Classifying conic sections in terms of differential forms

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Abstract

We explore classification of conics from a viewpoint of differential forms.

1 Introduction

Singularities play some roles in both mathematics and physics . Among other things, a conical singularity leads us to address the origin $O(0, 0, 0)$ in the double cone $A(x, y, z) = z^2 - x^2 - y^2 = 0$, where $\left.\frac{\partial A(x,y,z)}{\partial x}\right|_{x=0} = -2x|_{x=0} = -2 \cdot 0 = 0$, $\left.\frac{\partial A(x,y,z)}{\partial y}\right|_{y=0} = -2y|_{y=0} = -2 \cdot 0 = 0$, and $\left.\frac{\partial A(x,y,z)}{\partial z}\right|_{z=0} = 2z|_{z=0} = 2 \cdot 0 = 0$. Partial derivatives playing the above role in discerning a singular point , we wonder if the so-called differential form s can function similarly in distinguishing singularities . Viewing the double cone as an epitome, we derive differential form s from the conic sections in Euclidean geometry and investigate such possibility.

2 Obtaining *SING*, differential form -derived notion

We start from the following equation :

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$$\phi = ax^2+bxy+cy^2+ex+fy+g=0\ ,\ a,b,c,e,f,g\ \in\ \mathbb{R}^{\ 1\ ,\ 2\ ,\ 3}\ . \quad (1)$$

Differentiating ϕ wrt x gives

$$\frac{d\phi}{dx} = \frac{d}{dx}(ax^2+bxy+cy^2+ex+fy+g) \quad (2)$$

$$= {}^4\ a\frac{d}{dx}(x^2)+b\frac{d}{dx}(xy)+c\frac{d}{dx}(y^2)+e\frac{d}{dx}(x)+f\frac{d}{dx}(y)+\frac{d}{dx}(g)$$

$$= {}^5\ 2ax+b\{y\frac{d}{dx}(x)+x\frac{d}{dx}(y)\}+c\frac{dy}{dx}\cdot\frac{d}{dy}(y^2)+e\frac{d}{dx}(x)+f\frac{dy}{dx}\cdot\frac{d}{dy}(y)+\frac{d}{dx}(g)$$

$$= 2ax+by\frac{d}{dx}(x)+bx\frac{d}{dx}(y)+c\frac{dy}{dx}\cdot\frac{d}{dy}(y^2)+e\frac{d}{dx}(x)+f\frac{dy}{dx}\cdot\frac{d}{dy}(y)+\frac{d}{dx}(g)$$

$$= {}^6\ 2ax+by+bx\frac{dy}{dx}+2cy\frac{dy}{dx}+e+f\frac{dy}{dx}, \quad (3)$$

which we check using Maxima and Octave [7, 8, 9] :

```
% maxima

Maxima 5.41.0 http://maxima.sourceforge.net
using Lisp GNU Common Lisp (GCL) GCL 2.6.12
Distributed under the GNU Public License. See the file COPYING.
Dedicated to the memory of William Schelter.
The function bug_report() provides bug reporting information.

(%i1) diff(a*x^2+b*x*y(x)+c*y(x)^2+e*x+f*y(x)+g,x);
```

[1] We avoided writing 'dx' lest it should be confused with that in the differential operator $\frac{d}{dx}$, which will be used soon. *Cf.* footnote 3.

[2] Regarding the general binary quadratic form , Lagrange considered $ax^2+bxy+cy^2$ with integral coefficients, whereas Gauss restricted attention to $ax^2+2bxy+cy^2$ [1].

[3] Gauss treated the integral solutions to the equation $ax^2+2bxy+cy^2+2dx+2ey+f=0$ [2], and in this footnote, where because of the absence of the aforementioned $\frac{d}{dx}$, we are not so worried about mixing up dx's, we refrained from replacing 'd' in that equation by 'e' to be in conformity with the original text in [2]. *Cf.* footnote 1.

[4] Linearity of differentiation was used.

[5] Some differentiation rules were used.

[6] Ditto.

[7] See footnote 4 in [3] for how we verify our computations.

[8] Throughout this preprint, we use elementary OS ver. 5.0 (Juno) . Central processing units are the same as those indicated in footnote 3 of [4].

[9] Verbatim outputs of (on-line) softwares are sometimes edited. For instance, the Maxima output (%o1) on p3 doesn't always reflect the original one, which is not shown for simplicity.

```
                d                  d                d
(%o1) 2 c y(x) (-- (y(x))) + b x (-- (y(x))) + f (-- (y(x)))
                dx                 dx               dx

      + b y(x) + 2 a x + e

% octave -W

GNU Octave, version 4.2.2
Copyright (C) 2018 John W. Eaton and others.
This is free software; see the source code for copying conditions.
There is ABSOLUTELY NO WARRANTY; not even for MERCHANTABILITY or
FITNESS FOR A PARTICULAR PURPOSE.  For details, type 'warranty'.

Octave was configured for "x86_64-pc-linux-gnu".

Additional information about Octave is available at
http://www.octave.org.

Please contribute if you find this software useful.
For more information, visit http://www.octave.org/get-involved.html

Read http://www.octave.org/bugs.html to learn how to submit bug
reports.
For information about changes from previous versions, type 'news'.

octave:1> pkg load symbolic
octave:2> syms a b c e f g x y(x)
OctSymPy v2.6.0: this is free software without warranty, see source.
Initializing communication with SymPy using a popen2() pipe.
Some output from the Python subprocess (pid 21361) might appear next.
Python 2.7.15rc1 (default, Nov 12 2018, 14:31:15)
[GCC 7.3.0] on linux2
Type "help", "copyright", "credits" or "license" for more
information.
>>> >>>
OctSymPy: Communication established.  SymPy v1.1.1.
octave:3> diff(a*x^2+b*x*y+c*y^2+e*x+f*y+g,x)
```

```
ans(x) = (symfun)

              d                                   d
2*a*x + b*x*--(y(x)) + b*y(x) + 2*c*y(x)*--(y(x)) + e
              dx                                  dx

     d
+ f*--(y(x))
     dx
```

Having verified (3), we multiply both left-hand side (LHS) of (2) and right-hand side (RHS) of (3) by dx. Then, after some rearrangements, we get the $1-$form ω, *i.e.,*

$$d\phi = (2ax + by + e)dx + (bx + 2cy + f)dy, \tag{4}$$

in which $\frac{\partial(2ax+by+e)}{\partial y} = \frac{\partial(bx+2cy+f)}{\partial x} = b$ holds [10]. Rewriting (4) more generally yields

$$\omega = d\phi = f(x, y)dx + g(x, y)dy. \tag{5}$$

$\omega = 0$ [11] implying $d\omega = 0$ [12, 13], we try defining a *SING* to be a point at which $f(x, y) = g(x, y) = 0$ holds, and accordingly, such a $1-$form vanishes [14]. With regard to whereabouts, *SING*'s can exist [15]:

- **IN**(side) := *SING* is enclosed by a certain curve ;
- (up)**ON** := *SING* is on certain curve (s) [16];

[10] *Cf.* **Criterion 1.10** in [5].

[11] Since it follows from (1) that $\phi = 0$, $\frac{d\phi}{dx} = \frac{d}{dx}(0) = 0$. So $\frac{d\phi}{dx} = 0$. Multiplying both sides of it by dx, we get $d\phi = 0$. Hence, $\omega = 0$, too, since $\omega = d\phi$. See, *e.g.,* (5).

