Proof of the Twin Prime Conjecture

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Abstract: Let P_n be the n_th prime. For twin primes $P_n - P_{n-1} = 2$. We exhibit formula for calculating the number of twin primes in the closed interval $[P_n, P_n^2]$. We prove that as P_n increases the number of twin primes in the interval $[P_n, P_n^2]$ also increases.

Introduction: All primes greater than or equal to five are of the form $6j - 1$ or $6j + 1$.
$m = 1$ to n $\prod P_m = J_n$ is the product of the first n primes. The number of $(6j - 1, 6j + 1)$ pairs with no factor less than P_{n+1} in the closed interval $[1, J_n+1]$ is exactly $(1/6)(3/5)(5/7)\ldots((P_n - 2)/P_n)(J_n)$.

Closely related to this is the number of $(6j - 1, 6j + 1)$ pairs in $[P_n, P_n^2]$ with no factor less than P_{n+1}. They are all twin primes. Let X be the number of $(6j - 1, 6j+1)$ pairs in the interval $[P_n, P_n^2]$.

Using the formula for the number of $(6j - 1, 6j + 1)$ pairs with no factor less than P_{n+1} in the closed interval $[1, J_n+1]$ as a standard, the number of twin primes in $[P_n, P_n^2]$ can be approximated by the formula

$$(a_3/5)(a_4/7)(a_5/11)\ldots(a_n/P_n)(X) \text{ for } 3 \le m \le n,\ a_m = P_m - 2.$$

Table 1 shows the values of $3 \le m \le n$, a_m for twin primes in $[743, 743^2]$ and $[19993, 19993^2]$.

Table 2 shows the actual number of twin primes (TPA_n) versus the calculated number (TPC_n) in $[P_n, P_n^2]$ for $347 \le n \le 31153$.

The number of twin primes is always greater than $(3/5)(5/7)(7/9)\ldots(P_n-2)/P_n)(X) = 3X/P_n$.

Let $P_n - P_{n-1} = c$. Table 3 shows that for all n

$$(TPA_{n-1})(1+(2c - 2)/2P_{n-1}+(c^2-2c)/2P_{n-1}^2) < TPA_n$$

Section 1

Calculating the number of $(6j - 1, 6j + 1)$ pairs (F_n) with no factor $< P_{n+1}$ in $[1, J_n+1]$.

For each $(6j - 1, 6j + 1)$ pair with no factor less than P_n in $[1, J_{n-1}+1]$ there are pairs $(6j - 1 + mJ_{n-1}, 6j+1+mJ_{n-1})$ for $m = 0$ to $P_n - 1$ in $[1, J_n+1]$. P_n and J_{n-1} are relatively prime. Thus, P_n divides $6j - 1 + mJ_{n-1}$ and $6j + 1 + mJ_{n-1}$ each for exactly one different value of m.

$P_3=5$, $P_4=7$. $\underline{F_3 = (1/6)(3/5)(J_3)}$. $F_4 = (5)(F_3)$. $J_4 = (7)(J_3)$ $F_4 / F_3 = (5/7)(J_4 / J_3)$. $\underline{F_4 = (1/6)(3/5)(5/7)(J_4)}$.

The number of $(6j - 1, 6j + 1)$ pairs with no factor less than P_{n+1} in the interval $[1, J_n+1]$ is exactly $(1/6)(3/5)(5/7)\ldots((P_n - 2)/P_n)(J_n)$.

This occurs because J_n is divisible by all primes in the interval $[P_3, P_n]$.
All the $(6j - 1, 6j+1)$ pairs with no factor less than P_{n+1} in which $6j < P_{n+1}^2$ are twin primes.

Determining the number of twin primes pairs (TPA_n) in the closed interval $[P_n, P_n^2]$.

Let X be the number of $(6j - 1, 6j+1)$ pairs in the interval $[P_n, P_n^2]$. The number of twin prime pairs in $[P_n, P_n^2]$ is $(a_3/5)(a_4/7)(a_5/11)\ldots(a_n/P_n)(X)$. For $3 \le m \le n$ $P_m - 4 < a_m \le P_m - 0$.

