Collatz Conjecture Proof

Abstract. Collatz sequences are formed by applying the Collatz algorithm to any positive integer. If it is even repeatedly divide by two until it is odd, then multiply by three and add one to get an even number and vice versa. If the Collatz conjecture is true eventually you always get back to one. A connected Collatz Structure is created, which contains all positive integers exactly once. The terms of the Collatz Structure are joined together via the Collatz algorithm. Thus, every positive integer forms a Collatz sequence with unique terms terminating in the number one.

Introduction.

The Collatz Structure (displayed in the diagram below) consists of horizontal branches and vertical towers. Vertical arrows ↓ represent descending Collatz towers, where each term is half the previous term. Horizontal arrows ← indicate the Collatz algorithm is applied to move from term to term in the branch.

To prove the Collatz Conjecture it is necessary to show that the Collatz Structure is connected **(Section 7)**. Otherwise there could be circular or unending Collatz sequences. We need to show that every positive integer is in the Collatz Structure **(Section 1)** exactly once **(Section 2)**. The heart of the proof is in **Sections 3,4,5,** where we develop the properties of the Collatz Structure**. Section 6** discusses the relation between towers and branches.

Collatz Structure Branches and Towers ↓ indicates a descendng Collatz tower

		↓		↓				↓				↓				↓		↓							↓
↓		84		160	←	53	←	106	←	35	←	70	←	23	←	46	←	15							36
128		42		80					↓				↓				↓						↓		18
64	←	21		40←	13	←	26	←	52	←	17	←	34	←	11	←	22	←	7	←	14	←	28	←	9
32				20	↓																				
16	←	5	←	10←	3																				
8																									
4																									
2																									
1																									

Section 1

Defining and populating the Collatz Structure

The Trunk Tower is the left-most tower, where each term is a power of two 2^s, $s=0,1,2,3\dots$. A Collatz sequence can begin anywhere within the Collatz Structure and eventually by applying the Collatz algorithm a 2^s term in the Trunk Tower will be reached. From there we repeatedly divide by two until the base term 1 is reached. Every Collatz sequence terminates at the Trunk Tower base term 1.

Notice that every **red tower** base term is of the form $24m+4$, $24m+10$, or $24m+22$. The rest of the **red tower** terms alternate between $12k+8$ terms $20, 80$ in **blue** and $24k+16$ terms $40, 160$ in **brown.**

We trace a **red tower** from its ***n-th*** term $24k_n+16 \rightarrow 12k_n+8 \rightarrow 6k_n+4 = 24k_{n-1}+16$ $(k_n = 4k_{n-1}+2)\dots$to its ***first*** term. $24k_2+16 \rightarrow 12k_2+8 \rightarrow 6k_2+4 = 24k_1+16 \rightarrow 12k_1+8 \rightarrow \underline{6k_1+4}.$ $(k_2 = 4k_1+2)$

If $k_1=4m$, $\underline{6k_1+4} = 24m+4$. **If** $k_1=4m+1$, $\underline{6k_1+4} = 24m+10$. **If** $k_1=4m+3$, $\underline{6k_1+4} = 24m+22$. In a branch $6k_1+4$ is always the immediate successor of an odd term. $2k_1+1 \rightarrow 6k_1+4$ is an even number that has a remainder of one when divided by three as are $24m+4$, $24m+10$, and $24m+22$.

Note that all $24k+16$ terms, which are all divisible by eight are the last term in a branch. All the other even terms that appear in the middle of a branch $24m+4 \rightarrow 12m+2$ $(24j+2, m=2j, 24j+14, m=2j+1)$, $24m+10$, or $24m+22$, are divisible by at most four or two. In appendix 1 we show there can be no more than two consecutive even terms in a branch. Since they are divisible by eight, $24k+16$ terms must appear at the end of a branch. We will show in section 5 that there are no unending branches.

Collatz Structure Branches and Towers ↓ indicates an ascendng Collatz tower

		↓		↓				↓				↓				↓		↓							↓
↓		84		160	←	53	←	106	←	35	←	70	←	23	←	46	←	15							36
128		42		80					↓				↓				↓						↓		18
64	←	21		40←	13	←	26	←	52	←	17	←	34	←	11	←	22	←	7	←	14	←	28	←	9
32				20	↓																				
16	←	5	←	10←	3																				
8																									
4																									
2																									
1																									

The **green** first terms in a branch are of the form $6j+3$**.** They all have a remainder of zero when divided by three. All other terms in a **green tower** are of the form $(2^s)(6j+3)$ $s=1,2,3\ldots$double their previous term.

The successor of any odd term is an even term $2j+1\rightarrow 6j+4$ that leaves a reminder of one when divided by three. Since no even term that leaves a reminder of one when divided by three appears above the $6j+3$ terms, no odd term can appear above a $6j+3$ term in a **green tower.** $6j+3$ terms can only appear at the beginning of a branch. $(2^s)(6j+3)$ equals $24k$, $24k+6$, $24k+12$, or $24k+18$**.**

$$24k\rightarrow 12k\rightarrow 6k\rightarrow 3k=6j+3\ (k=2j+1),$$
$$24k+6\rightarrow 12k+3=6j+3\ (j=2k),$$
$$24k+12\rightarrow 12k+6\rightarrow 6j+3,\ (j=k),$$
$$24k+18\rightarrow 12k+9=6j+3\ (j=2k+1).$$

Since all terms in towers have been accounted for and $6j+3$ terms are at the beginning of a branch, we will prove that all $6j+1$ and $6j+5$ terms appear in the middle of a branch.

We have accounted for all terms in the Collatz Structure

$24k$ **green tower**
$24k+2$ **successor of** $24j+4, j=2k$
$24k+4$ **middle of a branch**
$24k+6$ **green tower**
$24k+8$ **red tower successor of** $24j+16, j=2k$
$24k+10$ **middle of a branch**
$24k+12$ **green tower**
$24k+14$ **successor of** $24j+4, j=2k-1$
$24k+16$ **red tower end of a branch**
$24k+18$ **green tower**
$24k+20$ **red tower successor of** $24j+16, j=2k-1$
$24k+22$ **middle of a branch**
$6j+1$ **middle of a branch**
$6j+3$ **beginning of a branch**
$6j+5$ **middle of a branch**

$24k_n+16\rightarrow 12k_n+8\rightarrow 24k_{n-1}+16\rightarrow\ldots 24k_2+16\rightarrow 12k_2+8\rightarrow 24k_1+16\rightarrow 12k_1+8\rightarrow 6k_1+4=24m+s$, $s=4,10$, **or** 22.