[12] When $d\omega = 0$, ω is regarded as closed [6].

[13] *Cf.* [7].

[14] The fact that all the partial derivatives simultaneously vanish at the singular points has inspired us. See also **1**.

[15] Henceforth, a line is regarded as a kind of a curve . See footnotes 19, 20, and 52.

[16] *SING* can be on the intersection point of curve s. See **3.5**, **3.6**, and **4**. *Cf.* footnote 53.

- **OUT**(side) := *SING* is neither enclosed by a certain curve nor on certain curve (s);
- **NO**(where) := *SING* is nonexistent.

3 *SING*-based classification of conic section s

We derive five examples from (1), apply the notion of *SING* to them, and classify the conic s into the above four categories.

3.1 The case where $a = 1$, $b = 0$, $c = 1$, $e = -8$, $f = -8$, and $g = 31$

In this case, we consider

$$\phi = x^2 + y^2 - 8x - 8y + 31 = (x-4)^2 + (y-4)^2 - 1^2 = 0, \qquad (6)$$

a circle . So

$$\frac{d\phi}{dx} = \frac{d}{dx}(x^2 + y^2 - 8x - 8y + 31) = 2x + 2y\frac{dy}{dx} - 8 - 8\frac{dy}{dx},$$

and we get the 1−form $\omega = d\phi = 2xdx + 2ydy - 8dx - 8dy = 2(x-4)dx + 2(y-4)dy$. Thus, *SING* is the point (4, 4), or the center of the circle . The *SING* lies inside the circle , and the circle is therefore classified into the category **IN**.

3.2 The case where $a = 4$, $b = 0$, $c = 1$, $e = 32$, $f = -8$, and $g = 79$

In this case, we consider

$$\phi = 4x^2 + y^2 + 32x - 8y + 79 = 4(x+4)^2 + (y-4)^2 - 1 = 0, \qquad (7)$$

an ellipse . So

$$\frac{d\phi}{dx} = \frac{d}{dx}(4x^2 + y^2 + 32x - 8y + 79) = 8x + 2y\frac{dy}{dx} + 32 - 8\frac{dy}{dx},$$

and we get the 1−form $\omega = d\phi = 8xdx + 2ydy + 32dx - 8dy = 8(x+4)dx + 2(y-4)$ dy. Thus, *SING* is the point (−4, 4), or the center of the ellipse . The *SING* lies inside the ellipse , and likewise, the ellipse is classified into the category **IN**.

3.3 The case where $a = 1$, $b = 0$, $c = 0$, $e = 0$, $f = -1$, and $g = 1$

In this case, we consider

$$\phi = x^2 - y + 1 = 0, \tag{8}$$

a parabola . So

$$\frac{d\phi}{dx} = \frac{d}{dx}(x^2 - y + 1) = 2x - \frac{dy}{dx},$$

and we get the $1-$form $\omega = d\phi = 2xdx - dy$. This time, even if we set $x = 0$, $-dy$ remains, which means that ω doesn't vanish. The parabola is therefore classified into the category **NO**.

3.4 The case where $a = 1$, $b = 0$, $c = -1$, $e = 0$, $f = 0$, and $g = -61$

In this case, we consider

$$\phi = x^2 - y^2 - 61 = 0, \tag{9}$$

a hyperbola . So

$$\frac{d\phi}{dx} = \frac{d}{dx}(x^2 - y^2 - 61) = 2x - 2y\frac{dy}{dx},$$

and we get the $1-$form $\omega = d\phi = 2xdx - 2ydy$. Thus, *SING* is the point $(0, 0)$, or the center of the hyperbola . The hyperbola cannot encircle the *SING*, and the hyperbola is therefore classified into the category **OUT**.

3.5 The case where $a = 1$, $b = 0$, $c = -1$, $e = 0$, $f = -4$, and $g = -4$

In this case, we consider

$$\phi = x^2 - y^2 - 4y - 4 = x^2 - (y+2)^2 = 0, \tag{10}$$

two intersecting lines $y = \pm x - 2$ [17], [18] . So

---

[17] *Cf.* here .

[18] We are interested more in *SING*-based classification of conic section s than in pondering on whether to exclude degenerate cases , including two intersecting lines and a double line , which is why we consider them for now.

$$\frac{d\phi}{dx} = \frac{d}{dx}(x^2 - y^2 - 4y - 4) = 2x - 2y\frac{dy}{dx} - 4\frac{dy}{dx},$$

and we get the 1−form $\omega = d\phi = 2xdx - 2ydy - 4dy = 2xdx - 2(y+2)dy$. Thus, *SING* is the point $(0, -2)$. The *SING* lies on the intersection point of those line s, and those intersecting lines are therefore classified into the category **ON**.

3.6 Visualizing (6) – (10)

We visualize (6) – (10) using SageMath and Xcas (browser version) :

```
% more Fig1.sage

var('x y')
C1=implicit_plot((x-4)^2+(y-4)^2-1^2,(x,-10,10),(y,-10,10),
                 color='blue')
C2=implicit_plot(4*(x+4)^2+(y-4)^2-1,(x,-10,10),(y,-10,10),
                 color='red')
C3=implicit_plot(x^2-y+1,(x,-10,10),(y,-10,10),
                 color='green')
C4=implicit_plot(x^2-y^2-61,(x,-10,10),(y,-10,10),color='orange')
C5=implicit_plot(x^2-(y+2)^2,(x,-10,10),(y,-10,10),color='black')
t1=text("(x-4)^2\n               +(y-4)^2-1^2=0",(3.7,6.0),
        color='blue')
t2=text("4*(x+4)^2\n      +(y-4)^2-1=0",(-5.4,5.8),color='red')
t3=text("x^2-y+1=0",(0.0,8.4),color='green')
t4=text("x^2-y^2-61=0",(5.3,0.7),color='orange')
t5=text("x^2-(y+2)^2=0",(0.0,-5.4),color='black')
(C1+C2+C3+C4+C5+t1+t2+t3+t4+t5).show(xmax=10,xmin=-10,ymax=10,
ymin=-10,axes=true)

% sage

SageMath version 8.1, Release Date: 2017-12-07
Type "notebook()" for the browser-based notebook interface.
Type "help()" for help.

sage: load("Fig1.sage")
```

```
Launched png viewer for Graphics object consisting of 10
graphics primitives
```

