A parametric equation of the equation
\[ a^5 + b^5 = 2c^2 \]

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Abstract

The equation \( a^5 + b^5 = c^2 \) has no solution in integer. However, related to Fermat-Catalan conjecture, the equation \( a^5 + b^5 = 2c^2 \) has a solution in integer. In this article, we give a parametric equation of the equation \( a^5 + b^5 = 2c^2 \)

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The parametric equations of equation above:

\[ a = \frac{5s^4t^4[10s^2t^2 - (5s^4 + t^4)]^2}{4} - \frac{[5s^4 - t^4]^4}{4} + \frac{5s^4 - \frac{10s^2t^2 - (5s^4 + t^4)}{4}}{2} \sqrt{\frac{10s^2t^2 - (5s^4 + t^4)}{4}} \frac{5s^4 - t^4}{4} \]

\[ b = \frac{5s^4t^4[10s^2t^2 - (5s^4 + t^4)]^2}{4} - \frac{[5s^4 - t^4]^4}{4} - \frac{5s^4 - \frac{10s^2t^2 - (5s^4 + t^4)}{4}}{2} \sqrt{\frac{10s^2t^2 - (5s^4 + t^4)}{4}} \frac{5s^4 - t^4}{4} \]

\[ c = c_1c_2 \]

\[ c_1 = \pm \sqrt{\left| \frac{5s^4t^4[10s^2t^2 - (5s^4 + t^4)]^2}{4} - \frac{[5s^4 - t^4]^4}{4} \right|} \]

\[ c_2 = \pm 16\left[ \frac{5s^4t^4 - \frac{10s^2t^2 - (5s^4 + t^4)}{4}}{4} \right]^4 + 5s^4t^4\left[ \frac{10s^2t^2 - (5t^4 + s^4)}{4} \right]^2 \frac{5s^4 - t^4}{4} \]

s and t are odd coprime.

The smallest solution is that \( s = 1, t = 1 \) then \( a = 3, b = -1, c = \pm 11 \)
\( 3^5 - 1 = 211^2 \)
or \( 122^2 - 11^4 = 3^5, \) [1]

Basing on exponent of the parametric equations above, if the equation \( a^5 + b^5 = 2c^2 \) have other solutions, they must be large, even very large.
Note that, the equation \( \frac{10s^2t^2 - (5s^4 + t^4)}{4} = r^2 \) and the equation \( 5x^4 - y^4 = z^2 \) (general expression of \( c_1^2 \)) have many infinite solutions in integer.

Ex.

For \( \frac{10s^2t^2 - (5s^4 + t^4)}{4} = r^2 \)
\( s = 5, \ t = 7, \ r = 41 \)

For \( 5x^4 - y^4 = z^2 \)
\( x = 13, \ y = 11, \ z = 358 \)

Related to the Fermat - Catalan conjecture, the equations below .

\( a^5 + b^5 = c^2 \) has no solution in integer.

\( a^8 - b^2 = c^3 \) (\( a^2 - b^8 = c^3 \)) has no solution in integer with \( c \) odd .

If \( c \) is even , it is known : \( 30042907^2 - 43^8 = 962223^3 \) [1]

References

1. Fermat- Catalan conjecture , Wikipedia
2. Quang N V, Theorem for \( W^n \) and Fermat’s Last theorem Vixra:1811.0072 v2(NT)
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