Proof of the Twin Prime Conjecture

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Abstract: Let P_n be the n_th prime. For twin primes $P_n - P_{n-1} = 2$. We exhibit formula for calculating the number of twin primes in the closed interval $[P_n, P_n^2]$. We prove that as P_n increases the number of twin primes in the interval $[P_n, P_n^2]$ also increases.

Introduction: All primes greater than or equal to five are of the form $6j - 1$ or $6j + 1$.
$\prod_{m=1 \text{ to } n} P_m = J_n$ is the product of the first n primes. The number of $(6j-1, 6j+1)$ pairs with no factor less than P_{n+1} in the closed interval $[1, J_n+1]$ is exactly $(1/6)(3/5)(5/7)\ldots((P_n-2)/P_n)(J_n)$.

Closely related to this is the number of $(6j-1, 6j+1)$ pairs in $[P_n, P_n^2]$ with no factor less than P_{n+1}. They are all twin primes. Let X be the number of $(6j-1, 6j+1)$ pairs in the interval $[P_n, P_n^2]$.

Using the formula for the number of $(6j-1, 6j+1)$ pairs with no factor less than P_{n+1} in the closed interval $[1, J_n+1]$ as a standard, the number of twin primes in $[P_n, P_n^2]$ can be approximated by the formula

$$(a_3/5)(a_4/7)(a_5/11)\ldots(a_n/P_n)(X) \text{ for } 3 \le m \le n,\ a_m = P_m - 2.$$

Table 1 shows the values of the $3 \le m \le n$, a_m for twin primes in the interval $[19993, 19993^2]$.

Table 2 shows the actual number of twin primes (TPA_n) versus the calculated number (TPC_n) in $[P_n, P_n^2]$ for $347 \le n \le 31153$.

The number of twin primes is always greater than $(3/5)(5/7)(7/9)\ldots(P_n-2)/P_n)(X) = 3X/P_n$.

Let $P_n - P_{n-1} = c$. **Table 3** shows that for all n

$$(TPA_{n-1})(1+(2c-2)/2P_{n-1}+(c^2-2c)/2P_{n-1}^2) < TPA_n$$

Section 1

Calculating the number of $(6j-1, 6j+1)$ pairs (F_n) with no factor $< P_{n+1}$ in $[1, J_n+1]$.

For each $(6j-1, 6j+1)$ pair with no factor less than P_n in $[1, J_{n-1}+1]$ there are pairs $(6j-1+mJ_{n-1}, 6j+1+mJ_{n-1})$ for $m = 0$ to $P_n - 1$ in $[1, J_n+1]$. P_n and J_{n-1} are relatively prime. Thus, P_n divides $6j-1+mJ_{n-1}$ and $6j+1+mJ_{n-1}$ each for exactly one different value of m.

$P_3=5$, $P_4=7$. $\underline{F_3 = (1/6)(3/5)(J_3)}$. $F_4 = (5)(F_3)$. $J_4 = (7)(J_3)$ $F_4 / F_3 = (5/7)(J_4 / J_3)$. $\underline{F_4 = (1/6)(3/5)(5/7)(J_4)}$.

The number of $(6j-1, 6j+1)$ pairs with no factor less than P_{n+1} in the interval $[1, J_n+1]$ is exactly $(1/6)(3/5)(5/7)\ldots((P_n-2)/P_n)(J_n)$.

This occurs because J_n is divisible by all primes in the interval $[P_3, P_n]$.
All the $(6j-1, 6j+1)$ pairs with no factor less than P_{n+1} in which $6j < P_{n+1}^2$ are twin primes.

Determining the number of twin primes pairs (TPA_n) in the closed interval $[P_n, P_n^2]$.

Let X be the number of $(6j-1, 6j+1)$ pairs in the interval $[P_n, P_n^2]$. The number of twin prime pairs in $[P_n, P_n^2]$ is $(a_3/5)(a_4/7)(a_5/11)\ldots(a_n/P_n)(X)$. For $3 \le m \le n$ $P_m - 4 < a_m \le P_m$.

$P_m - 4$ is the lower bound for the a_m. The absolute value of $P_m - 2$ for the number of pairs in $[1, J_m+1]$ with no factor less than P_{m+1} sets a range for the possible values of a_m at $P_m - (2+2) < a_m \le P_m - (2-2)$.

Table 1 illustrates this formula for the interval $[19993, 19993^2]$. For selected P_m it shows the actual number $(6j-1, 6j+1)$ pairs with no factor less than P_{m+1} and the value of a_m $(P_m - a)$. a_m starts at $5-2$ and rises slightly to $571 - 1.87$. It declines unevenly but steadily to a low value of $6563 - 3.07$. It then rises unevenly but steadily to $19991 - 0.00$, finishing at $19993 - 0.01$. This same a_m $(P_m - a)$ pattern is in all primes between **500** and **31000**.