Every $24k+16$ term can be written as $4^j a$, $j=1,2,3\ldots$ $a=24m+4$, $24m+10$, **or** $24m+22$, $m=0,1,2,3\ldots$

The Collatz Structure starts with the **Trunk Tower.** Each $(4^j)(4)$, $j=1,2,3$...Trunk Tower term is the last term in a branch. At every $a=24m+4$, $24m+10$, and $24m+22$ base term in the Trunk Tower branches is a $4^j a$, $j=1,2,3\ldots$ secondary **red tower**. Each of these $4^j a$ terms in the secondary **red towers** is the last term in a branch. At every $a=24m+4$, $24m+10$, and $24m+22$ base term in these secondary branches is a $4^j a$ secondary **red tower**. Each $4^j a$ is the last term in a branch. This process is repeated indefinitely.

Section 2

Showing that every positive integer is in a branch or a tower exactly once.

Every ***6j+3*** term is at the beginning of a branch, and every ***24k+16*** term is the last term in a branch.

Multiplying each of ***24m+4, 24m+10,*** and ***24m+22*** by four gives a ***24k+16*** **red tower** term so all ***24m+4, 24m+10,*** and ***24m+22*** terms are **red tower** bases.

As discussed above, all ***12j+2 (24k+2, 24k+14)*** terms are in branches. In section 5 we will prove all ***6j+1,*** and ***6j+5*** are in branches. All ***(2^s)(6j+3) 24k, 24k+6, 24k+12,*** and ***24k+18*** terms appear above ***6j+3*** terms in **green towers**. Since all ***24j+16*** are in **red towers** (as well as being the last term in a branch), all ***12j+8 (24k+8, 24k+20)*** terms are in **red towers**.

There can be no duplicate terms in a branch. All the predecessors of a duplicate pair of terms would be duplicates. This would require ***24h+3, 24h+9,*** or ***24h+15*** to be a duplicate term, and those terms only appear at the beginning of a branch. ***24h+21*** **have a** ***24(3h+2)+16*** term as an immediate successor without duplicates. There can be no duplicates in a tower. They are strictly increasing sequences. Since they all start with a different base, no duplicates can appear in different towers. Finally, no duplicate terms can appear in different branches. From the second term forward until the last term is reached all terms in branches have unique predecessors and successors.

Thus, if it is connected, all positive integers appear in the Collatz Structure exactly once. We will use the concept of a branch "binary series" to show that the Collatz Structure is connected, and that every branch has a beginning ***6j+3*** term and an ending ***24k+16*** term, and every ***6j+1*** and ***6j+5*** term is in the middle of a branch.

Section 3

Defining the "binary series" of a branch.

The ***6n+3*** branch first terms are sub-divided into four types: ***24h+3, 24h+9, 24h+15*** **and** ***24h+21, $h \geq 0$.*** A branch **binary series** counts the number of divisions by two on its **red tower** base terms: ***24m+4 (2), 24m+10 (1),*** and ***24m+22 (1)***. The binary series will be used to show that there are no unending branches. Only ***24h+3, 24h+9,*** **and** ***24h+15*** first terms appear in branches with binary series. These three groups of branches are characterized by their first term ***24h+3, 24h+9*** or ***24h+15*** and a binary series of ***1***'s and ***2***'s (see ***2,1,1,2*** below) counting the divisions by two on their **red tower** base terms ***24m+4 (2), 24m+10 (1),*** or ***24m+22 (1)*** and a last term ***24k+16.*** The length ***r*** of its binary series is the number of **red tower** base terms in a branch. If the sum of ***r 1's*** and ***2's*** in the binary series is ***s,*** there are three groups of branches with each branch in a group having the same binary series.

The first terms are: $24h+3+(p-1)(24)(2^s)$,
$24h+9+(p-1)(24)(2^s)$
$24h+15+(p-1)(24)(2^s)$, $p=1,2,3\ldots 2^s > h \geq 0$.

Each individual value of ***h*** is part of a different group with the same binary series.

All groups end with $24k+16+(p-1)(24)(3^{r+1})$, $k \geq 0$, $r > 0$, $p=1,2,3\ldots$.

24h+21 has no binary series. However, there are branches that begin with ***24h+21*** followed immediately by the branch last term ***(24)(3h+2)+16, $h \geq 0$.***

Using the formula ***$24h+9+(p-1)(24)(2^s)$,*** **with** ***p=0,1,2*** **and** ***s=6.*** We have 3 branches with the binary series ***(2,1,1,2)*** counting **divisions by two** on their **red tower** base terms ***24m+4 (2), 24m+10 (1),*** and ***24m+22 (1).***

The first branch is ***9, 28(2), 14, 7, 22(1), 11, 34(1), 17, 52(2), 26, 13, 40.***
The second branch is ***1545, 4636(2), 2318, 1159, 3478(1), 1739, 5218(1), 2609, 7828(2), 3914 1957, 5872.***
The third branch is ***3081, 9244(2), 4622, 2311, 6934(1), 3467, 10402(1), 5201, 15604(2), 7802, 3901, 11704.***

The sum of this binary series is six. These are a series of branches whose first terms differ by ***$(24)(2^6)=1536$.***
The first term sequence is ***$9+(p-1)(24)(2^6)$ 9, 1545, 3081…*** The last terms differ by ***$(24)(3^5)=5832$***.
The last term sequence is ***$40+(p-1)(24)(3^5)$ 40, 5872, 11704…***

Section 4

Proving the formula for branches with the same binary series.

Let r be the length of the binary series. If the sum of r *1's* and *2's* in a $24h+9$ binary series is s,

The first terms are: $24h+9+(p-1)(24)(2^s)$,
All groups end with $24k+16+(p-1)(24)(3^{r+1})$, $p=1,2,3\ldots$ $k \geq 0$.

We have two branches with the binary series *(2,1,1,2)* counting **divisions by two** on their **red tower** base terms *24m+4 (2)*, *24m+10 (1)*, and *24m+22 (1)*.

