Relation between
The Euler Totient,
the counting prime formula
and the prime generating Functions

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1/ Introduction:

The theory of numbers is an area of mathematics which deals with the properties of whole and rational numbers... In this paper I will introduce relation between Euler phi function and prime counting and generating formula, as well as a concept of the possible operations we can use with them. There are four propositions which are mentioned in this paper and I have used the definitions of these arithmetical functions and some Lemmas which reflect their properties, in order to prove them.

2/ Definitions:

Here are some definitions to illustrate how the functions work and describe some of their most useful properties.

The Euler totient:

The Euler totient function is defined to be the number of positive integers which are less or equal to an integer and are relatively prime to that integer: for \( n \geq 1 \), the Euler totient \( \varphi(n) \) is:

\[
\varphi(n) = \sum_{k=1}^{n} 1
\]

There is a formula for the divisor sum which is one of the most useful properties of the Euler totient:

Lemma:

for \( n \geq 1 \) we have \( \sum_{d \mid n} \varphi(d) = n \). Since the Euler totient is the number of positive integers relatively prime to \( n \).

\[
\sum_{d \mid n} \varphi(d) = \sum_{P \in \mathbb{P}} \varphi(P) + \sum_{Q \in \mathbb{P}^c} \varphi(Q) = n \quad ; \quad (1)
\]

\[
d = \sum p + \sum q, P \in \mathbb{P}, Q \in \mathbb{P}^c, Q \in \mathbb{N} - |P|
\]

\[
\sum \varphi(p) = \sum p - 1 \quad ; (2)
\]

From (1) & (2):
\[
\sum_{d|n} \phi(d) = \sum_{p \leq n} p - 1 + \sum_{\emptyset(Q)} = n
\]

And \[\pi(n) = \sum_{p \leq n} 1 = \sum_{p \leq n} 1 + \sum_{\emptyset(Q)} \]
(symbol // p is not divisor of n)

\[
\sum_{p \leq n} p - \sum_{p \leq n} 1 + \sum_{\emptyset(Q)} = \sum_{p \leq n} p - \sum_{p \leq n} 1 + \sum_{\emptyset(Q)}
\]

\[
\sum_{p \leq n} p - \pi(n) + \sum_{\emptyset(Q)} = n
\]

So:

\[
\pi(n) = \sum_{p \leq n} p + \sum_{\emptyset(Q)} - n
\]

Exp:

\[
\pi(9) = \sum_{p \leq 9} p + \sum_{\emptyset(Q)} - 9
\]

and divisors of 9: \{1,3,9\} and 3 is prime, there 3 primes are not divisors of 9: \{2,5,7\} ≤ 9

= 3 + 3 + \phi(1) + \phi(9) - 9 = 3 + 3 + 1 + 6 - 9 = 4

And there are 4 primes less than 9: \{2,3,5,7\}