Table 2 shows the number of twin primes calculated (TPC_n) in $[P_n, P_n^2]$ for $347 \le n \le 31153$, when a_m equals $P_m - 2.04$, $P_m - 2.06$, and $P_m - 2.08$ for $3 \le m \le n$. Comparing TPC_n with TPA_n shows the average value for $a_m = P_m - a$ starts out at approximately $P_m - 2.02$ for $n = 347$ and gradually decreases to slightly less than $P_m - 2.06$ for $n = 31153$. There are proportionally fewer twin prime pairs in $[P_n, P_n^2]$ than there are $(6j-1, 6j+1)$ pairs with no factors less than P_{n+1} in $[1, J_n+1]$. The number of twin prime pairs in $[P_n, P_n^2]$ is always greater than $(3/5)(5/7)(7/9)\dots(P_n-2)/P_n)(X) = 3X/P_n$, where X is the number of $(6j-1, 6j+1)$ pairs in $[P_n, P_n^2]$.

Section 2

Establishing a lower bound for the ratio TPA_n / TPA_{n-1}

The number of twin primes in the interval $[P_n, P_n^2]$ is $(a_3/5)(a_4/7)(a_5/11)\dots(a_n/P_n)(X)$

$m=1$ to n $\prod P_m = J_n$. Using $(1/6)(3/5)(5/7)\dots((P_n-2)/P_n)(J_n)$ the number of $(6j-1, 6j+1)$ pairs with no factor less than P_{n+1} in the closed interval $[1, J_n+1]$ as a standard, the average value of a_m, $3 \le m \le n$ can be approximated by $P_m - 2$. This can be verified by comparing TPC_n with TPA_n in **table 2.** X is close to $(P_n^2 - P_n)/6$. $m=3$ to n $\prod P_m - 2 = F_n$

The number of twin prime pairs in $[P_n, P_n^2]$ is approximately $(F_n)(P_n^2)/J_n$

TPA_n is approximately $(TPA_{n-1})((F_n)(P_n^2)/J_n) / ((F_{n-1})(P_{n-1})^2/J_{n-1})$.

TPA_n is greater than $(TPA_{n-1})(((F_n)(P_n^2)/J_n) / ((F_{n-1})(P_{n-1})^2/J_{n-1}))+1)/2$.

Calculating $(((F_n)(P_n^2)/J_n) / ((F_{n-1})(P_{n-1})^2/J_{n-1}))+1)/2$.

Let $P_n - P_{n-1} = c$.

$$((F_n)(P_n^2)/J_n) / ((F_{n-1})(P_{n-1})^2/J_{n-1}) =$$

$$((F_{n-1})(P_{n-1}+c-2)(P_{n-1}+c)^2/((J_{n-1})(P_{n-1}+c))) / ((F_{n-1})(P_{n-1})^2/J_{n-1}) =$$

$$(P_{n-1}+c-2)(P_{n-1}+c) / P_{n-1}^2 =$$

$1+(2c-2)/P_{n-1}+(c^2-2c)/P_{n-1}^2$ **table 3 (column D) / (column C)**

For all n, $(TPA_{n-1})(1+(2c-2)/2P_{n-1}+(c^2-2c)/2P_{n-1}^2) < TPA_n$

Table 3 (column B)((column D/column C)+1)/2=(column F)

For all n, $TPA_{n-1} < TPA_n$

Table 1 - Twin Primes in the interval [*19993*, *19993*^{2}]

X_m*_start* is the number of *(6j –1, 6j+1)* pairs starting with no factor less than P_m

X_{m+1}*_remain* is the number of *(6j –1, 6j+1)* pairs remaining with no factor less than P_{m+1}