The absolute value of $P_m - 2$ for the number of pairs in $[1, J_m+1]$ with no factor less than P_{m+1} sets a range for the possible values of a_m at $P_m - (2 + 2) < a_m \le P_m - (2 - 2)$.

Table 1 illustrates this formula for the intervals $[743, 743^2]$ and $[19993, 19993^2]$. For selected P_m it shows the actual number $(6j - 1, 6j+1)$ pairs with no factor less than P_{m+1} and the value of a_m $(P_m - a)$. Filtering out factors less than P_{m+1} is a linear process. Each P_m divides $6j - 1$ and $6j+1$ terms at regular intervals within $[P_n, P_n^2]$. Thus, simular a $(P_m - a)$ patterns are found in all $[P_n, P_n^2]$ intervals of primes greater than 500.

Table 2 shows the number of twin primes calculated (TPC_n) in $[P_n, P_n^2]$ for $347 \leq n \leq 31153$, when a_m equals $P_m - 2.04$, $P_m - 2.06$, and $P_m - 2.08$ for $3 \leq m \leq n$. Comparing TPC_n with TPA_n shows the average value for $a_m = P_m - a$ starts out at approximately $P_m - 2.02$ for $n = 347$ and gradually decreases to slightly less than $P_m - 2.06$ for $n = 31153$ (see the ratio *TPA / 2.06*). There are proportionally fewer twin prime pairs in $[P_n, P_n^2]$ than there are $(6j - 1, 6j + 1)$ pairs with no factors less than P_{n+1} in $[1, J_n+1]$. The number of twin prime pairs in $[P_n, P_n^2]$ is always greater than $(3/5)(5/7)(7/9)\ldots(P_n - 2)/P_n)(X) = 3X/P_n$, where X is the number of $(6j - 1, 6j+1)$ pairs in $[P_n, P_n^2]$.

Section 2

Establishing a lower bound for the ratio TPA_n / TPA_{n-1}

The number of twin primes in the interval $[P_n, P_n^2]$ is $(a_3/5)(a_4/7)(a_5/11)\ldots(a_n/P_n)(X)$

$m = 1$ *to* $n \prod P_m = J_n$. Using $(1/6)(3/5)(5/7)\ldots((P_n - 2)/P_n)(J_n)$ the number of $(6j - 1, 6j + 1)$ pairs with no factor less than P_{n+1} in the closed interval $[1, J_n+1]$ as a standard, the average value of a_m, $3 \leq m \leq n$ can be approximated by $P_m - 2$. This can be verified by comparing TPC_n with TPA_n in table 2. X is close to $(P_n^2 - P_n)/6$. $m = 3$ *to* $n \prod P_m - 2 = F_n$

The number of twin prime pairs in $[P_n, P_n^2]$ is approximately $(F_n)(P_n^2) / J_n$

TPA_n is approximately $(TPA_{n-1})((F_n)(P_n^2) / J_n) / ((F_{n-1})(P_{n-1})^2 / J_{n-1})$.

TPA_n is greater than $(TPA_{n-1})(((F_n)(P_n^2) / J_n) / ((F_{n-1})(P_{n-1})^2 / J_{n-1})) + 1)/2$.

Calculating $(((F_n)(P_n^2) / J_n) / ((F_{n-1})(P_{n-1})^2 / J_{n-1})) + 1)/2$.

Let $P_n - P_{n-1} = c$.

$$((F_n)(P_n^2) / J_n) / ((F_{n-1})(P_{n-1})^2 / J_{n-1}) =$$

$$((F_{n-1})(P_{n-1}+c-2)(P_{n-1}+c)^2 / ((J_{n-1})(P_{n-1}+c))) / ((F_{n-1})(P_{n-1})^2 / J_{n-1}) =$$

$$(P_{n-1}+c-2)(P_{n-1}+c) / P_{n-1}^2 =$$

$1+(2c-2)/P_{n-1}+(c^2-2c)/P_{n-1}^2$ **table 3 (column D) / (column C)**

For all n, $(TPA_{n-1})(1+(2c-2)/2P_{n-1}+(c^2-2c)/2P_{n-1}^2) < TPA_n$

Table 3 (column B)((column D/column C)+1)/2=(column F)

For all n, $TPA_{n-1} < TPA_n$

Table 1 - Twin Primes in the interval [$743, 743^2$] and [$19993, 19993^2$]

