Algebraic Invariants of Gravity

Hans Detlef Hüttenbach

Abstract. Newton's mechanics is simple. His equivalence principle is simple, as is the inverse square law of gravitational force. A simple theory should have simple solutions to simple models. A system of n particles, given their initial speed and positions along with their masses, is such a simple model. Yet, solving for $n > 2$ is not simple.
This paper discusses, why that is a difficult problem and what could be done to get around that problem.

1. Problem Statement

Classical mechanics is essentially a linear, "first order" theory in which the dynamical quantities describe properties of the particles themselves, such as the law of inertia, $F = ma$, as well as energy and momentum conservation etc.
The graviational force, $F = (const)\nabla\frac{m_1 m_2}{|x_1 - x_2|}$, is the exception to that theory: it is a product of quantities, namely the mutual interaction the masses, disguised as a linear first order quantity F. That makes it complicated to even deal with a gravitational interaction of two particles, necessitating elliptic integrals, Legendre polynoms, Bessel functions, and all that, in order to derive its solutions. But it can be done, and it involves some beautiful mathematics and calculations, which explains, why it's done in physics first hand up to this day. The result is that the particles move (with their reduced masses) around the center of mass in all curves given by the intersection of a plane with a cone.
That is mathematically interesting, as it allows to describe the set of solutions through a hyperbolic, quadratic equation, namely that of the cone itself. And it straight leads to the question, if not a quadratic approach to the dynamics might be simpler to describe gravitational interaction.

2. The Cone

The picture of that cone is always that of a two-dimensional surface in three dimensions, because it is easy to visualize, but, even given the fact that one angular, cyclic coordinate can be eliminated, this is still inconvenient:
it is mathematically the product of two non-parallel intersecting lines and a circle. And the circle is well-known to yield us the conservation of angular momentum, which the Hamiltonean does then not depend on. So, let's drop it. We are then left with the two non-parallel lines in a two dimensional (Euclidean) coordinate system, which by proper scaling, transform into the diagonals of $\mathbb{R}^2$, that intersect in the origin. Let a, b denote horizontal and vertical axes. Then the original condition that the path of motion is to be the intersection of a plane with the cone reduces to the intersection of a line with the diagonals, i.e.: $a^2 - b^2 = 0$. That is an invariant, in fact *the* invariant of a mass point moving in a constant gravitational field. I am now free to scale a^2, which I choose to be the square of total energy E^2 of the system, up to an additive constant of motion. Then $E^2 - b^2 = (Const)$ is a constant of motion. And, as E is a constant of motion, E^2 is conserved, so b^2, and b must be conserved, too. Since a is now measured in units of energy, b will have to be of the same dimension. Now, in any closed system, there always exists at least one other energy quantity that must be conserved, namely the energy of the system, in which all particles are at rest with respect to eachother. Let me call it a rest system. That system generally is not uniquely defined, though: At any instance of time t_0 I can freeze the system, be it stable or not, and calulate its energy $E_{rest}(t_0)$ at this frozen state, disregarding its internal kinetic motion. Then, up to an invariant, additive constant, I get for the two-particle system

$$E^2 - E_{rest}(t_0)^2 = \pm T(t_0)^2,$$

where $T(t_0) = c|p|$ is c times the absolute value momentum of the particles' relative motion around the center of mass with reduced mass $\mu := \frac{m_1 m_2}{m_1 + m_2}$. In the following I'll laxly drop the distinction between reduced mass and mass (which means that I'll tacitly assume the moving mass to be small compared with the total mass).

Next, to get rid of the additive constant, I define the energy E to converge to zero at large spatial distances. So, the Hamiltonean function H boils down to

$$H^2(\vec{x}, \vec{p}) = E_{rest}^2 + sign(m)c^2p^2,$$

where E_{rest} includes the rest energy including all of its potential energy (which is a function of $\vec{x}$ and $sign(m)$ could be ± 1, at least in principle. And it takes a sheer convention to fix the sign to be always positive: It is agreed that the energy it takes to speed up a mass, must be positive. So, we end up with

$$H^2(\vec{x}, \vec{p}) = E_{rest}^2 + c^2p^2. \tag{2.1}$$

Now, it is generally overlooked in classical mechanics, that time inversion $\mathcal{T} : t \mapsto -t$ is equivalent to energy inversion $E \mapsto -E$: this is, because energy, that is gained along a path $\gamma : t \mapsto \gamma(t) \in \mathbb{R}^3$ from start time t_0 to end time t_1 is equivalent to an energy loss from t_1 to t_0 in the opposite time direction, and, by having E be zero outside the system, we fixed the value of a possible constant to be zero. Likewise, parity $\mathcal{P} : \vec{x} \mapsto -\vec{x}$ inverts momentum. So, the fact that the above equation is in terms of squares, just states the local conservation of energy and momentum (for each pair $(t, \vec{x})$).