The **first branch** is *9, 28(2), 14, 7, 22(1), 11, 34(1), 17, 52(2), 26, 13, 40.*
The **second branch** is *1545, 4636(2), 2318, 1159, 3478(1), 1739, 5218(1), 2609, 7828(2), 3914 1957, 5872.*

Start with the **first branch** $24h+9$, *9* and **second branch** $24h+9+(p)(24)(2^s)$ $9+(24)(2^6)=1545$. Multiplying by three and adding one $(2j+1\rightarrow 6j+4)$ gives two terms that differ by $(p)(24)(2^s)(3)$.
$72h+28$ *28* and $72h+28+(p)(24)(2^s)(3)$. $28+(24)(2^6)=4636$.

A total of $r+1$ applications of $2j+1\rightarrow 6j+4$ to $24h+9$, and its odd successors, $r+1=5$ applications of $2j+1\rightarrow 6j+4$ to *9*, and its **green odd** successors, and s divisions by two on $72h+28$, $s=6$ **(See brown *(2),(1),(1),(2)* above)** divisions by two on *28*, and its **red even** successors, which cause $24k+16$ *40* term to appear, are mirrored in $24h+9+(p)(24)(2^s)$ and its odd successors, $9+(2^6)(24)=1545$ and its **green odd** successors, and $72h+28+(p)(24)(2^s)(3)$ and its even successors, $28+(24)(2^6)(3)=4636$ and its **red even** successors, so that a $24k+16+(p)(24)(3^{r+1})$ $40+(24)(3^5)=5872$ term appears.

The same proof holds for groups with the first terms $24h+3+(p-1)(24)(2^s)$, and $24h+15+(p-1)(24)(2^s)$,

Section 5

Showing there are no unending branches or unending branch segments.

A **branch segment** has a first term of the form $24h+1$, $24h+7$, $24h+13$, $24h+19$, $24h+5$, $24h+11$, $24h+17$, or $24h+23$ and a $24k+16$ last term.

Section 5.1 ***24k+16* are the last terms of branches with binary series of every combination of *1's* and *2's* for every value of *r*.**

Put all $24k+16$ terms in a sequence $24k+16$, $k=0,1,2,3\ldots$

Theorem 5.1: The proportion of $24k+16$ terms in branches with a binary series of length r is $2^r/3^{r+1}$.

Lemma 5.1.1: The proportion of $24k+16$ terms in branches without a binary series is $1/3$.

A branch with no binary series has the form: $24h+21\rightarrow 72h+64 = 24(3h+2)+16$.
$24(3h+2)+16$, $h=0,1,2,3\ldots$ $24(2)+16$, $24(5)+16$, $24(8)+16$ *2,5,8… is 1/3* of the terms in the sequence $24k+16$, $k=0,1,2,3\ldots$

Lemma 5.1.2: The proportion of all $24k+16$ terms with the same binary series of length r is $1/3^{r+1}$ of the terms in the sequence.

The formula for the last term in a group of branches with the same binary series of length r is $24k+16+(p-1)(24)(3^{r+1})$ $p=1,2,3\ldots$ $0 \leq k < 3^{r+1}$. They comprise $1/3^{r+1}$ of the terms in the $24k+16$ sequence.

By lemma 5.1.2 The proportion of all $24k+16$ terms with the same binary series of length r is $1/3^{r+1}$ of the terms in the sequence. There are 2^r different binary series of length r. Thus, the proportion of $24k+16$ terms in branches with a binary series of length r is $2^r/3^{r+1}$.

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Summing the geometric series for $r=0,1,2,3\ldots$gives $1/3+2/9+4/27+\ldots. = (1/3)(1-2/3) = 1$. This accounts for all terms in the sequence $24k+16$, $k=0,1,2,3\ldots$ There are branches with $24k+16$ last terms with binary series of every combination of *1's* and *2's* for every value of r.

Section 5.2 ***24h+3, 24h+9,* and *24h+15* are the first terms of branches with binary series of every combination of *1's* and *2's* for every value of *r.***

There are three groups of branches whose binary series sums to ***s*:**

$$24h+3+(p-1)(24)(2^s),$$
$$24h+9+(p-1)(24)(2^s)$$
$$24h+15+(p-1)(24)(2^s),\ p=1,2,3\ldots\ 0 \le h < 2^s.$$

If all ***24h+3, 24h+9,*** and ***24h+15*** terms are put in three separate ascending sequences, terms with the same binary series occur every 2^s terms: $1/2^s$ proportion of the sequence terms. We show by induction arguments that each of ***24h+3, 24h+9,*** and ***24h+15*** have formulas for the proportion of terms that are in branches with a binary series of length ***r.*** We show that collectively all ***24h+3, 24h+9,*** and ***24h+15*** terms are in branches with binary series of every combination of ***1's*** and ***2's*** for every value of ***r.***

Theorem 5.2.1: The proportion of ***24h+3*** terms in branches with a binary series of length $r \ge 2$ is $3^{r-2}/2^{2r-1}$.

Lemma 5.2.1.1: The first two ***24h+3*** binary series are ***(1)*** if ***h=2*** and ***(1,2)*** if ***h=3.***

$24h+3 \to 72h+10 \to 36h+5 \to 108h+16.$
For $h=2,\ 51 \to 154(1) \to 77 \to 232=24(9)+16.$
For $h=3,\ 75 \to 226(1) \to 113 \to 340(2) \to 85 \to 256=24(10)+16$

For $r=2,\ 3^{r-2}/2^{2r-1} = 1/2^3$. By Lemma 5.2.1.1 The binary series for ***r=2*** is $(1,2) = 1/2^3$.

Assume the proportion of ***24h+3*** terms in branches with a binary series of length ***r*** is $3^{r-2}/2^{2r-1}$.

The ***r+1*** position of every binary series of that length contains either ***(1)*** one or ***(2)*** two divisions by two. This increases the distance between ***24h+3*** terms with the same binary series by a factor of ***2*** for ***(1)*** and 2^2 for ***(2),*** decreasing the proportion by a factor of ***1/2*** for ***(1)*** and $1/2^2$ for ***(2).***

The proportion of ***24h+3*** terms of binary series length ***r+1*** is
$(1/2)(3^{r-2}/2^{2r-1})+(1/2^2)(3^{r-2}/2^{2r-1}) = (3/4)(3^{r-2}/2^{2r-1}) = 3^{r-1}/2^{2(r+1)-1}.$

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Starting with ***1/2*** for length one and summing the geometric series $3^{r-2}/2^{2r-1}$ for length $r = 2,3,4,\ldots$ gives

$$1/2+1/8+3/32+9/128+\ldots= 1/2+(1/8)/(1-3/4)=1.$$

The first two ***24h+3*** binary series are ***(1)*** for ***h* even** and ***(1,2)*** if ***h=3,11,19,…***
All other binary series with ***h* odd** begin with ***(1,2,…).***

Theorem 5.2.2: The proportion of ***24h+9*** terms in branches with a binary series of length ***r*** is $3^{r-1}/2^{2r}$.

