A new look inside the theory of the linear approximation: Gravity assists and Flybys

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Abstract

Whitehead's description of the gravitational field of a static point mass source is equivalent to Schwarzshild's solution of Einstein's equations. Conveniently generalized in the framework of Special relativity, I proved that it leads to a new description of the linear approximation of General relativity with a color gage group symmetry. Here I introduce a new line of thought to discuss the problem of spacecrafts orbiting a planet taking into account its motion around the Sun or its proper rotation.

1 Introduction

In 1922 Whitehead [1] presented a theory of gravity that was supposed to be a possible alternative to Einstein's theory of General relativity. It was simpler, being a theory in the framework of Special relativity, and predicted the correct values for the advance of the peri-helium of Mercury and the deviation of light by the Sun. The reason for that is that both theories propose in particular the same exact description of the gravitational field of a static point mass source. More restrictive but also more generally it can be said that the two theories coincide in describing stationary weak fields. Beyond this approximation, as discussed in [3], Whitehead's theory is not a match to Einstein's theory.

Despite of this I believe that Whitehead's approach to gravity is remarkable because in the simplest possible way it assumes from the very beginning that gravitation is a retarded interaction, and with very few steps Newton's theory follows thus solving the action at a distance problem that so much bothered Newton himself and so many people since then.

A few years ago, [4], I generalized Whitehead's formalism in a way that it describes gravity as a true gage theory, this gage being distinguished from coordinate changes that so much obscure metrology problems in General relativity, while remaining equivalent to it at the linear approximation in vacuum, and predicting the same observational values to the classical tests. The formalism is remarkable also in the sense that the velocity of a source point mass is explicit in the exterior solution and this paper makes an essential use of this feature. Also it is worthwhile mentioning the pedagogical interest that it has avoiding to the beginner the most difficult aspects of General relativity.

The purpose of this paper is to use this generalized Whitehead formalism to discuss qualitatively a genuine relativistic effect that might have something to say about what in [5] is described as "Anomalous Orbital-Energy Changes

Observed during Spacecraft Flybys of the Earth", as a "Puzzle" in [6], as a "Paradox" in [7], as "Unexpected" in [8], and as a suggestion that gravitation is a retarded interaction. [9].

The principal character of the play is a spacecraft of negligible mass compared to the mass of the planet, coming in and going out of stage. The planet has a negligible mass compared to that of the Sun and is peacefully orbiting it as always does. And the Sun is there as a standard of rest. The essence of the result that I derive is the following: if the Sun is taken to be the standard of rest then the system planet-Spacecraft can not be simplified assuming that the planet is at rest and the Sun is moving. Similarly, if the center of the planet is the standard of rest and the planet is rotating then the linear velocity of each element of mass can not be ignored. In both cases this introduces a relativistic effect of order $1/c$ while the classical relativistic tests are of order $1/c^2$

2 Extended Whitehead formalism

Let us consider, in the frame-work of special relativity, an idealized point source, a planet in this case, moving around the Sun with a velocity $v^i,\ i = 1,2,3$ with respect to a galilean frame of reference where the sun is at rest. Let $\hat{x}^\alpha, \alpha, \dots = 0,1,2,3$ be a particular event in the trajectory of the planet and x^α an event in the future of $\hat{x}^\alpha$, and let u^α be the unit 4-velocity of the planet $u^\alpha u_\alpha = -1$. In [4] I introduced the symmetric tensor:

$$g_{\beta\lambda} = \eta_{\beta\lambda} + h_{\beta\lambda}, \quad h_{\beta\lambda} = \frac{1}{r}\left(A_0 u_\beta u_\lambda + A_1 \eta_{\beta\lambda} + A_2(l_\beta u_\lambda + l_\lambda u_\beta) - A_3 l_\lambda l_\beta\right) \tag{1}$$

where:

$$l^\alpha = \frac{L^\alpha}{r}, \quad r = -u_\alpha L^\alpha, \quad L^\alpha = x^\alpha - \hat{x}^\alpha \tag{2}$$

and A_0, A_1, A_2, A_3 are for the time being four constants, to be latter chosen proportional to the mass of the Sun or the planet depending on the problem to be considered.