Let's now see, how we get from 2.1 to the well-known 1^{st} order Hamiltonian $H(\vec{x}, \vec{p}) = mV(x) + \frac{1}{2m}p^2$:
We know that by taking the root of E^2, we can determine H or E only up to an additive constant, i.e. an invariant of the motion, and we know that the (square of the) kinetic energy is such an invariant. So, we can solve for $H^2 - E_{rest}^2 = 0$ and add add the kinetic energy later. But $H^2 - E_{rest}^2 = (H - E_{rest})(H + E_{rest})$. To get rid of one of the factors of the r.h.s. it is (for simplicity) postulated that H and E are of the same sign for all mass locations $\vec{x}$. Then $H^2 - E_{rest}^2 = 0$ if and only if $H - E_{rest} = 0$. Defining $mV(\vec{x}) := (\frac{1}{2mc^2}E_{rest}(\vec{x}))^2 - M^2c^4$, where $m > 0$ is the mass of the (moving) particle outside the center of mass with total mass M, $H^2 - (M^2 - m^2)c^4 = m^2c^4 + 2mc^2mV(\vec{x}) + p^2c^2$, so H is up to a constant equal to $\sqrt{m^2c^4 + 2mc^2mV(\vec{x}) + p^2c^2}$, which for $c \to \infty$ goes to $mc^2 + mV(\vec{x}) + \frac{1}{2m}p^2$, if only $mV(\vec{x})/c^4$ gets small for $c \to \infty$, (which we assume). This would give the desired relation up to an additive constant.
There is however at least one grain of salt into it: in order to deliver contraction, given the positivity of m, V must be negative, which will break the time-inversion symmetry of the above square equation: let's rewrite it into $H^2 - (M^2 - m^2)c^4 = m^2c^4 + W^2 + p^2c^2$, where W^2 is the square of potential energy between the location of m and the center of mass in the origin. As an additive square of the potential, introduces an additive force to the mass m. And if that is to be given by $F = -\nabla V$, for small velocities, then $dp/dt = -F$ would enforce an inverse force $+F$ for the time inverted solution, and only for $W = V \equiv 0$ time-inversion symmetry is maintained. We can however interpret all the squared additive quantities to be the product of the product of the quantity itself and its time-inverted quantity: $H^2 = H^*\dot{H}, M^2 = M^*\dot{M}, p^2 = p^*\dot{p}$, and so forth, where q^{ast} denotes the time-inversion of quantity q. All involved quantities other than W^2 are already time inversion symmetric, and $W^*\dot{W}$ now also is. So, $H^2 - W^2$ would rewrite into $H^2 - W^2 = (H^* - W^*)(H - W) = (H + W)(H - W)$, and, if we drop the time-inverted factor from $H^2 - W^2$, then we would come out with $H - W$ as the "proper square root for $H^2 - W^2$, and we would set $W := mV$ (see: **??**).

3. Stability of the 2-particle system

In order that 2-particle system at rest is to be stable, at least all forces in the system must add to zero, so that the acceleration vanishes. And if other forces than gravity were missing, then the only stable solution at rest, is within the center of mass. So it appears to be natural, to define E_{rest} to be taken as the energy of all particles is at rest at their (mutual) center mass. However, the particle generally won't stably reside in there, because quantum physics forbids it: without these addititional short ranged forces, the mass density of systems would be allowed to become unbounded, which is not what we observe. The particles will therefore align next to eachother. While the progress of this alignment itself implies an unelastic collision, which is outside the bounds of our model, it well decribes small perturbations around that stable state.

4. Gravity in a closed n-particle system

So far, we have found another dynamic invariant of the two-body problem. Albeit not a cyclic coordinate for the Lagrangian or Hamiltonian mechanics, but in terms of squares of energy: it's $E^2 - V_1^2$. And because it is a cyclic coordinate for two particles, it is for any n-particle system with gravitational interaction, which is invariant over time, such as e.g. a rock or a bar of iron:

In the above two-particle problem, the center of mass at rest could be itself a system of $n-1$ pointwise particles, of which the particles are allowed to be in motion, while the n-th particle, of which we extracted its kinetic energy, is at rest w.r.t. the center of mass. As long as the particles don't collide with this n-th particle, given that one of the moving particles, say the k-th particle does not collide with all the others, I can freeze that system at any time and extract the square of kinetic energy of the k-th particle as a constant of motion. So, if the system is such that all particles do not collide with eachother, I end up with $E^2 - E_{rest}^2 = \sum_{1\leq j\leq n} T_j^2$, where the T_j^2 is the square of the of the kinetical energy of the j-th particle (for the reduced mass).