Lemma 5.2.2.1: For ***h=3*** the ***24h+9*** branch binary series binary series is ***(2)***.

$24h+9 \to 72h+28 \to 18h+7 \to 54h+22.$
For $h=3,\ 81 \to 244(2) \to 61 \to 184=24(7)+16.$

For $r=1,\ 3^{r-1}/2^{2r} = 1/2^2$. By Lemma 5.2.2.1 The binary series for ***r=1*** is $(2) = 1/2^2$.

Assume the proportion of ***24h+9*** terms in branches with a binary series of length ***r*** is $3^{r-1}/2^{2r}$.

The proportion of ***24h+9*** terms of binary series length ***r+1*** is
$(1/2)(3^{r-1}/2^{2r})+(1/2^2)(3^{r-1}/2^{2r}) = (3/4)(3^{r-1}/2^{2r}) = 3^r/2^{2(r+1)}.$

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Summing the geometric series $3^{r-1}/2^{2r}$ for length $r = 1,2,3,\ldots$ gives $1/4+3/16+9/64+\ldots= (1/4)/(1-3/4)=1.$

For ***h=3,7,11,…*** the ***24h+9*** branch binary series binary series is ***(2)***.
All other binary series begin with ***(2,…).***

Theorem 5.2.3: The proportion of ***24h+15*** terms in branches with a binary series of length $r \ge 2$ is $3^{r-2}/2^{2r-2}$.

Lemma 5.2.3.1: For ***h=3*** the ***24h+15*** branch binary series binary series is ***(1,1)***.

$24h+15 \to 72h+46 \to 36h+23 \to 108h+70 \to 54h+35 \to 162h+106.$
$87 \to 262(1) \to 131 \to 394(1) \to 197 \to 592=24(24)=16.$

For $r=2$, $3^{r-2}/2^{2r-2} = 1/2^2$. By Lemma 5.2.3.1 The binary series for $r=2$ is $(1,1) = 1/2^2$.

Assume the proportion of ***24h+15*** terms in branches with a binary series of length r is $3^{r-2}/2^{2r-2}$.

The proportion of ***24h+15*** terms of binary series length $r+1$ is
$(1/2)(3^{r-2}/2^{2r-2})+(1/2^2)(3^{r-2}/2^{2r-2}) = (3/4)(3^{r-2}/2^{2r-2}) = 3^{r-1}/2^{2(r+1)-2}$.

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Summing the geometric series $3^{r-2}/2^{2r-2}$ for length $r = 2,3,4,\ldots$ gives $1/4+3/16+9/64+\ldots= (1/4)/(1-3/4)=1$.

For $h=3,7,11,\ldots$ the ***24h+15*** branch binary series binary series is ***(1,1)***.
All other binary series begin with ***(1,1,…)***.

Collectively all ***24h+3, 24h+9,*** and ***24h+15*** are first terms in branches with binary series of all 2^r combinations of ***1's*** and ***2's*** for every value of r. All ***24h+16*** are last terms in branches with binary series of all 2^r **combinations of** ***1's*** and ***2's*** for every value of r. **Thus, there are no unending branches.**

Section 5.3 *24k+16* are the last terms of branches segments beginning with *24h+1, 24h+7, 24h+13* and *24h+19* with binary series of every combination of *1's* and *2's* for every value of *r*.

Put all ***24k+16*** terms in a sequence ***24k+16, k=0,1,2,3…***

Theorem 5.3: The proportion of ***24k+16*** terms in branch segments with a binary series of length r is $2^r/3^{r+1}$.

Lemma 5.3.1: The proportion of ***24k+16*** terms in branches without a binary series is ***1/3.***

A branch segment with no binary series has the form: $24h+13 \rightarrow 72h+40 = 24(3h+1)+16$. $24(3h+1)+16$, $h=0,1,2,3\ldots$ $24(1)+16$, $24(4)+16$, $24(7)+16$ $1,4,7\ldots$ ***is 1/3*** of the terms in the sequence ***24k+16, k=0,1,2,3…***

Lemma 5.3.2: The proportion of all ***24k+16*** terms with the same binary series of length r is $1/3^{r+1}$ of the terms in the sequence.

The formula for the last term in a group of ***24h+1, 24h+7,*** and ***24h+19*** branch segments with the same binary series of length r is
$24k+16+(p-1)(24)(3^{r+1})$ $p=1,2,3\ldots$ $0 \le k < 3^{r+1}$. They comprise $1/3^{r+1}$ of the terms in the ***24k+16*** sequence.

By lemma 5.3.2 The proportion of all ***24k+16*** terms with the same binary series of length r is $1/3^{r+1}$ of the terms in the sequence. There are 2^r different binary series of length r. Thus, the proportion of ***24k+16*** terms in branches with a binary series of length r is $2^r/3^{r+1}$.

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Summing the geometric series for $r=0,1,2,3\ldots$gives $1/3+2/9+4/27+\ldots. = (1/3)(1 - 2/3) = 1$. This accounts for all terms in the sequence ***24k+16, k=0,1,2,3…*** There are ***24h+1, 24h+7, 24h+13*** and ***24h+19*** branch segments with ***24k+16*** last terms with binary series of every combination of ***1's*** and ***2's*** for every value of r.

Section 5.4 *24h+1, 24h+7,* and *24h+19* are the first terms of branch segments with binary series of every combination of *1's* and *2's* for every value of *r*.