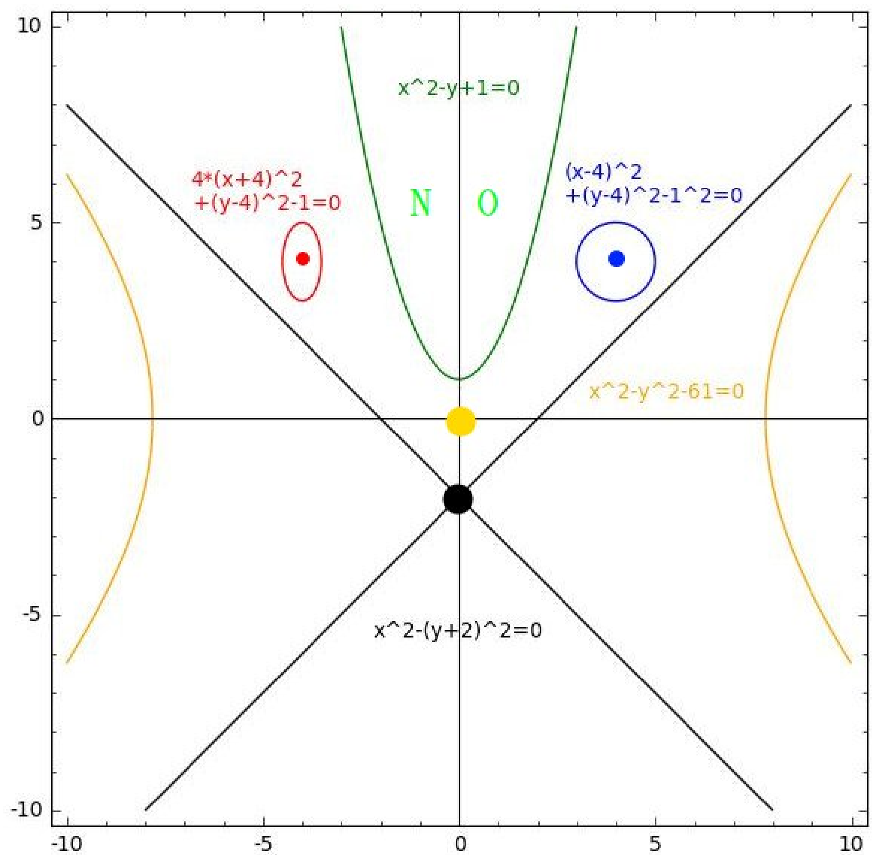

Fig. 1. (6) – (10) visualized by SageMath . Four dots were insetted later by using Pinta ver. 1.6 and correspond to the *SING*'s of the curve s except for the parabola [19] . 'NO' was insetted in a similar manner and denotes the category **NO**(where).

[19]As mentioned in footnote 15, the two intersecting lines in this Fig. are regarded as certain curve s.

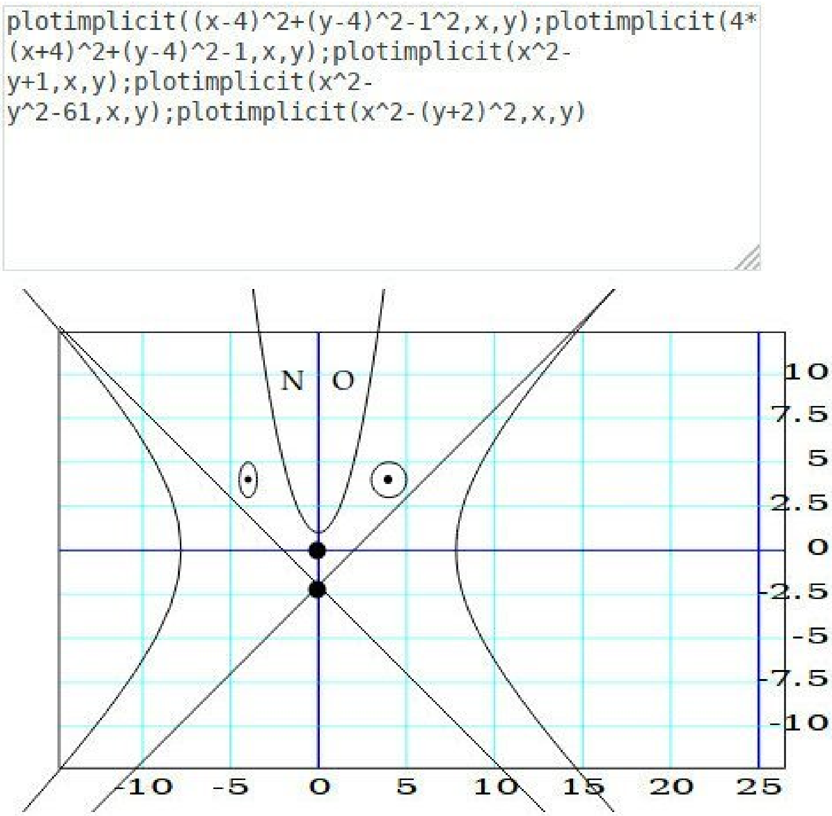

Fig. 2. (6) – (10) visualized by Xcas . Four dots were insetted in a manner similar to Fig. 1 and likewise indicate the *SING*'s of the curve s except for the parabola [20] . 'NO' was also insetted in a similar manner and likewise denotes the category **NO**(where).

[20] Ditto.

3.7 The case where $a = 1$, $b = 2$, $c = 1$, $e = 0$, $f = 0$, and $g = 0$

In addition to the aforementioned five cases, we consider

$$\phi = x^2 + 2xy + y^2 = (x+y)^2 = 0, \tag{11}$$

a double line [21] . So

$$\frac{d\phi}{dx} = \frac{d}{dx}(x^2 + 2xy + y^2) = 2x + 2y + 2x\frac{dy}{dx} + 2y\frac{dy}{dx},$$

and we get the $1-$form $\omega = d\phi = (2x+2y)dx + 2xdy + 2ydy = 2(x+y)dx + 2(x+y)$ dy. Thus, *SING* is the line $x + y = 0$. As we haven't treated such a 1-dimensional *SING* yet [22] , taking this opportunity, we would like to visualize (11) and its *SING* by using SageMath and Xcas (browser version) :

```
% more Fig3.sage

var('x y')
C1=implicit_plot((x+y)^2-0.00007,(x,-0.4,1.4),(y,-1.05,0.05),
color='black')
# Actually, the term -0.00007 is a "dummy". If we simply write
# (x+y)^2, the double line (x+y)^2=0 fails to show up.
C2=implicit_plot(x+y,(x,-0.4,1.4),(y,-1.05,0.05),color='cyan')
t1=text("(x+y)^2=0",(0.50,-0.30),color='black')
t2=text("x+y=0",(0.25,-0.45),color='cyan')
(C1+C2+t1+t2).show(xmax=1.4,xmin=-0.4,ymax=0.05,ymin=-1.05,
axes=true)

% sage

SageMath version 8.1, Release Date: 2017-12-07

sage: load("Fig3.sage")
Launched png viewer for Graphics object consisting of 4
graphics primitives
```

[21] *Cf.* here .

[22] See **3.1**-**3.6**.