I have some trouble, though to determine E^2 and E_{rest}^2, but the r.h.s. is a conserved quantity, and it qualifies as square of heat or temperature, T^2.

If only we could extend that relation to closed n-particle systems that allow for collisions, then we would have shown that the temperature of a closed gravitational system of particles was a dynamic invariant, which would prove that heat would be freely tranferrable from one gravitational system to another!

So, let's do that:

Mathematically, a free 1-particle system is a particle of some mass $m > 0$, that moves in time along a straight line at constant speed, an as such has a constant momentum and energy. With this, a free n-partical system can mathematically easily defined as the n-fold product of 1-particle systems.

Physically, even a 1-particle system is constrained over time t to the inner of a 3-dimensional box with elastic walls, namely the walls of the laboratory, which shield to the outside, and the mathematical line becomes a segmented path of lines with edges at the boundary, where the collisions with the box happen. Because at the collision with the boundary the ingoing angle is equal to outgoing angle and the absolute value of speed of the particle before and after collision with the box are the same, the whole path still qualifies as a line, if only taken modulo inversion $\mathcal{P}$ of the location coordinates: the outgoing line is bent parallel to the ingoing line by a suitable reflection of location coordinates: the principle of momentum conservation is equivalent to the symmetry of space-inversion. So, physically, the reflected path is dynamically equivalent to the path, that just passes straight through the boundary without any impact.

Now let's overlay n such pathes within the container. Then the paths will probably intersect. Let k paths of masses intersect in a point $\vec{x}_0$ at a given time t_0. Then, knowing the momenta of these particle paths, we know the momentum of the center of mass of the colliding particles and can transform to the system where the center of mass is at rest. In this frame all colliding particles bounce back at the collision point in opposite direction (given elasticity). That is: outgoing paths are the location inverted ingoing paths. So, again we get that momentum and energy conserving collisions are invariants, not affecting the dynamics.

In the presence of gravitational interaction, the only change to the above is the addition of a velocity-independent acceleration to the center of mass of the colliding particles, and we get the same result locally again, because gravity is invariant w.r.t. space-reflection.

What will happen, when collisions are allowed to be inelastic? Then the flux of energy within the system will not be constant in time, contradicting the assumption made that the system is to be invariant over time.

Remark 4.1 (Hamitonian of free n-particle systems). In a free n-particle system, each particle obeys the square Hamiltonian $H_k^2 = E_k^2 = m_k^2 c^4 + p_k^2 c^2$, $(1 \leq k \leq n)$, each particle living in its own, independent coordinate system. And by projection to a single coordinate system, these square Hamiltonians sum up as a vector $H^2 = \sum_k m_k^2 c^4 + p_k^2 c^2$, or, if we assume distinctness of the $\vec{x}_k$ at each time t, we can write it as a function $H^2(t, \vec{x}) = \sum_k 1_{(t,\vec{x_k})}(m_k^2 c^4 + p_k^2 c^2)$, where $1_{(t,\vec{x})}$ is the characteristic function of $(t, \vec{x})$, defined to be 1 for $(t, \vec{x_k})$ and 0 elsewhere. In either view, elastic collisions, which amount to space reflections, are symmetries, and we get symmetry problems only, if we consider the symmetry-breaking square root. In both views, H^2 not being a scalar, but a tuple of scalars, taking the square-root means taking it for each particle component, or individually for each mass point, so $H(t, \vec{x_k}) = 1_{(t,\vec{x})}(m_k^2 c^4 + p_k^2 c^2)^{1/2}$, which for $c \to \infty$ goes to $1_{(t,\vec{x})}(m_k c^2 + \frac{1}{2} m_k v^2)$. It is straight forward to extend the n-particles system to a continuum of particles and to introduce densities or square-root densities into that theory. More important seems to me to stress that this free n-particle system is nothing but the model of an ideal gas. And that is interessant: whether the particles go criss-cross and collide or go altogether in one or two directions, that is irrelevant: the heat and the pressure is the same:

In fact, starting with the n coordinate systems for the individual particles, we could have rotated and displaces each of them, until all particles were moving parallel in time. The choice of these coordinate system is essentially what is called a gauge, and the Galilei transformations then define a gauge symmetry. In particular, the notion of entropy in an ideal gas appears to get lost.