There are three groups of branches whose binary series sums to s:

$$24h+1+(p-1)(24)(2^s),$$
$$24h+7+(p-1)(24)(2^s)$$
$$24h+19+(p-1)(24)(2^s),\ p=1,2,3\ldots\ 0 \le h < 2^s.$$

If all ***24h+1, 24h+7,*** and ***24h+19*** terms are put in three separate ascending sequences, terms with the same binary series occur every 2^s terms: $1/2^s$ proportion of the sequence terms. We show by induction arguments that each of ***24h+1, 24h+7,*** and ***24h+19*** have formulas using length r for the proportion of terms that are in branches with a binary series of length r. We show that collectively all ***24h+1, 24h+7,*** and ***24h+19*** terms are in branches with binary series of every combination of ***1's*** and ***2's*** for every value of r.

Theorem 5.4.1: The proportion of $24h+19$ terms in branches with a binary series length $r \geq 2$ is $3^{r-2}/2^{2r-1}$.

Lemma 5.4.1.1: The first two $24h+19$ binary series are (1) if $h=2$ and $(1,2)$ if $h=5$.

$24h+19 \rightarrow 72h+58 \rightarrow 36h+29 \rightarrow 108h+88.$
For $h=2,\ 67 \rightarrow 202(1) \rightarrow 101 \rightarrow 304=24(12)+16$
For $h=5,\ 139 \rightarrow 418(1) \rightarrow 209 \rightarrow 628(2) \rightarrow 157 \rightarrow 472=24(19)+16$

For $r=2,\ 3^{r-2}/2^{2r-1} = 1/2^3$. By Lemma 5.4.1.1 The binary series for $r=2$ is $(1,2) = 1/2^3$.

Assume the proportion of $24h+19$ terms in branches with a binary series of length r is $3^{r-2}/2^{2r-1}$.

The proportion of $24h+19$ terms of binary series length $r+1$ is
$(1/2)(3^{r-2}/2^{2r-1})+(1/2^2)(3^{r-2}/2^{2r-1}) = (3/4)(3^{r-2}/2^{2r-1}) = 3^{r-1}/2^{2(r+1)-1}$.

***

Starting with $1/2$ for length one and summing the geometric series $3^{r-2}/2^{2r-1}$ for length $r = 2,3,4,\dots$ gives

$$1/2+1/8+3/32+9/128+\dots= 1/2+(1/8)/(1-3/4)=1.$$

The first two $24h+19$ binary series are (1) for h **even** and $(1,2)$ if $h=5,13,21,\dots$
All other binary series with h **odd** begin with $(1,2,\dots)$.

Theorem 5.4.2: The proportion of $24h+1$ terms in branches with a binary series of length r is $3^{r-1}/2^{2r}$.

Lemma 5.4.2.1: For $h=2$ the $24h+1$ branch binary series binary series is (2).

$24h+1 \rightarrow 72h+4 \rightarrow 18h+1 \rightarrow 54h+4.$
For $h=2,\ 49 \rightarrow 148(2) \rightarrow 37 \rightarrow 112=24(4)+16.$

For $r=1,\ 3^{r-1}/2^{2r} = 1/2^2$. By Lemma 5.4.2.1 The binary series for $r=1$ is $(2) = 1/2^2$.

Assume the proportion of $24h+1$ terms in branches with a binary series of length r is $3^{r-1}/2^{2r}$.

The proportion of $24h+1$ terms of binary series length $r+1$ is
$(1/2)(3^{r-1}/2^{2r})+(1/2^2)(3^{r-1}/2^{2r}) = (3/4)(3^{r-1}/2^{2r}) = 3^r/2^{2(r+1)}$.

***

Summing the geometric series $3^{r-1}/2^{2r}$ for length $r = 1,2,3,\dots$ gives $1/4+3/16+9/64+\dots= (1/4)/(1-3/4)=1.$

For $h=2,6,10,\dots$ the $24h+1$ branch binary series binary series is (2).
All other binary series begin with $(2,\dots)$.

Theorem 5.4.3: The proportion of $24h+7$ terms in branches with a binary series of length $r \geq 2$ is $3^{r-2}/2^{2r-2}$.

Lemma 5.4.3.1: For $h=2$ the $24h+7$ branch binary series binary series is $(1,1)$.

$24h+7 \rightarrow 72h+22 \rightarrow 36h+11 \rightarrow 108h+34 \rightarrow 54h+17 \rightarrow 108h+52.$
$55 \rightarrow 166(1) \rightarrow 83 \rightarrow 250(1) \rightarrow 125 \rightarrow 376=24(15)+16.$

For $r=2,\ 3^{r-2}/2^{2r-2} = 1/2^2$. By Lemma 5.4.3.1 The binary series for $r=2$ is $(1,1) = 1/2^2$.

Assume the proportion of $24h+7$ terms in branches with a binary series of length r is $3^{r-2}/2^{2r-2}$.

The proportion of $24h+7$ terms of binary series length $r+1$ is
$(1/2)(3^{r-2}/2^{2r-2})+(1/2^2)(3^{r-2}/2^{2r-2}) = (3/4)(3^{r-2}/2^{2r-2}) = 3^{r-1}/2^{2(r+1)-2}$.

***

Summing the geometric series $3^{r-2}/2^{2r-2}$ for length $r = 2,3,4,\dots$ gives $1/4+3/16+9/64+\dots= (1/4)/(1-3/4)=1.$

For $h=2,6,10,\dots$ the $24h+7$ branch binary series binary series is $(1,1)$.
All other binary series begin with $(1,1,\dots)$.

Collectively all $24h+1$, $24h+7$, and $24h+19$ are first terms in branches with binary series of all 2^r combinations of *1's* and *2's* for every value of r. All $24h+16$ are last terms in branch segments with binary series of all 2^r **combinations of** *1's* and *2's* for every value of r. **There are no unending branch segments.**

Section 5.5 ***24k+16* are the last terms of branches segments beginning with *24h+5, 24h+11, 24h+17* and *24h+23* with binary series of every combination of *1's* and *2's* for every value of *r*.**

Put all ***24k+16*** terms in a sequence ***24k+16, k=0,1,2,3…***

Theorem 5.5: The proportion of ***24k+16*** terms in branch segments with a binary series of length ***r*** is $2^r/3^{r+1}$.