X_m_start is the number of $(6j-1, 6j+1)$ pairs starting with no factor less than P_m

X_{m+1}_remain is the number of $(6j-1, 6j+1)$ pairs remaining with no factor less than P_{m+1}

$a_m = P_m - a$ $(X_m\_start)(P_m - a) / P_m = X_{m+1}\_remain$

743 P_m	X_m_start	X_{m+1}_remain	a	19993 P_m	X_m_start	X_{m+1}_remain	a
5	91885	55130	2.0001	5	66616677	39970006	2.0000
13	32217	27262	1.9994	263	5274325	5234610	1.9804
23	21524	19648	2.0046	601	4069608	4056904	1.8761
37	17115	16195	1.9889	971	3543773	3536233	2.0660
47	14699	14083	1.9697	1361	3203847	3198695	2.1886
59	13563	13118	1.9358	1747	2958682	2954671	2.3684
73	11972	11649	1.9695	2161	2768592	2765443	2.4579
89	11078	10824	2.0406	2617	2616137	2613615	2.5228
103	10368	10159	2.0763	3023	2489007	2486884	2.5785
113	9740	9542	2.2971	3491	2381553	2379728	2.6752
137	9207	9045	2.4106	3917	2288951	2287347	2.7449
151	8756	8621	2.3281	4363	2207533	2206130	2.7729
167	8361	8244	2.3369	4831	2135963	2134725	2.8000
181	8015	7905	2.4841	5309	2072410	2071250	2.9716
197	7701	7602	2.5325	5779	2015597	2014628	2.7783
223	7408	7321	2.6189	6563	1932497	1931593	3.0701
233	7155	7082	2.3772	6709	1917577	1916765	2.8409
251	6943	6867	2.7475	7207	1875356	1874590	2.9437
269	6732	6668	2.5573	7681	1837295	1836635	2.7592
277	6605	6536	2.8937	8191	1802389	1801785	2.7449
307	6366	6312	2.6041	8689	1770544	1769971	2.8120
317	6215	6170	2.2953	9161	1741327	1740812	2.7094
347	6086	6043	2.4517	9631	1714403	1713940	2.6010
359	5966	5924	2.5273	10139	1690075	1689647	2.5676
379	5843	5807	2.3351	10639	1667653	1667255	2.5391
397	5732	5701	2.1471	11161	1647452	1647112	2.3034
419	5638	5605	2.4525	11717	1629036	1628722	2.2585
433	5552	5521	2.4177	12211	1612271	1611971	2.2721
449	5466	5442	1.9715	12671	1597277	1597031	1.9515
463	5396	5369	2.3167	13177	1583504	1583257	2.0554
487	5324	5299	2.2868	13711	1571065	1570856	1.8240
503	5254	5232	2.1062	14281	1559772	1559590	1.6664
523	5191	5170	2.1158	14767	1549911	1549719	1.8293
557	5140	5122	1.9506	15277	1541111	1540979	1.3085
571	5097	5087	1.1203	15761	1533344	1533215	1.3260
593	5064	5054	1.1710	16301	1526754	1526634	1.2812
607	5027	5021	0.7245	16879	1521076	1520977	1.0986
619	5002	4994	0.9900	17389	1516362	1516288	0.8486
643	4981	4975	0.7745	17921	1512581	1512526	0.6516
659	4964	4959	0.6638	18401	1509727	1509698	0.3535
701	4948	4941	0.9917	19031	1507754	1507721	0.4165
727	4937	4935	0.2945	19991	1506428	1506428	0.0000
743	4934	4934	0.0000	19993	1506428	1506428	0.0000

Table 2 – Twin Primes in the interval [P_n, P_n^2] for $347 \le n \le 31153$

$(a_3/5)(a_4/7)(a_5/11)\ldots(a_n/P_n)(X)$. For $3 \le m \le n$

a_m **is replaced by** $P_m - 2.04$ $P_m - 2.06$ $P_m - 2.08$ **for** $m = 3$ *to* n.