5. The mass problem

In all, we showed that for a closed gravitational system of N masses $(m_j)_{1\leq j\leq N}$, which is invariant over time, the quantity

$$E^2 - E_{rest}^2 = T^2 := \sum_j p_j^2 c^2, \tag{5.1}$$

is conserved, where E^2 is the square of its total energy, E_{rest}^2 the square of energy of the system, for which the particles' motion is frozen out at a given time t_0, and T^2, its square temperature, is sum of the squares of the kinetic energy of the particles. As above, we may deliberately restrict the system to be deviating slightly from a stable frozen system with all particles at rest.

So, given the masses and intial motions of these, we determine and extract the kinetical energy of the closed system, which leaves us a system with all particles at rest. And of this system we can then calculate its potential gravitational energy V, say. Ok, we can set $E_{rest}^2(\vec{x}_k) = E_0^2(\vec{x}_k) - m_k^2 V_k^2(\vec{x}_k)$, where and extend the above remark to include the perturbating potential $m_k V_k$, for each k then write it as a product of $E(\vec{x}_k) - m_k V_k$ and its time-inverse, and proceed as for the two-particle problem. But neither do we know E^2, nor E_{rest}^2: Because with this, we could perhaps calculate, how much of the potential energy maximally may be transformed into motion and therefore into mechanical work!
Because the gravitational potential is negative and proportional to $\frac{1}{r}$, where r is the distance of the mass to the center of mass, the potential can be minimized by rearranging the body to a densely and packed ball of radius r. But, because that potential energy is also proportional to the mass m of the body itself, we can deliberately partition that mass into fractions that all add up to the same mass m again, and the field itself, representing the potential energies of these pieces must weigh 0.
Not astonishingly, General Relativity says the same: If we extract the energies $m_k c^2$ from E_{rest} and arrange these with their kinetic energy $|p_k|\, c$ then what remains of E_{rest} is a massless field, which is to be replaced with the corresponding space-time curvature, caused by the masses m_k.

And now test it against the inert energy: We have from 5.1 above: $E^2 = E_{rest}^2 + T^2$, and because the center of mass is at rest for the total system, T^2 being only its internal kinetic energy, E^2 and not E_{rest}^2 is the energy of its rest mass! And this makes $M = \sqrt{E_{rest}^2 + T^2}/c^2$ its rest mass. So, inert and gravitational mass should differ. And the straightforward, perhaps difficult way to test that (because of heat dissipation), will be to to heat up the mass of a mathematical pendulum (preferrably shielding it by a tube in vacuum): From $F = \sqrt{m_0^2 + T^2/c^4} = -m_0 g\phi$, where g is the gravitational constant and ϕ a small angle with the vertical rest position, the period Π becomes $\Pi = \frac{\sqrt[4]{m_0^2+T^2/c^4}}{\sqrt{m_0}} 2\pi\sqrt{\frac{l}{g}}$, ($l$ being the rod's length), which raises the the period for $T = 0$ by the factor $\frac{\sqrt[4]{m_0^2+T^2/c^4}}{\sqrt{m_0}}$.

In order to get at the equivalence of inert and gravitational mass for $T > 0$, it would then be necessary to allow gravity to increase with temperature. But that will result into a temperature-dependent blue-shift of the atomic spectrum, which cannot be observed. So, the equivalence of the masses is lost.

In view of the inequality of inert and gravitational mass for $T > 0$, what can be done to align with General Relativity, is to express the inert mass in terms of curvature of space and time. That way, the masses m_k would again be extracted from the from the "frozen system", described by E^2_{rest}, and assembled with the momenta p_k. And the rest of E_{rest} would then go into the space-time curvature.

So, again there is strong evidence, that only a fraction of the total energy of a rigid body can be transformed into kinetic energy, which might appear possible from the equation $E^2 = E^2_{rest} + T^2$: If only E is to be constant, then we can lower E_{rest}, whilst increasing T^2: However, at a minimum, E_{rest} must still contain all the individual rest masses of the particles; so, not all of E_{rest} could be transformed into T.

6. Anti-Gravity

Take a step back and return to the 2-dimensional unit cone in three dimensions, the product of the diagonals with the unit circle, in which the vertical axis is taken to be the gravitational potential. Hyperplanes parallel to the horizonal plain then intersect the cone in a circle (where the potential is constant), at polar angles from $0 < \theta < \frac{\pi}{4}$ the interections are ellipes, at $\theta = \frac{\pi}{4}$ these are parabolae, and then up to $\theta \leq \frac{\pi}{2}$ the intersections are hyperbolae. Rotation around the vericle axis the covers the possible intersections in the upper halfpane. The lower halfplane, which is symmetrical, is left out: why? - because, if the upper vertical axis is to be understood to be negative (in order to gain at the concept of attraction, then the lower, inverted axis will represent a positive potential, which would represent the opposite of attraction.