> **Lemma 5.5.1:** The proportion of ***24k+16*** terms in branches without a binary series is ***1/3.***
>
> A branch segment with no binary series has the form: ***24h+5→72h+16 = 24(3h)+16. 24(3h)+16, h=0,1,2,… 24(0)+16, 24(3)+16, 24(6)+16 0,3,6… is 1/3*** of the terms in the sequence ***24k+16, k=0,1,2,3…***
>
> **Lemma 5.5.2:** The proportion of all ***24k+16*** terms with the same binary series of length ***r*** is $1/3^{r+1}$ of the terms in the sequence.
>
> The formula for the last term in a group of ***24h+11, 24h+17,*** and ***24h+23*** branch segments with the same binary series of length ***r*** is
> $24k+16+(p-1)(24)(3^{r+1})$ ***p=1,2,3…*** $0 \le k < 3^{r+1}$. They comprise $1/3^{r+1}$ of the terms in the ***24k+16*** sequence.

By lemma 5.5.2 The proportion of all ***24k+16*** terms with the same binary series of length ***r*** is $1/3^{r+1}$ of the terms in the sequence. There are 2^r different binary series of length ***r.*** Thus, the proportion of ***24k+16*** terms in branches with a binary series of length ***r*** is $2^r/3^{r+1}$.

***

Summing the geometric series for ***r=0,1,2,3…***gives ***1/3+2/9+4/27+…. = (1/3)(1 – 2/3) = 1.*** This accounts for all terms in the sequence ***24k+16, k=0,1,2,3…*** There are ***24h+5, 24h+11, 24h+17*** and ***24h+23*** branch segments with ***24k+16*** last terms with binary series of every combination of ***1's*** and ***2's*** for every value of ***r.***

Section 5.6 ***24h+11, 24h+17,* and *24h+23* are the first terms of branch segments with binary series of every combination of *1's* and *2's* for every value of *r*.**

There are three groups of branches whose binary series sums to ***s*:**

$$24h+11+(p-1)(24)(2^s),$$
$$24h+17+(p-1)(24)(2^s)$$
$$24h+23+(p-1)(24)(2^s),\ p=1,2,3\ldots\ 0 \le h < 2^s.$$

If all ***24h+11, 24h+17,*** and ***24h+23*** terms are put in three separate ascending sequences, terms with the same binary series occur every 2^s terms: $1/2^s$ proportion of the sequence terms. We show by induction arguments that each of ***24h+11, 24h+17,*** and ***24h+23*** have formulas using length ***r*** for the proportion of terms that are in branches with a binary series of length ***r.*** We show that collectively all ***24h+11, 24h+17,*** and ***24h+23*** terms are in branches with binary series of every combination of ***1's*** and ***2's*** for every value of ***r.***

Theorem 5.6.1: The proportion of ***24h+19*** terms in branches with a binary series length $r \ge 2$ is $3^{r-2}/2^{2r-1}$.

> **Lemma 5.6.1.1:** The first two ***24h+11*** binary series are ***(1)*** if ***h=1*** and ***(1,2)*** if ***h=8.***
>
> ***24h+11→72h+34→36h+17→108h+52.***
> For ***h=1, 35→106(1)→53→160=24(6)+16***
> For ***h=8, 203→610(1)→305→916(2)→229→688=24(28)+16***

For $r=2$, $3^{r-2}/2^{2r-1} = 1/2^3$. By Lemma 5.6.1.1 The binary series for ***r=2*** is $(1,2) = 1/2^3$.

Assume the proportion of ***24h+11*** terms in branches with a binary series of length ***r*** is $3^{r-2}/2^{2r-1}$.

The proportion of ***24h+11*** terms of binary series length ***r+1*** is
$(1/2)(3^{r-2}/2^{2r-1})+(1/2^2)(3^{r-2}/2^{2r-1}) = (3/4)(3^{r-2}/2^{2r-1}) = 3^{r-1}/2^{2(r+1)-1}$.

***

Starting with ***1/2*** for length one and summing the geometric series $3^{r-2}/2^{2r-1}$ for length ***r = 2,3,4,…*** gives

1/2+1/8+3/32+9/128+…= 1/2+(1/8)/(1-3/4)=1.

The first two ***24h+11*** binary series are ***(1)*** for ***h =1,3,5,…*** and ***(1,2)*** if ***h=8,16,24,…***

All other binary series with ***h*** **even** begin with ***(1,2,…).***

Theorem 5.6.2: The proportion of $24h+17$ terms in branches with a binary series of length r is $3^{r-1}/2^{2r}$.

Lemma 5.6.2.1: For $h=4$ the $24h+17$ branch binary series binary series is (2).

$$24h+17 \to 72h+52 \to 18h+13 \to 54h+40.$$
For $h=4,\ 113 \to 340(2) \to 85 \to 256=24(10)+16.$

For $r=1,\ 3^{r-1}/2^{2r} = 1/2^2$. By Lemma 5.6.2.1 The binary series for $r=1$ is $(2) = 1/2^2$.

Assume the proportion of $24h+17$ terms in branches with a binary series of length r is $3^{r-1}/2^{2r}$.

The proportion of $24h+17$ terms of binary series length $r+1$ is
$(1/2)(3^{r-1}/2^{2r})+(1/2^2)(3^{r-1}/2^{2r}) = (3/4)(3^{r-1}/2^{2r}) = 3^r/2^{2(r+1)}$.

***

Summing the geometric series $3^{r-1}/2^{2r}$ for length $r = 1,2,3,\ldots$ gives $1/4+3/16+9/64+\ldots= (1/4)/(1-3/4)=1$.

For $h=4,8,12,\ldots$ the $24h+17$ branch binary series binary series is (2).
All other binary series begin with $(2,\ldots)$.

Theorem 5.6.3: The proportion of $24h+23$ terms in branches with a binary series of length $r \geq 2$ is $3^{r-2}/2^{2r-2}$.

Lemma 5.6.3.1: For $h=4$ the $24h+23$ branch binary series binary series is $(1,1)$.

$$24h+23 \to 72h+70 \to 36h+35 \to 108h+106 \to 54h+53 \to 108h+160.$$
$$119 \to 358(1) \to 179 \to 538(1) \to 269 \to 808=24(33)+16.$$

For $r=2,\ 3^{r-2}/2^{2r-2} = 1/2^2$. By Lemma 5.6.3.1 The binary series for $r=2$ is $(1,1) = 1/2^2$.

Assume the proportion of $24h+23$ terms in branches with a binary series of length r is $3^{r-2}/2^{2r-2}$.

The proportion of $24h+23$ terms of binary series length $r+1$ is
$(1/2)(3^{r-2}/2^{2r-2})+(1/2^2)(3^{r-2}/2^{2r-2}) = (3/4)(3^{r-2}/2^{2r-2}) = 3^{r-1}/2^{2(r+1)-2}$.