Whitehead [1] and later Synge [2] introduced directly the following particular case:

$$g_{\alpha\beta} = \eta_{\alpha\beta} - \frac{2m}{r} l_\alpha l_\beta. \tag{3}$$

while I decided to use Einstein's vacuum equations to first order:

$$R_{\alpha\beta} = 0 \tag{4}$$

that turns out to be equivalent to [1]:

$$A_0 = 2(A_1 - A_2) \tag{5}$$

Let us consider a time-like world-line with parametric equations $\hat{x}^\alpha = \hat{x}^\alpha(\tau)$ and let u^α be its future pointing unit vector. Let x^α be an event in the future of an event x^α corresponding to any particular value of the proper time τ. By

[1] Complete details in the Appendix

definition, a concomitant variation of x^α, $x^\alpha + \delta x^\alpha$, and τ, $\tau + \delta\tau$ is causal if $x^\alpha + \delta x^\alpha$ is in the future of $\hat{x}^\alpha + u^\alpha \delta\tau$. And for this to happen the condition is:

$$\delta\tau = -\frac{1}{r} L_\beta \delta x^\beta, \quad \delta\hat{x}^\alpha = u^\alpha \delta\tau \tag{6}$$

Let us now consider any function $f(L^\alpha)$ and its causal variation:

$$\delta f = \frac{\partial f}{\partial x^\alpha}\delta x^\alpha - \frac{\partial f}{\partial \hat{x}^\alpha}\delta\hat{x}^\alpha \tag{7}$$

Since the two partial derivatives are equal the preceding result can be written:

$$\delta f = \hat{\partial}_\alpha f \delta x^\alpha \quad \text{with} \quad \hat{\partial}_\alpha = \left(\delta^\beta_\alpha + \frac{1}{r} u^\beta L_\alpha\right)\partial_\beta \tag{8}$$

The substitution $\partial_\alpha \to \hat{\partial}_\alpha$ is instrumental in the development of Whitehead's formalism and the reader is invited to muse about it.

To start with we get:

$$\hat{\partial}_\alpha u_\beta = 0, \quad \hat{\partial}_\alpha r = -u_\alpha + l_\alpha, \quad \hat{\partial}_\alpha l_\beta = \frac{1}{r}(\eta_{\alpha\beta} + u_\alpha l_\beta + l_\alpha u_\beta - l_\alpha l_\beta) \tag{9}$$

Proceeding now with the first derivatives of the $h's$ we obtain:

$$\begin{aligned} \hat{\partial}_\alpha h_{\beta\lambda} = &\frac{A_1}{r^2}(2u_\beta u_\lambda + \eta_{\beta\lambda})(-l_\alpha + u_\alpha) \\ &+ \frac{2A_2}{r^2}\left((l_\beta u_\lambda + u_\beta l_\lambda)(-l_\alpha + u_\alpha) + 2u_\beta u_\lambda(2l_\alpha - u_\alpha) + (\eta_{\alpha\beta}u_\lambda + \eta_{\alpha\lambda}u_\beta)\right) \\ &+ \frac{A_3}{r^2}\left((3l_\beta l_\lambda - l_\beta u_\lambda - l_\lambda u_\beta)l_\alpha - 3l_\beta l_\lambda u_\alpha - \eta_{\alpha\lambda}l_\beta - \eta_{\alpha\beta}l_\lambda\right) \end{aligned} \tag{10}$$

from where we get directly the Christoffel symbols of the connection to first order with the corresponding substitutions:

$$\hat{\Gamma}^\mu_{\alpha\beta} = \eta^{\mu\lambda}\hat{\Gamma}_{\alpha\beta\lambda}, \quad \hat{\Gamma}_{\alpha\beta\lambda} = \frac{1}{2}(\hat{\partial}_\alpha h_{\beta\lambda} + \hat{\partial}_\beta h_{\alpha\lambda} - \hat{\partial}_\lambda h_{\alpha\beta}) \tag{11}$$

The equations of motion for a test particle with unit 4-velocity w^μ to be considered are then:

$$\frac{dw^\mu}{d\sigma} = -\hat{\Gamma}^\mu_{\alpha\beta} w^\alpha w^\beta \equiv f^\mu \tag{12}$$

where σ is the proper time along its trajectory and f^μ is:

$$\begin{aligned} f^\mu = &-\frac{A_1}{r^2}\left(\left(uw^2 - \frac{1}{2}\right) l^\mu + \left(-2lw\, uw + uw^2 + \frac{1}{2}\right) u^\mu + (uw - lw)w^\mu\right) \\ &- \frac{A_2}{r^2}(-2lw^2 + 4lw\, uw - uw^2 - 1)u^\mu \\ &- \frac{A_3}{r^2}\left(\left(\frac{3}{2}lw^2 - 3lw\, uw + 1\right) l^\mu + \frac{1}{2}lw^2 u^\mu\right) \end{aligned} \tag{13}$$

where:

$$lw = l_\alpha w^\alpha, \quad uw = u_\alpha w^\alpha \tag{14}$$

3 The low velocities approximation

In this section I assume that all the quantities u^i and w^i are small quantities and neglect any product of small quantities. And therefore I shall have

$$u^0 = 1, \quad w^0 = 1 \tag{15}$$

an approximation that means that the proper-time of the source of the gravitational field as well as the proper-time of the test particle flow at the same pace as that of the underlying Lorentz frame corresponding to the dominant standard of rest.