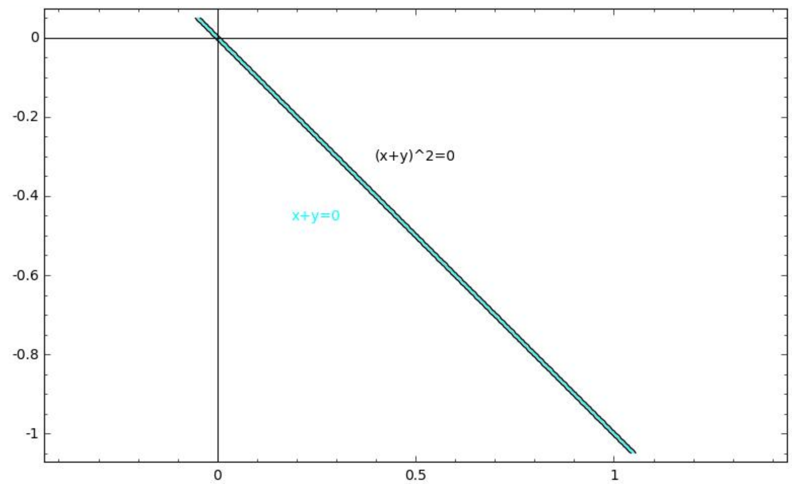

Fig. 3. (11) visualized by SageMath

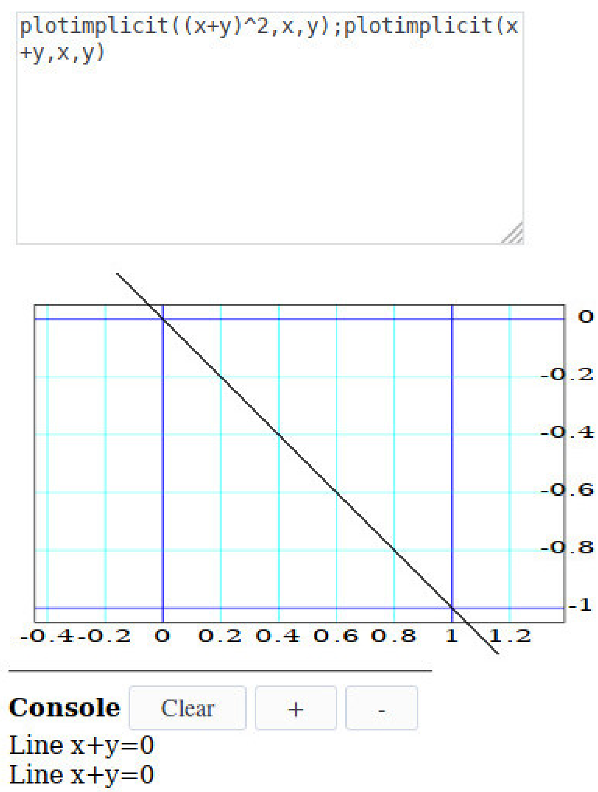

Fig. 4. (11) visualized by Xcas

These figures indicate that the double line $(x+y)^2=0$ and the line $x+y=0$ overlap. Now we learn that the *SING* $x+y=0$ is, in a sense, stuff *itself* we initially considered, which leads us to add **IT**(self) to the aforementioned four categories.

We have thus referred to five categories, which include **IN**, **IT**, **NO**, **ON**, and **OUT** [23] .

4 Wrap-up

We tabulate the results we have so far obtained as follows:

Table

Equation	Shape	1−form
$(x-4)^2+(y-4)^2-1^2=0$	Circle	$2(x-4)dx+2(y-4)dy$
$4(x+4)^2+(y-4)^2-1=0$	Ellipse	$8(x+4)dx+2(y-4)dy$
$x^2-y+1=0$	Parabola	$2xdx-dy$
$x^2-y^2-61=0$	Hyperbola	$2xdx-2ydy$
$x^2-(y+2)^2=0$	Two intersecting lines	$2xdx-2(y+2)dy$
$(x+y)^2=0$	Double line	$2(x+y)dx+2(x+y)dy$

Table (cont'd)

Whereabouts of *SING*	*SING*-based classification of equation
(4, 4)	**IN**
(−4, 4)	**IN**
Nonexistent	**NO**
(0, 0)	**OUT**
(0, −2)	**ON**
$x+y=0$	**IT**

Cf. determinant-based classification .

[23] As **IT** and **ON** have been embodied by the line $x+y=0$ and the point (0, −2), respectively, defining a line as a set of points might enable us to subsume **ON** under **IT**, thereby reducing such five categories to four. See also **4**.

5 Some generalizations

Having summarized (rather) concrete results we obtained, we wish to see if at least some of them generalize. Undertaking (10), which can be rewritten as $(x+y+2)(x-y-2)=0$, we consider

$$\phi = (hx+jy+k)(\ell x+my+n) = 0,\ h,\ j,\ k,\ \ell,\ m,\ n \in \mathbb{R}\ ,\ hm-j\ell \neq 0 \ ^{24}\ ,\ ^{25}\ . \quad (12)$$

So

$$\begin{aligned}\phi &= hx(\ell x+my+n)+jy(\ell x+my+n)+k(\ell x+my+n)\\ &= h\ell x^2+hmxy+hnx+j\ell xy+jmy^2+jny+k\ell x+kmy+kn\\ &= h\ell x^2+(hm+j\ell)xy+jmy^2+(hn+k\ell)x+(jn+km)y+kn.\end{aligned}$$

And

$$\begin{aligned}\frac{d\phi}{dx} &= \frac{d}{dx}\{h\ell x^2+(hm+j\ell)xy+jmy^2+(hn+k\ell)x+(jn+km)y+kn\}\\ &= 2h\ell x+(hm+j\ell)y+(hm+j\ell)x\frac{dy}{dx}+2jmy\frac{dy}{dx}+hn+k\ell+(jn+km)\frac{dy}{dx}. \quad (13)\end{aligned}$$

We check (13) using Maxima and Octave :

```
% maxima

Maxima 5.41.0 http://maxima.sourceforge.net
using Lisp GNU Common Lisp (GCL) GCL 2.6.12
(%i1) ratsimp(diff((h*x+j*y(x)+k)*(l*x+m*y(x)+n),x));
                                                 d
(%o1) (2 j m y(x) + (h m + j l) x + j n + k m) (-- (y(x)))
                                                 dx
```

[24] We substituted 'j' for 'i' lest 'i' should be confused with the imaginary unit $\imath$. See also footnote 76.

[25] Rewriting $\phi = 0$ as the product of $hx+jy+k=0$ and $\ell x+my+n=0$ yields two line s. In matrix notation, one gets

$$\begin{pmatrix} h & j \\ \ell & m \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}+\begin{pmatrix} k \\ n \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which we further rewrite as $A\vec{x}+\vec{a}=\vec{0}$. If $hm-j\ell \neq 0$, that is, the matrix A is invertible , intersection point (x,y) is obtained by computing $-A^{-1}\vec{a}$. Explicitly,

$$\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} h & j \\ \ell & m \end{pmatrix}^{-1}\begin{pmatrix} k \\ n \end{pmatrix} = \frac{1}{hm-j\ell}\begin{pmatrix} -m & j \\ \ell & -h \end{pmatrix}\begin{pmatrix} k \\ n \end{pmatrix} = \frac{1}{hm-j\ell}\begin{pmatrix} -km+jn \\ k\ell-hn \end{pmatrix}.$$

And the condition $hm-j\ell \neq 0$ will take effect again. See the denominator s in the RHS of (15).

```
       + (h m + j l) y(x) + 2 h l x + h n + k l

% octave -W

GNU Octave, version 4.2.2

octave:1> pkg load symbolic
octave:2> syms h j k l m n x y(x)
OctSymPy v2.6.0: this is free software without warranty, see
source.
Python 2.7.15rc1 (default, Nov 12 2018, 14:31:15)
[GCC 7.3.0] on linux2
>>> >>>
OctSymPy: Communication established.  SymPy v1.1.1.
octave:3> expand(diff((h*x+j*y(x)+k)*(l*x+m*y(x)+n),x))
ans = (sym)

                d                                 d
2*h*l*x + h*m*x*--(y(x)) + h*m*y(x) + h*n + j*l*x*--(y(x))
                dx                                dx

                        d              d
+ j*l*y(x) + 2*j*m*y(x)*--(y(x)) + j*n*--(y(x)) + k*l
                        dx             dx

      d
+ k*m*--(y(x))
      dx
```