$a_m = P_m$ - a (X_m_start)(P_m – a) / P_m = X_{m+1}_remain

P_m	X_m_start	X_{m+1}_remain	a	P_m	X_m_start	X_{m+1}_remain	a
5	66616678	39970007	2.00	8219	1801175	1800557	2.82
7	39970007	28550003	2.00	8221	1800557	1799962	2.72
107	7355655	7217859	2.00	8819	1761507	1760973	2.67
109	7217859	7085101	2.00	8821	1760973	1760428	2.73
269	5234611	5196203	1.97	9281	1734617	1734102	2.76
271	5196203	5158420	1.97	9283	1734102	1733622	2.57
569	4149357	4135665	1.88	9719	1710654	1710182	2.68
571	4135665	4122104	1.87	9721	1710182	1709724	2.60
857	3672390	3663859	1.99	10139	1690076	1689648	2.57
859	3663859	3655409	1.98	10141	1689648	1689193	2.73
1229	3298275	3292426	2.18	10709	1664961	1664565	2.55
1231	3292426	3286628	2.17	10711	1664565	1664172	2.53
1487	3115253	3110461	2.29	11171	1647113	1646763	2.37
1489	3110461	3105680	2.29	11173	1646763	1646383	2.58
1877	2897420	2893716	2.40	11939	1621083	1620785	2.19
1879	2893716	2890113	2.34	11941	1620785	1620458	2.41
2129	2787812	2784561	2.48	12377	1607307	1607030	2.13
2131	2784561	2781375	2.44	12379	1607030	1606751	2.15
2591	2623801	2621171	2.60	13337	1579669	1579428	2.03
2593	2621171	2618644	2.50	13339	1579428	1579203	1.90
2999	2497680	2495491	2.63	13877	1567332	1567120	1.88
3001	2495491	2493373	2.55	13879	1567120	1566909	1.87
3371	2405949	2404010	2.72	14387	1558080	1557896	1.70
3373	2404010	2402127	2.64	14389	1557896	1557687	1.93
3671	2338764	2337030	2.72	15269	1541408	1541266	1.41
3673	2337030	2335307	2.71	15271	1541266	1541112	1.53
4049	2263783	2262217	2.80	15737	1533801	1533651	1.54
4051	2262217	2260674	2.76	15739	1533651	1533486	1.69
4271	2221853	2220432	2.73	16361	1526151	1526028	1.32
4273	2220432	2218988	2.78	16363	1526028	1525904	1.33
4721	2153736	2152444	2.83	17027	1519492	1519407	0.95
4723	2152444	2151174	2.79	17029	1519407	1519311	1.08
5231	2082560	2081470	2.74	17597	1514618	1514536	0.95
5233	2081470	2080308	2.92	17599	1514536	1514467	0.80
5651	2030615	2029630	2.74	17957	1512342	1512279	0.75
5653	2029630	2028630	2.79	17959	1512279	1512220	0.70
6089	1980674	1979692	3.02	18287	1510337	1510286	0.62
6091	1979692	1978760	2.87	18289	1510286	1510231	0.67
6563	1932498	1931594	3.07	19181	1507414	1507400	0.18
6691	1920045	1919232	2.83	19183	1507400	1507376	0.31
6959	1895941	1895144	2.93	19751	1506526	1506519	0.09
6961	1895144	1894360	2.88	19753	1506519	1506512	0.09
7547	1848874	1848215	2.69	19991	1506429	1506429	0.00
7549	1848215	1847507	2.89	19993	1506429	1506428	0.01

Table 2 – Twin Primes in the interval [P_n, P_n^2] for $347 \leq n \leq 31153$
$(a_3/5)(a_4/7)(a_5/11)\ldots(a_n/P_n)(X)$. For $3 \leq m \leq n$
a_m is replaced by $P_m - 2.04$ $P_m - 2.06$ $P_m - 2.08$ for $m = 3$ to n.

P_n	TPA_n	TPC_n 2.04	TPC_n 2.06	TPC_n 2.08	$3X/P_n$
347	1405	1360	1321	1282	173
349	1419	1368	1328	1290	174
1151	10387	10294	9958	9633	575
1153	10408	10311	9975	9649	576
1997	26735	26690	25783	24905	998
1999	26777	26716	25808	24929	999
2969	52817	53125	51267	49470	1484
2971	52877	53161	51301	49502	1485
3851	82712	83885	80901	78017	1925
3853	82802	83928	80942	78055	1926
4649	114842	116843	112636	108572	2324
4651	114919	116892	112683	108617	2325
5849	171367	175132	168737	162561	2924
5851	171471	175191	168793	162614	2925
6947	231582	237533	228770	220312	3473
6949	231708	237600	228834	220373	3474
8387	322646	331826	319452	307514	4193
8389	322805	331903	319526	307585	4194
9677	415267	427606	411530	396024	4838
9679	415417	427693	411613	396103	4839
10937	515723	531882	511751	492342	5468
10939	515884	531977	511842	492428	5469
12251	630469	651581	626775	602859	6125
12253	630646	651685	626874	602954	6126
13997	798218	826113	794435	763902	6998
13999	798427	826228	794545	764006	6999
15731	982287	1017750	978483	940646	7865
15733	982497	1017877	978604	940761	7866
17291	1162662	1206394	1159636	1114583	8645
17293	1162911	1206531	1159767	1114707	8646
18251	1280482	1328116	1276491	1226753	9125
18253	1280728	1328259	1276627	1226882	9126
19991	1506151	1564964	1503866	1445014	9995
19993	1506427	1565117	1504011	1445152	9996
21191	1671686	1737182	1669161	1603649	10595
21193	1671950	1737343	1669314	1603795	10596
22541	1866304	1940784	1864560	1791151	11270
22543	1866615	1940953	1864721	1791304	11271
23831	2061886	2144230	2059785	1978464	11915
23833	2062203	2144407	2059953	1978623	11916
26111	2428375	2528479	2428472	2332181	13055
26113	2428739	2528668	2428652	2332353	13056
27689	2697588	2811333	2699839	2592502	13844
27691	2697935	2811532	2700028	2592683	13845
29207	2968309	3093224	2970220	2851812	14603
29209	2968674	3093432	2970418	2852000	14604
31151	3333028	3476151	3337515	3204082	15575
31153	3333407	3476370	3337724	3204280	15576