Using (12), a straightforward simple calculation leads to the following equations of motion of the test particle:

$$\frac{dw^i}{dt} = -\frac{A1-A3}{2L^3}L^i + \frac{A1-A3}{2c\,L^2}\left(u^i - \frac{3Lu\,L^i}{L^2}\right), \quad L^2 = L^iL_i, \quad Lu = L^iu_i \tag{16}$$

where:

$$L^i = x^i - \hat{x}^i(t), \quad u^i = d\hat{x}^i(t)/dt \tag{17}$$

and where I have included c in arbitrary units to have the formula dimensionally correct. The first term is the Newtonian term with the identification:

$$A1 - A3 = 2Gm \tag{18}$$

It is the purpose of this paper to point out that the velocity dependent term of the right-hand-side of (16) might have something to say about gravity assists and the "Anomalous Orbital-Energy Changes Observed during Spacecraft Flybys of Earth", [5]. To show it I consider the Newtonian orbital energy of the test particle:

$$H = \frac{1}{2}w^iw_i - Gm/L \tag{19}$$

Differentiating H with respect to t and keeping products of pairs of small quantities yields the simple formula:

$$\frac{dH}{dt} = \frac{Gm}{cL^2}\left(uw - 3\frac{LuLw}{L^2}\right), \quad Lw = L^iw_i, \; uw = u^iw_i, \tag{20}$$

Noteworthy is the fact that contrary to the classical tests that are of order $1/c^2$ the second term of the r-h-s is of order $1/c$.

4 Rotating planets

This section examines the case where an aircraft byflies a planet, assuming that the velocity of the center of mass of this planet with respect to the Sun can be neglected but not its proper rotation, i.e. the linear velocity of its elements of mass.

To coordinate points exteriors to the planet I consider a system of coordinates with origin at the center of the planet where α is the right ascension and

δ is the declination. To coordinate elements of mass $\delta m = \rho r^2 \sin\theta$, with ρ constant, I consider polar spherical coordinates r, θ, ϕ such that $\phi = 0$ is the same plane as $\alpha = 0$.

I now consider a point of space E at a distance D from the center of the planet, with right ascension $\alpha = 0$, and declination δ. And correspondingly I consider an element of mass δm of the planet located at a point P with coordinates r, θ, ϕ. If the planet is rotating with angular velocity ω the corresponding element of mass will have a linear velocity with components:

$$u^1 = -r\sin(\theta)\sin(\phi)\omega,\ u^2 = r\sin(\theta)\cos(\phi)\omega,\ u^3 = 0. \tag{21}$$

So that the components of the vector with origin P and extremity E are:

$$L^1 = D\cos(\delta) - r\sin(\theta)\cos(\phi),\ L^2 = -r\sin(\theta)\sin(\phi),\ L^3 = D\sin(\delta) - r\cos(\theta) \tag{22}$$

With these data and (16) we can calculate the components of the force that each element δm of mass of the planet exerts on the aircraft. Assuming that D is larger than the radius R of the planet the leading terms of f^i can be approximated as follows:

$$f^1 = \frac{-G\cos\delta}{D^2} + \frac{2Gr\omega\sin\theta\sin\phi(3\cos^2\delta - 1)}{cD^2} \tag{23}$$

$$f^2 = \frac{Gr\sin\theta\sin\phi}{D^3} + \tag{24}$$

$$\frac{2G\sin\theta(5\sin\theta\cos^2\phi - 3\sin\theta)r^2\omega\cos\delta}{cD^3} + \frac{4G\sin\theta r^2\omega\sin\delta\cos\phi\cos\theta}{cD^3} \tag{25}$$

$$f^3 = \frac{-G\sin\delta}{D^2} + \frac{6Gr\omega\sin\theta\sin\phi\cos\delta\sin\delta}{cD^2} \tag{26}$$

The components of the force acting on a aircraft

$$F^i = \int_V \rho f^i r^2 \sin(\theta) dr d\theta d\phi \tag{27}$$

that in the particular case in which it is located on the plane corresponding to right-ascension $\alpha = 0$ they are:

$$F^1 = -\frac{GM}{D^2}\cos\delta,\ F^2 = -\frac{GM}{D^2}\left(\frac{\omega R^2}{5Dc}\right)\cos\delta,\ F^3 = -\frac{GM}{D^2}\sin\delta \tag{28}$$

where M is the total mass of the planet, and R is its radius. F^1 and F^3 are pure Newtonian contributions. F^2 instead is due to the rotation of the planet.