We have thus verified (13) and multiply it by dx. After some rearrangements, one gets

$$\omega = d\phi = \{2h\ell x + (hm + j\ell)y + hn + k\ell\}dx + \{(hm + j\ell)x + 2jmy + jn + km\}dy.$$

Thus, we are meant to solve

$$\begin{cases} 2h\ell x + (hm + j\ell)y + hn + k\ell = 0, \\ (hm + j\ell)x + 2jmy + jn + km = 0 \end{cases} \tag{14}$$

for x, y and get

$$(x, y) = \left(\frac{jn-km}{hm-j\ell}, \frac{k\ell-hn}{hm-j\ell}\right), \tag{15}$$

which is the *SING*. Let us check it using Giac and SageMath .

```
% giac -v

1.2.3

% giac

// Using locale /usr/share/locale/
// ja_JP.UTF-8
// /usr/share/locale/
// giac
// UTF-8
// Maximum number of parallel threads 4
Help file /usr/share/giac/doc/local/aide_cas not found
Added 0 synonyms
Welcome to giac readline interface
(c) 2001,2016 B. Parisse & others
Homepage http://www-fourier.ujf-grenoble.fr/~parisse/giac.html
Released under the GPL license 3.0 or above
See http://www.gnu.org for license details
May contain BSD licensed software parts (lapack, atlas, tinymt)
-------------------------------------------------
Press CTRL and D simultaneously to finish session
Type ?commandname for help
0>> linsolve([2*h*l*x+(h*m+j*l)*y+h*n+k*l=0,
              (h*m+j*l)*x+2*j*m*y+j*n+k*m=0],[x,y])
[(j*n-k*m)/(h*m-j*l),(-h*n+k*l)/(h*m-j*l)]
// Time 0.01

% sage

SageMath version 8.1, Release Date: 2017-12-07

sage: h,j,k,l,m,n,x,y=var('h,j,k,l,m,n,x,y')
sage: solve([2*h*l*x+(h*m+j*l)*y+h*n+k*l==0,
            (h*m+j*l)*x+2*j*m*y+j*n+k*m==0],x,y)
```

```
[[x == (k*m - j*n)/(j*l - h*m), y == -(k*l - h*n)/(j*l - h*m)]]
```

We have thus verified (15). What about its whereabouts, then? As the point (0, -2) lies on the intersection point of (10) [26] , we infer that the RHS of (15) lies on the intersection point of (12). Sure enough, $(\frac{jn-km}{hm-j\ell}, \frac{k\ell-hn}{hm-j\ell})$, or the RHS of (15), coincides with $\frac{1}{hm-j\ell}\begin{pmatrix} -km+jn \\ k\ell-hn \end{pmatrix}$ mentioned in footnote 25. Moreover, substituting (15) into (12) yields $(h\cdot\frac{jn-km}{hm-j\ell}+j\cdot\frac{k\ell-hn}{hm-j\ell}+k)\cdot(\ell\cdot\frac{jn-km}{hm-j\ell}+m\cdot\frac{k\ell-hn}{hm-j\ell}+n) = \{\frac{h(jn-km)+j(k\ell-hn)+k(hm-j\ell)}{hm-j\ell}\}\cdot\{\frac{\ell(jn-km)+m(k\ell-hn)+n(hm-j\ell)}{hm-j\ell}\} = (\frac{hjn-hkm+jk\ell-jhn+khm-kj\ell}{hm-j\ell}) \cdot(\frac{\ell jn-\ell km+mk\ell-mhn+nhm-nj\ell}{hm-j\ell}) = \frac{0}{hm-j\ell}\cdot\frac{0}{hm-j\ell} = 0\cdot 0 = 0$ [27] . Thus, the point (0, -2) is to (10) what the RHS of (15) is to (12). Therefore, like (10), (12) is classified into the category **ON** [28] , and we note that such categorization is immutable following a certain generalization.

Now that we seem to be able to achieve some generalizations, we proceed to deduce the two intersecting lines $x^2-(y+2)^2 = 0$ [29] from (12). Replacing h, j, k, ℓ, m, and n in (12) by 1, 1, 2, 1, -1, and -2, respectively, we get $(x+y+2)(x-y-2)$, which amounts to $\{x+(y+2)\}\{x-(y+2)\} = x^2-(y+2)^2$, one of the cases we have already described [30] . Can we further proceed to deduce their *SING*, or the point (0, -2) [31] , from something more general, then? Likewise, replacing $h,\ldots,$ n in the RHS of (15) with $1,\ldots,-2$, respectively, we get $(x, y) = (\frac{1\times(-2)-2\times(-1)}{1\times(-1)-1\times 1}, \frac{2\times 1-1\times(-2)}{1\times(-1)-1\times 1})$, the RHS of which amounts to the point (0, -2), the very *SING* we obtained in **3.5**. We have thus managed to deduce the two intersecting lines $x^2-(y+2)^2 = 0$ and their *SING* (0, -2) from (12) and (15), respectively.

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[26] See **3.5**, **3.6**, and **4**.

[27] By the way, substituting (15) into the LHS 's of (14) also yields 0's, which in turn confirms that we solved (14) correctly.

[28] See **3.5** and **4**.

[29] See **3.5**, **3.6**, and **4**.

[30] See, *e.g.,* **3.5**.

[31] See **3.5**, **3.6**, and **4**.

Next, we try to generalize the double line $(x+y)^2 = 0$ [32] in a similar manner. Let us consider

$$\phi = (ox+py+q)^2 = 0, \qquad o, p, q \in \mathbb{R} . \tag{16}$$

Expanding $(ox+py+q)^2$, one gets

$$\phi = o^2x^2 + p^2y^2 + q^2 + 2opxy + 2pqy + 2oqx.$$

So

$$\begin{aligned}\frac{d\phi}{dx} &= \frac{d}{dx}(o^2x^2 + p^2y^2 + q^2 + 2opxy + 2pqy + 2oqx)\\ &= 2o^2x + 2p^2y\frac{dy}{dx} + 2opy + 2opx\frac{dy}{dx} + 2pq\frac{dy}{dx} + 2oq,\end{aligned}$$

and we get the $1-$form

$$\begin{aligned}\omega = d\phi &= 2o^2xdx + 2p^2ydy + 2opydx + 2opxdy + 2pqdy + 2oqdx\\ &= 2o(ox+py+q)dx + 2p(ox+py+q)dy.\end{aligned} \tag{17}$$

Though it is clear that it follows from (16) that $ox+py+q = 0$, we try resorting to *reductio ad impossibilem* . Specifically, we venture to suppose $ox+py+q \neq 0$. Then, for us to obtain a *SING* from (17), the relation $2o = 2p = 0$ must hold, that is, we have $o = p = 0$. Now we plug $o = 0$ and $p = 0$ into (16) to get $q^2 = 0$, which means that $q = 0$, too. Hence, we have $o = p = q = 0$, from which it follows that $ox+py+q = 0\cdot x+0\cdot y+0 = 0$. But this contradicts our supposition $ox+py+q \neq 0$. So we have to admit that $ox+py+q = 0$. In any event, we have $ox+py+q = 0$, and consequently *SING* is the line $ox+py+q = 0$. We now imagine (16) and its *SING* overlap like the line s in Fig. 3 and/or Fig. 4. And like (11), (16) is classified into the category **IT** [33] . Again, we note that such categorization is immutable following a certain generalization. What about deduction, then? We can deduce the double line $(x+y)^2 = 0$ and the corresponding $1-$form $\omega = 2(x+y)dx+2(x+y)dy$, together with the *SING* $x+y = 0$ [34] , from plugging $o = 1$, $p = 1$, and $q = 0$ into (16) and (17). This means that (11) intended as a sheer example has generalized at least slightly [35] . Taken together, we could generalize two cases at least

[32] See **3.7** and **4**.

