^{r+1}.
$(24k+16)(4^{(x)(p-1)} - 1) / 3^{r+1}$ is a multiple 24. The same is true for the forms beginning with $24h+9\dots$, $24h+15\dots$, and $24h+21\dots$ Each tower's branch binary series structure is a microcosm of the total branch

binary series structure. $4^{(x)(p-1)}$, $x=3^{r+1}$ replaces 3^{r+1}. In each case the last terms of tower branches with the same binary series occur in intervals of 3^{r+1}. $2^r/3^{r+1}$ is the proportion of the 2^r last terms of tower branches with a binary series of **length** r.

For length $r \geq 0$ $1/3+2/9+4/27\ldots= 1$ is the total proportion.

There are tower branches with binary series of all 2^r combinations of r *1's* and *2's* for every value of r. The first branch with a binary series of length r comes within the first 3^{r+1} branches in the tower.

Appendix 3. Collatz Structure Details.

Groups of similar Collatz sequence segments. If a Collatz sequence segment has a first term a and a last term b with r, $2j+1\rightarrow 6j+4$ and s divisions by two, there is a series of Collatz sequence segments containing the same number of terms and the same number of adjoining L_8 of the same size and structure with a first term $a+(p-1)(24)(2^s)$ and last term $b+(p-1)(24)(3^r)$, $p=1,2,3\ldots$

The average branch binary series length: $3r=(1)(3/4)+(2)(9/16)+(3)(27/64)+\ldots$ $3r-(3)(3/4)r = 3$, $r=4$. The binary series usage factor is three. Three lengths are being calculated. $3/4$ is the proportion of length one. $9/16$ of length two…Multiply the equation by $3/4$ and subtract. $3r-(3)(3/4)r = 3/4 + 9/16 + \ldots = 3$.

The average branch binary series sum: $((2,1,1,1)+(2,2,1,1)+(2,1,1,1))/3 = (5+6+5)/3 = 4.333\ldots$
There are twice as many binary series components with one division by two $24j+10$ (1), $24j+22$ (1) than there are components with two divisions by two $24j+4$ (2). Three binary series of length four with twice as many ***1's*** as ***2's*** make up the computation.

Calculating the decrease in term size for L_8 with the fewest $24k+16$ terms.
$2/3$ $(1-1/3)$ of the branches in a tower have binary series of length one or more. $4/9$ $(1-1/3-2/9)$ have binary series of length two or more. The geometric series terms are increased by $3/2$ to base the calculation on the branches that have binary series. The average length of the L_8 binary series is:
$(1)(3/2)(2/3)+(2)(3/2)(4/9)+(3)(3/2)(8/27)+\ldots$
$(1+(2)(2/3)+(3)(4/9)+\ldots-(2/3)(1+(2)(2/3)+(3)(4/9)+\ldots)=1+2/3+4/9+\ldots=3$ $\quad(3)(3)=9$
Adjusting the proportion of branches with binary series from three to one. $9/3=3$.

The average L_8 binary series sum is $(1,1,2)=4$. $1/3$ of all branches have no binary series. The average number of divisions by two to reach the tower base term is $2+4+2=2.67$. Let $2j+1\rightarrow 6j+4$ be represented by an increase of 1.56 multiples of two. The average decrease in L_8 term values is

$-2.67-2+1.56-1+1.56-1+1.56=-2$. The ratio between the initial $24j+16$ term in an L_8 with minimum number of tower terms and the last $24j+16$ term is on average $4/1$.

A circular sequence $1\rightarrow 4\rightarrow 2\rightarrow 1$ can be used to generate a sequence of arbitrary length with the same number and positions of $2j+1\rightarrow 6j+4$ and divisions by two. The binary series of length s is $(2,2,2,\ldots)$

$1+(2^{2s})(24)(p-1)$ is the beginning term and $1+(3^s)(24)(p-1)$ end term.
For $s=3$, $p=2$, $1537\rightarrow 4612\rightarrow 2306\rightarrow 1153\rightarrow 3460\rightarrow 1730\rightarrow 865\rightarrow 2596\rightarrow 1298\rightarrow 649$

$24k+16$ first term sequence segments

s=1	2	3	4	5	6	$(2^{s-1}-1)(24)+16+(p-1)(24)(2^s)$ The binary series is $(1,1,1,\ldots)$ The **length** $r=s-3$.
k=0	1	3	7	15	31	
	2	5	11	23	47 95	
	4	9	19	39	79 159	

first term → last term					last term formula	
16 →8	40→10	88→11	184→35	376→107	$s=1,2,3$	$8,10,11+(24)(p-1)$
64→32	136→34	280→35	528→107	1144→323	$s\geq 4$	$11+s=4$ to $m\sum(24)(3^{s-4})+(24)(3^{s-3})(p-1)$

Thanks for your interest in this paper. If you wish to make comments send them to Jim Rock at collatz3106@gmail.com.