P_n	TPA_n	TPA / 2.06	TPC_n 2.04	TPC_n 2.06	TPC_n 2.08	$3X/P_n$
347	1405	1.0637	1360.2	1320.8	1282.4	173
349	1419	1.0683	1368.0	1328.2	1289.6	174
1151	10387	1.0430	10293.6	9958.4	9633.4	575
1153	10408	1.0434	10311.2	9975.2	9649.5	576
1997	26735	1.0369	26690.2	25783.3	24905.1	998
1999	26777	1.0375	26716.5	25808.3	24929.1	999
2969	52817	1.0302	53125.5	51267.4	49470.3	1484
2971	52877	1.0307	53160.6	51300.9	49502.3	1485
3851	82712	1.0224	83885.3	80901.1	78016.6	1925
3853	82802	1.0230	83928.1	80941.9	78055.5	1926
4649	114842	1.0196	116843	112636	108572	2324
4651	114919	1.0198	116892	112683	108617	2325
5849	171367	1.0156	175132	168737	162561	2924
5851	171471	1.0159	175191	168793	162614	2925
6947	231582	1.0123	237533	228770	220312	3473
6949	231708	1.0126	237600	228834	220373	3474
8387	322646	1.0100	331826	319452	307514	4193
8389	322805	1.0103	331903	319526	307585	4194
9677	415267	1.0091	427606	411530	396024	4838
9679	415417	1.0092	427693	411613	396103	4839
10937	515723	1.0078	531882	511751	492342	5468
10939	515884	1.0079	531977	511842	492428	5469
12251	630469	1.0059	651581	626775	602859	6125
12253	630646	1.0060	651685	626874	602954	6126
13997	798218	1.0048	826113	794435	763902	6998
13999	798427	1.0049	826228	794545	764006	6999
15731	982287	1.0039	1017750	978483	940646	7865
15733	982497	1.0040	1017877	978604	940761	7866
17291	1162662	1.0026	1206394	1159636	1114583	8645
17293	1162911	1.0027	1206531	1159767	1114707	8646
18251	1280482	1.0031	1328116	1276491	1226753	9125
18253	1280728	1.0032	1328259	1276627	1226882	9126
19991	1506151	1.0015	1564964	1503866	1445014	9995
19993	1506427	1.0016	1565117	1504011	1445152	9996
21191	1671686	1.0015	1737182	1669161	1603649	10595
21193	1671950	1.0016	1737343	1669314	1603795	10596
22541	1866304	1.0009	1940784	1864560	1791151	11270
22543	1866615	1.0010	1940953	1864721	1791304	11271
23831	2061886	1.0010	2144230	2059785	1978464	11915
23833	2062203	1.0011	2144407	2059953	1978623	11916
26111	2428375	1.0000	2528479	2428472	2332181	13055
26113	2428739	1.0000	2528668	2428652	2332353	13056
27689	2697588	0.9992	2811333	2699839	2592502	13844
27691	2697935	0.9992	2811532	2700028	2592683	13845
29207	2968309	0.9994	3093224	2970220	2851812	14603
29209	2968674	0.9994	3093432	2970418	2852000	14604
31151	3333028	0.9987	3476151	3337515	3204082	15575