[33] See **3.7** and **4**.

[34] Ditto.

[35] In **7.2.3**, we deal with the two parallel lines $(x+y)^2 = 1$. This is a special case of $(ox+py+q)^2 = r^2$, $o, p, q, r \in \mathbb{R}$, which is a further generalization of (16) and will be discussed elsewhere.

slightly, while keeping intact the *SING*-based categories to which they belong, and deduce such cases from something more general.

6 Discussion

At the outset, we note that for a point to be called a *SING*, we need not restrict ourselves to homogeneous polynomial s such as $x^2+2xy+y^2$, x^2+xz, and so on, though such polynomial s have been known to play a certain role in the field of algebraic geometry [36] . Actually, we were able to derive the *SING* (4, 4) from $x^2 + y^2 - 8x - 8y + 31$, an inhomogeneous polynomial [37] .

Next, we deform some shape s in **Table** of **4** in order to know whether/how such deformations affect *SING*'s. We try doubling the radius 1 in (6) to obtain $\phi = (x-4)^2 + (y-4)^2 - (1\cdot 2)^2 = (x-4)^2 + (y-4)^2 - 4 = 0$. Then, we note that both $\frac{d\phi}{dx} = \frac{d}{dx}\{(x-4)^2 + (y-4)^2 - 4\} = \frac{d}{dx}(x^2 - 8x + y^2 - 8y + 28) = 2x - 8 + 2y\frac{dy}{dx} - 8\frac{dy}{dx} = 2x + 2y\frac{dy}{dx} - 8 - 8\frac{dy}{dx}$ and the resultant 1−form $\omega = d\phi = 2(x-4)dx + 2(y-4)dy$ remain the same, so does the *SING* (4, 4) [38] . Furthermore, we deform the ellipse (7) by replacing its term $(y-4)^2$ with $4(y-4)^2$ to get the circle $4(x+4)^2 + 4(y-4)^2 - 1^2 = 0$ [39] . Likewise, we get the 1-form $\omega = 8(x+4)dx + 8(y-4)dy$, which proves different from $8(x+4)dx + 2(y-4)dy$, the original one [40] , and the *SING* (−4, 4), which remains the same [41] . We thus notice at least in these two cases, deformation *can* affect the 1-form $\omega = d\phi$, but not the whereabouts of *SING*'s, which makes *SING*'s look like fixed point s [42] .

Thirdly, we wish to mention geometric interpretation of *SING*. In general, a point (x, y) on a Cartesian coordinate plane can be regarded as a column vector

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[36] *Cf.* **Definition 1.10.2** and **Exercise 1.10.8** in [8].

[37] See, *e.g.,* **3.1**.

[38] *Cf.* **3.1**, **3.6**, and **4**.

[39] This is not so surprising, since the circle is a special case of the ellipse .

[40] See **3.2** and **4**.

[41] See **3.2**, **3.6**, and **4**.

[42] However, we can 'move' *SING*'s and apply them to a problem on merging black holes , which will be discussed elsewhere.

$\begin{pmatrix} x \\ y \end{pmatrix}$ [43], which we rewrite as the following linear combination :

$$x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = x\mathbf{e_1} + y\mathbf{e_2}. \tag{18}$$

By the way, a 2×2 matrix B acts on such a column vector like this:

$$\begin{aligned} B \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} r & s \\ t & u \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx+sy \\ tx+uy \end{pmatrix} = \begin{pmatrix} rx \\ tx \end{pmatrix} + \begin{pmatrix} sy \\ uy \end{pmatrix} = x \begin{pmatrix} r \\ t \end{pmatrix} + y \begin{pmatrix} s \\ u \end{pmatrix} \\ &= x\mathbf{e_3} + y\mathbf{e_4}, \qquad r, s, t, u \in \mathbb{R}. \end{aligned} \tag{19}$$

Comparing (18) with (19), we observe that B acts on the standard bases $\mathbf{e_1}, \mathbf{e_2}$, which form a unit square, by transforming them into bases $\mathbf{e_3}, \mathbf{e_4}$, which now form a parallelogram [44, 45]. We relate the above comparison to (4) as follows:

Equating the RHS of (4) with 0, one gets

$$\begin{cases} 2ax + by + e = 0, \\ bx + 2cy + f = 0. \end{cases}$$

In matrix notation, we have

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{20}$$

We rewrite the above as $C\vec{x} + \vec{b} = \vec{0}$ and recall affine transformation on a unit square [9]. Then, we have

[43] See, *e.g.,* footnote 25.

[44] We take it for granted that $\mathbf{e_3}, \mathbf{e_4} \neq \vec{0}$. And we assume $\mathbf{e_3} \nparallel \mathbf{e_4}$, *i.e.,* $ru - st \neq 0$. See here . In other words, we assume that B is invertible . A concrete example is here .

[45] Under the assumptions that $\mathbf{e_3}, \mathbf{e_4} \neq \vec{0}$ and $\mathbf{e_3} \nparallel \mathbf{e_4}$, if $\mathbf{e_3} \perp \mathbf{e_4}$, we get a rectangle or a square . On the other hand, if $\mathbf{e_3} \not\perp \mathbf{e_4}$ and the length of $\mathbf{e_3}$ equals that of $\mathbf{e_4}$, we get a rhombus . We regard these three as derivable from (deforming) a parallelogram . See, *e.g.,* here .

$$\begin{cases} C\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0}, \\ C\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2a \\ b \end{pmatrix} = \vec{c}, \\ C\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ 2c \end{pmatrix} = \vec{d}, \\ C\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2a+b \\ b+2c \end{pmatrix} = \vec{e}. \end{cases}$$

Since $\vec{c}+\vec{d}=\vec{e}$, the parallelogram rule reminds us of a parallelogram P, in which $\vec{c} \nparallel \vec{d}$ [46] . And the 'transition' from the LHS of (20) to $\vec{0}$, or the RHS of (20), leads us to imagine P (dwindling and) ending up with the origin $O(0, 0)$ subsequent to translation . So geometrically, to get a *SING* seems a bit analogous to managing to efface such a parallelogram .