More generally, the system of differential equations governing the evolution of the aircraft are:

$$F^1 = -\frac{GM}{D^3}x^1 + \frac{GMR^2\omega}{5cD^4}x^2 P(r/D) \quad (29)$$

$$F^2 = -\frac{GM}{D^3}x^2 - \frac{GMR^2\omega}{5cD^4}x^1 P(r/D) \quad (30)$$

$$F^3 = -\frac{GM}{D^3}x^3 \quad (31)$$

and $P(r/D)$ is a function whose three first terms of its series expansion are:

$$P(r/D) = 1 + \frac{2}{7}\frac{R^2}{D^2} + \frac{1}{7}\frac{R^4}{D^4} \quad (32)$$

Notice also that for most planets, including the Earth, ω is a negative parameter.

The advance of the perihelion of Mercury, while puzzling several generations of astronomers, was never considered to be a paradox, but as a fact to be explained by a new theory or other as yet unknown facts.

Gravity assists have been observed and used extensively to lead humanity to the gorgeous wonder of space flights. Byflys of aircrafts around the Earth are observed and like gravity assists are considered paradoxical and sometimes anomalous .

What this paper claims is that while akin and inspired from General Relativity a different but simpler theory, without the dubious principle of general covariance, can fill a missing link between a theory of gravitation and the era of space-flights. If facts support it.

Appendix

These are the formulas that prove the two fundamental implications:

$$A_0 - 2(A_1 - A_2) = 0 \Rightarrow R_{\alpha\beta} = 0 \quad (33)$$

$$A_0 - 2(A_1 - A_2) = 0 \text{ and } A_1 - A_3 = 0 \Rightarrow R_{\alpha\lambda\beta\mu} = 0 \quad (34)$$

Defining:

$$R_{\alpha\lambda\beta\mu} = -\frac{1}{2}(\hat{\partial}_{\alpha\beta}h_{\lambda\mu} + \hat{\partial}_{\lambda\mu}h_{\alpha\beta} - \hat{\partial}_{\alpha\mu}h_{\lambda\beta} - \hat{\partial}_{\lambda\beta}h_{\alpha\mu}) \quad (35)$$

$$R_{\alpha\beta} = \eta^{\lambda\mu}R_{\alpha\lambda\beta\mu} \quad (36)$$

and:

$$X_{\alpha\lambda\beta\mu} = \frac{3}{2}(l_\alpha u_\beta + u_\alpha l_\beta - l_\alpha l_\beta)\eta_{\lambda\mu} \quad (37)$$

The Riemann tensor can be decomposed as:

$$R_{\alpha\lambda\beta\mu} = A_0 R^0_{\alpha\lambda\beta\mu} + A_1 R^1_{\alpha\lambda\beta\mu} + A_2 R^2_{\alpha\lambda\beta\mu} + A_3 R^3_{\alpha\lambda\beta\mu} \quad (38)$$

where:

$$R^0_{\alpha\lambda\beta\mu} = -\frac{3}{2r^3}(l_\alpha u_\lambda - u_\alpha l_\lambda)(l_\beta u_\mu - u_\beta l_\mu) \tag{39}$$

$$R^3_{\alpha\lambda\beta\mu} = \frac{1}{r^3}(-X_{\alpha\lambda\beta\mu} - X_{\lambda\mu\alpha\beta} + X_{\alpha\mu\lambda\beta} + X_{\lambda\beta\alpha\mu} + \eta_{\beta\lambda}\eta_{\alpha\mu} - \eta_{\mu\lambda}\eta_{\alpha\beta}$$

$$+3(l_\alpha l_\beta u_\lambda u_\mu + u_\alpha u_\beta l_\lambda l_\mu - l_\alpha u_\beta u_\lambda l_\mu - u_\alpha l_\beta l_\lambda u_\mu) \tag{40}$$

$$R^1_{\alpha\lambda\beta\mu} = -R^3_{\alpha\lambda\beta\mu} - 2R^0_{\alpha\lambda\beta\mu} \tag{41}$$

$$R^2_{\alpha\lambda\beta\mu} = 2R^0_{\alpha\lambda\beta\mu} \tag{42}$$

and the Ricci tensor is:

$$R_{\alpha\beta} = -\frac{1}{2}(A_0 - 2(A_1 - A_2))\frac{1}{r^3}(\eta_{\alpha\beta} - 3l_\alpha l_\beta - 2u_\alpha u_\beta + 3(l_\alpha u_\beta + u_\alpha l_\beta)) \tag{43}$$

Notice that the scalar curvature is always zero:

$$R = \eta^{\alpha\beta}R_{\alpha\beta} = 0 \tag{44}$$

But this is irrelevant in the context in which the formalism has been used in the main body of the paper.

The implication (33) is obvious from the above expression of the Ricci tensor. Assuming now the first assumption of (34) in (38) and using the definitions (39)-(40) above we get:

$$R_{\alpha\lambda\beta\mu} = -(A_1 - A_3)R^3_{\alpha\lambda\beta\mu} \tag{45}$$

that proves (34).

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