Table 3

A	B	C	D	E	F	G	H	I
prime	TPA_{n-1}	$(F_{n-1})(P_{n-1})^2/J_{n-1}$	$(F_n)(P_n)^2/J_n$	(D/C+1)/2	(B)(E)	TPA_n	F/G	B/G
71	120	109.0	112.1	1.01408	121.7	123	0.989483	0.97561
73	123	112.1	127.9	1.07047	131.7	138	0.954117	0.89130
1019	8420	8935.3	8952.8	1.00098	8428.2	8450	0.997425	0.99645
1021	8450	8952.8	9111.3	1.00885	8524.8	8586	0.992872	0.98416
2087	28819	30850.0	30879.6	1.00048	28832.8	28867	0.998816	0.99834
2089	28867	30879.6	31146.2	1.00432	28991.6	29106	0.996070	0.99179
3461	68804	74874.0	74917.3	1.00029	68823.9	68872	0.999302	0.99901
3463	68872	74917.3	75047.1	1.00087	68931.7	69019	0.998735	0.99787
4637	114316	125244.7	125298.7	1.00022	114340.6	114394	0.999534	0.99932
4639	114394	125298.7	125460.9	1.00065	114468.0	114580	0.999023	0.99838
6299	195208	215150.4	215218.7	1.00016	195239.0	195319	0.999590	0.99943
6301	195319	215218.7	215833.9	1.00143	195598.2	195879	0.998566	0.99714
8009	297317	329810.8	329893.1	1.00012	297354.1	297454	0.999664	0.99954
8011	297454	329893.1	330305.0	1.00062	297639.7	297851	0.999291	0.99867
9857	428957	476792.2	476889.0	1.00010	429000.5	429089	0.999794	0.99969
9859	429089	476889.0	477953.7	1.00112	429568.0	430004	0.998986	0.99787
11777	588001	656535.4	656646.9	1.00008	588050.9	588163	0.999809	0.99972
11779	588163	656646.9	656981.4	1.00025	588312.8	588502	0.999679	0.99942
13931	791507	885279.3	885406.4	1.00007	791563.8	791704	0.999823	0.99975
13933	791704	885406.4	889096.0	1.00208	793353.6	794778	0.998208	0.99613
16187	1033547	1158651.2	1158794.4	1.00006	1033610.9	1033796	0.999821	0.99976
16189	1033796	1158794.4	1159223.9	1.00019	1033987.6	1034307	0.999691	0.99951
18041	1254327	1408473.1	1408629.2	1.00006	1254396.5	1254586	0.999849	0.99979
18043	1254586	1408629.2	1409097.7	1.00017	1254794.6	1255094	0.999761	0.99960
20147	1527206	1717720.9	1717891.4	1.00005	1527281.8	1527479	0.999871	0.99982
20149	1527479	1717891.4	1719767.6	1.00055	1528313.1	1529106	0.999481	0.99894
21839	1763993	1985940.5	1986122.3	1.00005	1764073.7	1764289	0.999878	0.99983
21841	1764289	1986122.3	1987759.5	1.00041	1765016.2	1765719	0.999602	0.99919
23741	2047968	2308071.0	2308265.5	1.00004	2048054.3	2048281	0.999889	0.99985
23743	2048281	2308265.5	2308848.8	1.00013	2048539.8	2048899	0.999825	0.99970
26861	2555034	2883638.7	2883853.4	1.00004	2555129.1	2555371	0.999905	0.99987
26863	2555371	2883853.4	2887074.9	1.00056	2556798.3	2558027	0.999520	0.99896
28619	2861908	3233814.5	3234040.4	1.00003	2862008.0	2862279	0.999905	0.99987
28621	2862279	3234040.4	3235170.5	1.00017	2862779.1	2863372	0.999793	0.99962
31319	3365123	3806114.0	3806357.0	1.00003	3365230.4	3365489	0.999923	0.99989
31321	3365489	3806357.0	3807572.4	1.00016	3366026.3	3366653	0.999814	0.99965