Table 3

A	B	C	D	E	F	G	H	I
prime	*TPA_{n-1}*	*$(F_{n-1})(P_{n-1})^2/J_{n-1}$*	*$(F_n)(P_n)^2/J_n$*	*(D/C+1)/2*	*(B)(E)*	*TPA_n*	*F/G*	*B/G*
71	*120*	*109.0*	*112.1*	*1.01408*	*121.7*	*123*	*0.989483*	*0.97561*
73	*123*	*112.1*	*127.9*	*1.07047*	*131.7*	*138*	*0.954117*	*0.89130*
1019	*8420*	*8935.3*	*8952.8*	*1.00098*	*8428.2*	*8450*	*0.997425*	*0.99645*
1021	*8450*	*8952.8*	*9111.3*	*1.00885*	*8524.8*	*8586*	*0.992872*	*0.98416*
2087	*28819*	*30850.0*	*30879.6*	*1.00048*	*28832.8*	*28867*	*0.998816*	*0.99834*
2089	*28867*	*30879.6*	*31146.2*	*1.00432*	*28991.6*	*29106*	*0.996070*	*0.99179*
3461	*68804*	*74874.0*	*74917.3*	*1.00029*	*68823.9*	*68872*	*0.999302*	*0.99901*
3463	*68872*	*74917.3*	*75047.1*	*1.00087*	*68931.7*	*69019*	*0.998735*	*0.99787*
4637	*114316*	*125244.7*	*125298.7*	*1.00022*	*114340.6*	*114394*	*0.999534*	*0.99932*
4639	*114394*	*125298.7*	*125460.9*	*1.00065*	*114468.0*	*114580*	*0.999023*	*0.99838*
6299	*195208*	*215150.4*	*215218.7*	*1.00016*	*195239.0*	*195319*	*0.999590*	*0.99943*
6301	*195319*	*215218.7*	*215833.9*	*1.00143*	*195598.2*	*195879*	*0.998566*	*0.99714*
8009	*297317*	*329810.8*	*329893.1*	*1.00012*	*297354.1*	*297454*	*0.999664*	*0.99954*
8011	*297454*	*329893.1*	*330305.0*	*1.00062*	*297639.7*	*297851*	*0.999291*	*0.99867*
9857	*428957*	*476792.2*	*476889.0*	*1.00010*	*429000.5*	*429089*	*0.999794*	*0.99969*
9859	*429089*	*476889.0*	*477953.7*	*1.00112*	*429568.0*	*430004*	*0.998986*	*0.99787*
11777	*588001*	*656535.4*	*656646.9*	*1.00008*	*588050.9*	*588163*	*0.999809*	*0.99972*
11779	*588163*	*656646.9*	*656981.4*	*1.00025*	*588312.8*	*588502*	*0.999679*	*0.99942*
13931	*791507*	*885279.3*	*885406.4*	*1.00007*	*791563.8*	*791704*	*0.999823*	*0.99975*
13933	*791704*	*885406.4*	*889096.0*	*1.00208*	*793353.6*	*794778*	*0.998208*	*0.99613*
16187	*1033547*	*1158651.2*	*1158794.4*	*1.00006*	*1033610.9*	*1033796*	*0.999821*	*0.99976*
16189	*1033796*	*1158794.4*	*1159223.9*	*1.00019*	*1033987.6*	*1034307*	*0.999691*	*0.99951*
18041	*1254327*	*1408473.1*	*1408629.2*	*1.00006*	*1254396.5*	*1254586*	*0.999849*	*0.99979*
18043	*1254586*	*1408629.2*	*1409097.7*	*1.00017*	*1254794.6*	*1255094*	*0.999761*	*0.99960*
20147	*1527206*	*1717720.9*	*1717891.4*	*1.00005*	*1527281.8*	*1527479*	*0.999871*	*0.99982*
20149	*1527479*	*1717891.4*	*1719767.6*	*1.00055*	*1528313.1*	*1529106*	*0.999481*	*0.99894*
21839	*1763993*	*1985940.5*	*1986122.3*	*1.00005*	*1764073.7*	*1764289*	*0.999878*	*0.99983*
21841	*1764289*	*1986122.3*	*1987759.5*	*1.00041*	*1765016.2*	*1765719*	*0.999602*	*0.99919*
23741	*2047968*	*2308071.0*	*2308265.5*	*1.00004*	*2048054.3*	*2048281*	*0.999889*	*0.99985*
23743	*2048281*	*2308265.5*	*2308848.8*	*1.00013*	*2048539.8*	*2048899*	*0.999825*	*0.99970*
26861	*2555034*	*2883638.7*	*2883853.4*	*1.00004*	*2555129.1*	*2555371*	*0.999905*	*0.99987*
26863	*2555371*	*2883853.4*	*2887074.9*	*1.00056*	*2556798.3*	*2558027*	*0.999520*	*0.99896*
28619	*2861908*	*3233814.5*	*3234040.4*	*1.00003*	*2862008.0*	*2862279*	*0.999905*	*0.99987*
28621	*2862279*	*3234040.4*	*3235170.5*	*1.00017*	*2862779.1*	*2863372*	*0.999793*	*0.99962*
31319	*3365123*	*3806114.0*	*3806357.0*	*1.00003*	*3365230.4*	*3365489*	*0.999923*	*0.99989*
31321	*3365489*	*3806357.0*	*3807572.4*	*1.00016*	*3366026.3*	*3366653*	*0.999814*	*0.99965*