***

Summing the geometric series $3^{r-2}/2^{2r-2}$ for length $r = 2,3,4,\ldots$ gives $1/4+3/16+9/64+\ldots= (1/4)/(1-3/4)=1$.

For $h=4,8,12,\ldots$ the $24h+23$ branch binary series binary series is $(1,1)$.
All other binary series begin with $(1,1,\ldots)$.

Collectively all $24h+11$, $24h+17$, and $24h+23$ are first terms in branches with binary series of all 2^r combinations of *1's* and *2's* for every value of r. All $24h+16$ are last terms in branch segments with binary series of all 2^r **combinations of** *1's* and *2's* for every value of r. **There are no unending branch segments.**

Section 6

The repeating binary series structure of towers.

Within a tower if the sum of r *1's* and *2's* in the binary series of a branch is s, there are three groups of branches having the same binary series. The first begins with $24h+3+(2^s)(24k+16)(4^{(x)(p-1)} - 1) / 3^{r+1}$, and ends with $(24k+16)(4^{(x)(p-1)})$, $x=3^{r+1}$, $p=1,2,3\ldots$ The other two groups that begin with $24h+9\ldots$ and $24h+15\ldots$ have the same form as $24h+3\ldots$ $r+1$ **applications of** $2j+1 \to 6j+4$ applied to $24h+3$ and its odd successors and applied to $(2^s)(24k+16)(4^{(x)(p-1)} - 1) / 3^{r+1}$ and s divisions by two applied to $72h+10$ and its even successors and applied to $(2^s)(24k+16)(4^{(x)(p-1)} - 1) / 3^r$ gives

$$(24k+16)+(24k+16)(4^{(x)(p-1)} - 1) = (24k+16)(4^{(x)(p-1)}).$$

A branch with no binary series starts with $24h+21+((24)(3h+2)+16)(4^{(3)(p-1)} - 1)/3$ and ends with
$$((24)(3h+2)+16)(4^{(3)(p-1)}).$$

Link between the formulas for branch and tower first terms.

For some t, $24h+3+(t-1)(24)(2^s) = 24h+3+(2^s)(24k+16)(4^{(x)(p-1)} - 1) / 3^{r+1}$.

For $x=3^{r+1}$ every power of three in $4^{(x)(p-1)} - 1 = (3+1)^{(x)(p-1)} - 1$ has a coefficient divisible by 3^{r+1}.
$(24k+16)(4^{(x)(p-1)} - 1) / 3^{r+1}$ is a multiple 24. The same is true for the forms beginning with $24h+9\ldots$,

24h+15…, and ***24h+21…*** Each tower's branch binary series structure is a microcosm of the total branch binary series structure**.** $4^{(x)(p-1)}$**,** $x=3^{r+1}$ replaces 3^{r+1}. In each case the last terms of tower branches with the same binary series occur in intervals of 3^{r+1}**.** $2^r/3^{r+1}$ is the proportion of the 2^r last terms of tower branches with a binary series of length ***r***.

For length $r \geq 0$ $1/3+2/9+4/27\ldots= 1$ is the total proportion.

There are tower branches with binary series of all 2^r **combinations of *r 1's* and *2's* for every value of *r*. The first branch with a binary series of length *r* comes within the first** 3^{r+1} **branches in the tower.**

Section 7

The Collatz Structure containing all positive integers is a connected structure. There are no circular or unending Collatz sequences. To prove this we need to define a new item that is a part of all Collatz sequences. An L_8 begins with a ***24k+16*** ***424*** term in a secondary tower. The Collatz algorithm is applied until the **red tower** base terms appears ***106.*** The Collatz algorithm is applied to the branch segment until a ***24k+16*** term appears (in an adjoining tower) ***160***. Thus we have an L_8**.** It has an **L** shape and joins two ***24k+16*** terms both divisable by eight. The adjoining L_8 begins with ***160.*** The Collatz algorithm is applied until a **red tower** base term ***10*** appears. The Collatz algorithm is applied to the branch segment until a ***24k+16*** term appears (in an adjoining tower) ***16***. We have reached the Trunk Tower. The process stops.

424
212
160←53← 106 *(1)*
80
40
20
16 ←5 ←10 *(1)*

A chain of adjoining L_8 moves through Collatz Structure until reaching a ***24k+16*** Trunk Tower term. An L_8 chain binary series is built from the number of divisions by two on all the **red tower** base terms in the L_8 chain. The above L_8 chain has a binary series of ***(1,1).*** The usage factor for the L_8 chain binary series is calculated by inverting the powers of two in the even factors of the **red tower** base terms. We will prove by induction that the usage factor of all L_8 chains with a binary series of **length *r*** is $3^r/4^r$. The binary series of an L_8 chain with one tower base term is ***1****,* or ***2.*** The usage factor is $1/2^1+1/2^2= 3/4$ verifying for $r = 1$.
If the **length *r*** usage factor is $3^r/4^r$, the **length *r+1*** usage factor is $(1/2)(3^r/4^r)+(1/4)(3^r/4^r) = 3^{r+1}/4^{r+1}$**.**

Every *24m+4, 24m+10, and 24m+22* red tower base term in an L_8 chain is in a branch with a first term of *24h+3, 24h+9, or 24h+15.* The sum of the geometric series of the L_8 chain binary series is $3/4+9/16+27/64+\ldots= (3/4)/(1 – 3/4)=3.$ This equals the total proportion of ***24h+3, 24h+9,*** and ***24h+15*** terms in branches. This total proportion is the sum of three geometric series, which are based on the powers of two of even factors in **red tower** base terms. The equality between the total proportion of ***24h+3, 24h+9,*** and ***24h+15*** terms in branches and the L_8 chain usage factor shows that **every *24j+4, 24j+10,* and *24j+22* tower base term in all *24h+3, 24h+9,* and *24h+15* branches appears in an L_8 chain.**

Since every L_8 chain ends in a Trunk Tower term, no L_8 chain can be part of a circular or unending Collatz sequence. Since all ***24h+3, 24h+9,*** and ***24h+15*** branches are part of some L_8 chain, the Collatz Structure containing all positive integers is a connected structure. Thus, every positive integer forms a Collatz sequence with unique terms terminating in the number one.