The answer given in the beginning was that the fear is unfounded: The gravitational potential $V(r)$ is not defined as the potential energy of a mass m at distance r, but per mass m. Hence, the energies $mV(r)$ and $(-m) \cdot (-V(r))$ are the same in both regions.

Though, since the masses in $E^2 = E_{rest}^2 + T^2$ come in squares, allowing masses of either sign, then also $(-m) \cdot V(r)$ should be physically realizable, which would mean anti-gravity.

To get around the anti-gravity, General Relativity considers the whole physically experienciable world to live inside a closed and convexly curved space-time. Hence, all observers will just see the inside, but not the maliciously concavely curved outside.
Perhaps surprisingly, that argument overlooks the immaterial heat $T := \sqrt{T^2}$: Add heat to a sytem of masses: if not already moving (w.r.t. the center of mass), the particles will start to move around the center of mass (if not transfering that heat in an inelastic collision to other particles), and the moving ones will increase their distance to that center otherwise, in other words, the negative potential energy will become less negative. So, that heat is transformed into potential energy. No matter is known that may sustain heat: it rips even atoms apart and lifts molecular fractions as gas up in the sky (against the gravitational force), and conversely the lifted gas particles will fall down again towards that center, if their temperature becomes lower than that of their neighborhood: even Helium atoms fall down as $T \to 0$.
In terms of General Relativity: The immaterial heat is flattening out the convexly curved space-time again: In a non-expansive and non-contracting conservative system gravitational pressure and thermal expansive pressure add to zero. And that would be anti-gravity. We can even go a step further: We can define the gravitational mass as the maximal amount of heat that the system may withstand the thermal expansion (and vice versa).

7. Electrical mass

All that is simply 19^{th} century chemistry. Let's proceed with it: It is currently commonly said that the driving motor for the industrialization of mid 18^{th} till end of the 19^{th} century was the steam engine: I beg to differ: energy consumption would not be possible without energy delivery. And it was chemistry that showed how to get more out of matter than just gravitational energy, like throwing pebbles or pouring water downhill:
Take a chunk of coal, coke it for a day say, i.e.: add heat to it, until you get a porous solid piece of carbon (with other fluid an gaseous substances). Then pour the same amount of liquid O_2 over the carbon and ignite that immersion. The result will be gaseous carbon dioxide and millions of times more heat than could be gained from its gravitational energy: any gas already steadily expands, if not constrained by an external force. According to chemistry, that enormous amount of energy came from the binding energy that was released by the reaction $C + O_2 \to CO_2$.

This opens the next layer of $E^2 = E_{rest}^2 + T^2$: Because the force between charges is proportional to the graviational force, they still obey that equation. The only difference up to a constant, big factor is that charges are anti-symmetrical w.r.t. parity $\mathcal{P}$, whereas neutral masses are symmetrical w.r.t. parity. So, given the symmetry of time-inversion $\mathcal{T}$, the product $\mathcal{C} := \mathcal{T}\mathcal{P}$ is to be an anti-symmetry for charges, which is the charge inversion (see e.g. [1]). (Again, it is nothing new: Maxwell formulated his equations in terms of Hamilton's quaternions, which are mathematically equivalent to spin matrices; and he did that in view of the fact that the magnetic field of a flux of charges inverts under parity.) Today we also know that all particles that come with an a non-zero gravitational mass, are composed of charged particles.

Two unanswered problems for experimental physics: Is the weight of the gas CO_2 equal to the weight of O_2 plus the weight of carbon before? And the second: we know that a single electron has an inert mass greater zero. Does it also have a gravitational mass?

For now, missing experimental verification, what remains is to deduce the answers theoretically: Because the total energy, the total energy of all particles at rest, and the heat of a closed system of masses is to be the same as that of its charged particle composites, the heat or thermal energy created by the chemical process $O_2 + C \to CO_2$ must come from E_{rest}^2, which therefore decreases: a weight loss is to be expected on side of CO_2. Since electrical charges are supposed to be elementary and conserved - note that only the squares of the charges enter E_{rest}^2, so it's the square of charges that is conserved first hand, and from that the charge conservation is derived - charges cannot be converted into thermal energy, and one could expect the electron to weigh nothing. Gravitational energy would hence be conceivable as being caused by the chemical/electromagnetic binding of charged particles.

References

[1] H. D. Hüttenbach, *From Bernoulli to Laplace and Beyond*, `http://vixra.org/abs/1801.0025`, 2018.

Hans Detlef Hüttenbach