Having thus far discussed *SING*'s in 2D, we touch on their 3D version. We consider, *e.g.*, $\phi = x^3+y^3+z^3-3xyz+2 = 0$. So $\frac{d\phi}{dx} = \frac{d}{dx}(x^3+y^3+z^3-3xyz+2) = 3x^2+3y^2\frac{dy}{dx}+3z^2\frac{dz}{dx}-3yz-3zx\frac{dy}{dx}-3xy\frac{dz}{dx}$, and we get the 1−form $\omega = d\phi = 3x^2dx+3y^2dy+3z^2dz-3yzdx-3zxdy-3xydz = 3(x^2-yz)dx+3(y^2-zx)dy+3(z^2-xy)dz$, which we equate with 0 to obtain

$$\begin{cases} x^2-yz=0, & (21) \\ y^2-zx=0, & (22) \\ z^2-xy=0. & (23) \end{cases}$$

Manipulating $2\times\{(21)+(22)+(23)\}$ yields $x^2-2xy+y^2+y^2-2yz+z^2+z^2-2zx+x^2=0$, which becomes $(x-y)^2+(y-z)^2+(z-x)^2=0$, and we have $x-y=y-z=z-x=0$. Thus, *SING* is $x=y=z$, or the line $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$. Although we encountered *SING* as a line before [47] , there may well be the following question:

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[46] We take it for granted that $\vec{c}, \vec{d} \neq \vec{0}$. *Cf.* footnote 44. Again, a rectangle , a rhombus , and a square are regarded as special cases of a parallelogram . See footnote 45.

[47] See **3.7** and **4**.

Question 6.1. Does at least one singularity remain a point subsequent to a slightly different definition?

We answer this question in the affirmative [48] . Indeed, even the *SING* $x + y = 0$ [49] can be interpreted as point s on a line [50] . But since intersection of lines *does* yield a point [51] , it seems possible for proper combination(s) of *SING*'s to work behind the scenes of the so-called singularities , and another (insidious) question arises:

Question 6.2. What if we perceive the intersection of *SING* curve s [52] in, *e.g.,* a plane, a 2D object , as a (usual) singularity ?

At any rate, we wish to propose the notion of *SING*, which is able to vanish $1-$form s such as (4), (5), and so forth, although we put aside, *e.g.,* whether a ring singularity seen in a certain field of physics should be thought of as actually composed of aggregated point singularities and won't try to answer *Question* 6.2 and those raised in footnotes 79 and 82 for the time being.

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[48] See **7.1**, in which we deal with the case where singularity is identical to *SING*.

[49] See **3.7** and **4**. By the way, this is a special case of the line $ox + py + q = 0$. See **5**.

[50] *Cf.* **Exercise 1.10.7** in [8].

[51] See, *e.g.,* left part of Fig. 8.1 in [10].

[52] As mentioned in footnote 15, line s are regarded as curve s.

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7 Appendix

7.1 What about the (celebrated) cusp $(0, 0)$ of the semicubical parabola $y^2 = x^3$?

We consider $\phi = \ y^2 - x^3 = 0$. So $\frac{d\phi}{dx} = 2y\frac{dy}{dx} - 3x^2$, and we get the $1-$form $\omega = d\phi = -3x^2dx + 2ydy$. Thus, *SING* is the point (0, 0) on the curve . The curve is therefore classified into the category **ON** [53] . In this case, *SING* coincides with the singularity on the curve [54] , and we now learn that the curve is a ‘close rela-

[53] See **3.5** and **4**. *Cf.* footnote 16.

[54] See footnote 48.

tive' of the two intersecting lines $x^2 - (y+2)^2 = 0$ [55] in terms of *SING* [56] .

7.2 What about, *e.g.*, $x^2 + y^2 = 0$, $x + y - 1 = 0$, and so forth?

In this subsection, we deal with a few equation s we failed to mention.

7.2.1 $x^2 + y^2 = 0$: a point

This is obtained by plugging into (1) $a = 1$, $b = 0$, $c = 1$, $e = 0$, $f = 0$, and $g = 0$, and we consider $\phi = x^2 + y^2 = 0$. So $\frac{d\phi}{dx} = \frac{d}{dx}(x^2 + y^2) = 2x + 2y\frac{dy}{dx}$, and we get the $1-$form $\omega = d\phi = 2xdx + 2ydy$. Thus, *SING* is the point $(0, 0)$, which is also $x^2 + y^2 = 0$ itself [57] . The equation $x^2 + y^2 = 0$ is therefore classified into the category **IT** [58] , [59] , and we now learn such a point is a 'close relative' of the double line $(x+y)^2 = 0$ [60] in terms of *SING* [61] .

7.2.2 $x + y - 1 = 0$: a line

This is obtained by plugging into (1) $a = 0$, $b = 0$, $c = 0$, $e = 1$, $f = 1$, and $g = -1$, and we consider $\phi = x + y - 1 = 0$. So $\frac{d\phi}{dx} = \frac{d}{dx}(x + y - 1) = 1 + \frac{dy}{dx}$, and we get the $1-$form $\omega = d\phi = dx + dy$, which means ω doesn't vanish. The equation $x + y - 1 = 0$ is therefore classified into the category **NO** [62] , and we now learn such a line is a 'close relative' of the parabola $y = x^2 + 1$ [63] in terms of *SING* [64] .

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[55] See **3.5**, **3.6**, and **4**.

[56] By the way, a hyperbola can degenerate into two lines crossing at a point .

[57] We neglect $(1, \iota)$, $(-\iota, -1)$, etc satisfying the equation $x^2 + y^2 = 0$. See also here .

[58] See **3.7** and **4**.

[59] By the way, $x^2 + y^2 = r^2$ falls into the category **IN**, if $r > 0$, which is because computing $\frac{d}{dx}(x^2 + y^2 - r^2) = 2x + 2y\frac{dy}{dx}$ results in the $1-$form $\omega = 2xdx + 2ydy$ and the point $(0, 0)$, or the *SING* lying inside the circle $x^2 + y^2 = r^2$. *Cf.* **3.1**.

[60] See **3.7** and **4**.

[61] By the way, a circle and an ellipse can degenerate into a point .

[62] See **3.3**, **3.6**, and **4**.

[63] Ditto.

[64] By the way, a circle or a parabola can degenerate into a line .

7.2.3 $(x+y)^2-1=0$: two parallel lines

Replacing a, b, c, e, f, and g in (1) by 1, 2, 1, 0, 0, and -1, respectively, we get $x^2+2xy+y^2-1$, for which we complete the square to obtain $(x+y)^2-1=0$ [65]. This can be rewritten as $(x+y+1)(x+y-1)=0$, the product of the following equation s:

$$\begin{cases} x+y+1=0, \\ x+y-1=0. \end{cases}$$

These are parallel to each other. Now we consider $\phi = x^2+2xy+y^2-1=0$. So $\frac{d\phi}{dx} = \frac{d}{dx}(x^2+2xy+y^2-1) = 2x+2y+2x\frac{dy}{dx}+2y\frac{dy}{dx} = 2(x+y)+2(x+y)\frac{dy}{dx}$, and we get the 1−form $\omega = d\phi = 2(x+y)dx+2(x+y)dy$. Thus, *SING* is the line $x+y=0$. The *SING* lies between those parallel lines , which are therefore classified into the category **OUT** [66] . We now learn that such two parallel lines are a 'close relative' of the hyperbola $x^2-y^2-61=0$ [67] in terms of *SING* [68] .

7.3 Two kinds of rational function s

By plugging into (1) $a=0$, $b=1$, $c=0$, $e=-1$, $f=0$, and $g=-1$, we get $xy-x-1=0$, *i.e.,* $xy=x+1$. Dividing both sides of it by x yields the explicit function $y=\frac{x+1}{x}$ [69] , and henceforth, we call such stuff a rational function in explicit form (RFE). We now consider $\phi = y-\frac{x+1}{x}=0$. So $\frac{d\phi}{dx} = \frac{d}{dx}(y-\frac{x+1}{x}) = \frac{dy}{dx}+\frac{1}{x^2}$, and we get the 1−form $\omega = d\phi = \frac{dx}{x^2}+dy$ [70] . Even if we let $x \to +\infty$ (or $-\infty$) to vanish dx, dy remains, which means that ω doesn't vanish. The RFE $y=\frac{x+1}{x}$ is therefore classified into the category **NO** [71] , and we now learn that such an RFE is a 'close

[65] See footnote 35.