Appendix 1. A branch cannot have more than two consecutive even terms, and only the even terms *24m+4, 24m+10, 24m+16,* or *24m+22* are the immediate successors of odds terms.

6n+1 → 18n+4
If $n = 4j$, $18n+4 = 72j+4$ $(24m+4,\ m=3j) \rightarrow 36j+2 \rightarrow 18j+1$.
If $n = 4j+1$, $18n+4 = 72j+22$ $(24m+22,\ m=3j) \rightarrow 36j+11$.
If $n = 4j+2$, $18n+4 = 72j+40$ $(24m+16,\ m=3j+1)$ Last term in the branch.
If $n = 4j+3$, $18n+4 = 72j+58$ $(24m+10,\ m=3j+2) \rightarrow 36j+29$
6n+3 → 18n+10.
If $n = 4j$, $18n+10 = 72j+10$ $(24m+10,\ m=3j) \rightarrow 36j+5$.
If $n = 4j+1$, $18n+10 = 72j+28$ $(24m+4,\ m=3j+1) \rightarrow 36j+14 \rightarrow 18j+7$
If $n = 4j+2$, $18n+10 = 72j+46$ $(24m+22,\ m=3j+1) \rightarrow 36j+23$.
If $n = 4j+3$, $18n+10 = 72j+64$ $(24m+16,\ m=3j+2)$ Last term in the branch.
6n+5 → 18n+16.
If $n = 4j$, $18n+16 = 72j+16$ $(24m+16,\ m=3j)$ Last term in the branch.
If $n = 4j+1$, $18n+16 = 72j+34$ $(24m+10,\ m=3j+1) \rightarrow 36j+17$.
If $n = 4j+2$, $18n+16 = 72j+52$ $(24m+4,\ m=3j+2) \rightarrow 36j+26 \rightarrow 18j+13$.
If $n = 4j+3$, $18n+16 = 72j+70$ $(24m+22,\ m=3j+2) \rightarrow 36j+35$.

Appendix 2. Collatz structure details.

Groups of similar Collatz sequence segments. If a Collatz sequence segment has a first term ***a*** and a last term ***b*** with ***r***, $2j+1 \rightarrow 6j+4$ and ***s*** divisions by two, there is a series of Collatz sequence segments containing the same number of terms and the same number of adjoining L_8 of the same size and structure with a first term $a+(p-1)(24)(2^s)$ and last term $b+(p-1)(24)(3^r)$, $p=1,2,3\ldots$

The average branch binary series length: $3r=(1)(3/4)+(2)(9/16)+(3)(27/64)+\ldots$ $3r - (3)(3/4)r = 3$, $r=4$.
The binary series usage factor is three. Three lengths are being calculated. ***3/4*** is the proportion of length one. ***9/16*** of length two…Multiply the equation by ***3/4*** and subtract. $3r - (3)(3/4)r = 3/4 + 9/16 + \ldots = 3$.

The average branch binary series sum: $((2,1,1,1)+(2,2,1,1)+(2,1,1,1))/3 = (5+6+5)/3 = 5.333\ldots$
There are twice as many binary series components with one division by two ***24j+10 (1), 24j+22 (1)*** than there are components with two divisions by two ***24j+4 (2).*** Three binary series of length four with twice as many ***1's*** as ***2's*** make up the computation.

Calculating the decrease in term size for L_8 with the fewest *24k+16* terms.
$2/3\ (1 - 1/3)$ of the branches in a tower have binary series of length one or more. $4/9\ (1 - 1/3 - 2/9)$ have binary series of length two or more. The geometric series terms are increased by ***3/2*** to base the calculation on the branches that have binary series. The average length of the L_8 binary series is:
$$(1)(3/2)(2/3)+(2)(3/2)(4/9)+(3)(3/2)(8/27)+\ldots$$
$$(1+(2)(2/3)+(3)(4/9)+\ldots - (2/3)(1+(2)(2/3)+(3)(4/9)+\ldots)=1+2/3+4/9+\ldots=3 \quad (3)(3)=9$$
Adjusting the proportion of branches with binary series from three to one. ***9/3=3.***
The average L_8 binary series sum is ***(1,1,2)=4.***

1/3 of all branches have no binary series. The average number of divisions by two to reach the tower base term is ***2+4+2=2.67.*** Let ***2j+1→6j+4*** be represented by an increase of ***1.56*** multiples of two. The average decrease in L_8 term values is $-2.67 - 2 + 1.56 - 1 + 1.56 - 1 + 1.56 = -2$. The ratio between the initial ***24j+16*** term in an L_8 with minimum number of tower terms and the last ***24j+16*** term is on average ***4/1.***

A circular sequence *1→4→2→1* can be used to generate a sequence of arbitrary length with the same number and positions of *2j+1→6j+4* and divisions by two. The binary series of length *s* is *(2,2,2,…)*
$1+(2^{2s})(24)(p-1)$ is the beginning term and $1+(3^s)(24)(p-1)$ end term.
For $s=3$, $p=2$, $1537 \rightarrow 4612 \rightarrow 2306 \rightarrow 1153 \rightarrow 3460 \rightarrow 1730 \rightarrow 865 \rightarrow 2596 \rightarrow 1298 \rightarrow 649$.

***24k+16* first term sequence segments**

s=1 2 3 4 5 6 $(2^{s-1}-1)(24)+16+(p-1)(24)(2^s)$ The binary series is *(1,1,1,...)* The length $r = s - 3$.

k=0	1	3	7	15	31
2	5	11	23	47	95
4	9	19	39	79	159

first term → last term **last term formula**

16 →8 40→10 88→11 184→35 376→107 $s=1,2,3$ $8,10,11+(24)(p-1)$

64→32 136→34 280→35 528→107 1144→323 $s \geq 4$ $11 + \sum_{s=4}^{m}(24)(3^{s-4})+(24)(3^{s-3})(p-1)$

History The Collatz conjecture was made in 1937 by Lothar Collatz. Through 2017 the conjecture has been checked for all starting values up to $(87)(2^{60})$, but very little progress has been made toward proving the conjecture. Paul Erödos said about the Collatz conjecture: "Mathematics may not be ready for such problems." https://en.wikipedia.org/wiki/Collatz_conjecture

Thanks for your interest in this paper. If you wish to make comments send them to Jim Rock at collatz3106@gmail.com**.**

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