[66] See **3.4** and **4**.

[67] See **3.4**, **3.6**, and **4**.

[68] The pencil of ellipses of equations $ax^2+b(y^2-1)=0$ can degenerate into two parallel lines .

[69] This has a (conventional) singularity at $x=0$.

[70] $\frac{1}{x^2}$ cannot be defined at $x=0$. *Cf.* here .

[71] See **3.3**, **3.6**, and **4**.

relative' of the parabola $x^2 - y + 1 = 0$ [72] in terms of *SING*. On the other hand, what if we regard the equation $xy - x - 1 = 0$ as defining an implicit function ? Hereafter, we call such stuff a rational function in implicit form (RFI) and consider $\phi = xy - x - 1 = 0$. So $\frac{d\phi}{dx} = \frac{d}{dx}(xy - x - 1) = y + x\frac{dy}{dx} - 1$, and we get the 1-form $\omega = d\phi = (y-1)dx + xdy$. Contrary to the aforementioned RFE case, *SING* is identified as the point (0, 1), which coincides with the center of the hyperbola $x(y-1) = 1$ [73] . The RFI $xy - x - 1 = 0$ is therefore classified into the category **OUT** [74] . We now learn such an RFI is a 'close relative' of the hyperbola $x^2 - y^2 - 61 = 0$ [75] in terms of *SING*. We have thus dealt with two kinds of rational function s discernible by the presence (or absence) of *SING*.

7.4 Relationship between hyperbola and RFI

Inspired by the abovementioned presence of **OUT** which relates the hyperbola $x^2 - y^2 - 61 = 0$ to the RFI $xy - x - 1 = 0$, we try to know whether we can transform the hyperbola into RFI (or vice versa). To be specific, we seek the following transformation:

$$\frac{(x-h)^2}{j^2} - \frac{(y-k)^2}{\ell^2} = 1, \qquad h,\, j,\, k,\, \ell \in \mathbb{R}, \qquad j, \ell \neq 0 \tag{24}$$

[76]

$$\xrightarrow{\text{Some transformation}} (mX - n)(oY - p) = 1,\ \ m, n, o, p \in \mathbb{R},\ m, o \neq 0. \tag{25}$$

Eliminating 1 between (24) and (25) yields

$$\frac{(x-h)^2}{j^2} - \frac{(y-k)^2}{\ell^2} = (mX - n)(oY - p). \tag{26}$$

Applying the identity $\frac{(q+r)^2 - (q-r)^2}{4} = qr$ to the RHS of (26), we get

[72] Ditto.

[73] Relevance of hyperbola to RFI will be discussed in the next subsection.

[74] See **3.4** and **4**.

[75] See **3.4**, **3.6**, and **4**.

[76] Again, we substituted 'j' for 'i' lest 'i' should be confused with the imaginary unit $\imath$. See footnote 24.

$$\frac{1}{4}\cdot[\{(mX-n)+(oY-p)\}^2-\{(mX-n)-(oY-p)\}^2]=(mX-n)(oY-p). \qquad (27)$$

Eliminating $(mX-n)(oY-p)$ between (26) and (27), one gets

$$\frac{1}{4}\cdot[\{(mX-n)+(oY-p)\}^2-\{(mX-n)-(oY-p)\}^2]=\frac{(x-h)^2}{j^2}-\frac{(y-k)^2}{\ell^2}. \qquad (28)$$

We thus write

$$\begin{cases}\frac{x-h}{j}=\frac{1}{2}\cdot(mX-n+oY-p),\\ \frac{y-k}{\ell}=\frac{1}{2}\cdot(mX-n-oY+p),\end{cases} \qquad (29)$$ [77]

which we rewrite as

$$\begin{cases}x=\frac{j}{2}(mX-n+oY-p)+h=\frac{jmX+joY-jn-jp+2h}{2},\\ y=\frac{\ell}{2}(mX-n-oY+p)+k=\frac{\ell mX-\ell oY-\ell n+\ell p+2k}{2}.\end{cases} \qquad (30)$$

In matrix language, we have

$$\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\frac{jm}{2} & \frac{jo}{2}\\ \frac{\ell m}{2} & -\frac{\ell o}{2}\end{pmatrix}\begin{pmatrix}X\\ Y\end{pmatrix}+\begin{pmatrix}\frac{-jn-jp+2h}{2}\\ \frac{-\ell n+\ell p+2k}{2}\end{pmatrix}, \qquad (31)$$

an affine transformation . Incidentally, solving (31) for X, Y gives

$$\begin{pmatrix}X\\ Y\end{pmatrix}=\begin{pmatrix}\frac{jm}{2} & \frac{jo}{2}\\ \frac{\ell m}{2} & -\frac{\ell o}{2}\end{pmatrix}^{-1}\left\{\begin{pmatrix}x\\ y\end{pmatrix}-\begin{pmatrix}\frac{-jn-jp+2h}{2}\\ \frac{-\ell n+\ell p+2k}{2}\end{pmatrix}\right\}$$

[77] For simplicity's sake, we let the pair (29) represent pairs satisfying (28). Other ones than (29) include:

$$\begin{cases}\frac{x-h}{j}=\frac{1}{2}\cdot(mX-n+oY-p),\\ \frac{y-k}{\ell}=-\frac{1}{2}\cdot(mX-n-oY+p),\end{cases}\qquad \begin{cases}\frac{x-h}{j}=-\frac{1}{2}\cdot(mX-n+oY-p),\\ \frac{y-k}{\ell}=\frac{1}{2}\cdot(mX-n-oY+p),\end{cases}$$

and

$$\begin{cases}\frac{x-h}{j}=-\frac{1}{2}\cdot(mX-n+oY-p),\\ \frac{y-k}{\ell}=-\frac{1}{2}\cdot(mX-n-oY+p).\end{cases}$$

The interested reader is invited to substitute the LHS 's of these into the RHS of (28) and check.

$$= \begin{pmatrix} \frac{1}{jm} & \frac{1}{\ell m} \\ \frac{1}{jo} & -\frac{1}{\ell o} \end{pmatrix} \left\{ \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \frac{-jn-jp+2h}{2} \\ \frac{-\ell n+\ell p+2k}{2} \end{pmatrix} \right\}$$

$$= \begin{pmatrix} \frac{1}{jm} & \frac{1}{\ell m} \\ \frac{1}{jo} & -\frac{1}{\ell o} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \frac{1}{jm} & \frac{1}{\ell m} \\ \frac{1}{jo} & -\frac{1}{\ell o} \end{pmatrix} \begin{pmatrix} \frac{-jn-jp+2h}{2} \\ \frac{-\ell n+\ell p+2k}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{jm} & \frac{1}{\ell m} \\ \frac{1}{jo} & -\frac{1}{\ell o} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{j\ell n-h\ell-jk}{j\ell m} \\ \frac{j\ell p-h\ell+jk}{j\ell o} \end{pmatrix},$$

which is an affine transformation , too. Rewriting the above yields

$$\begin{cases} X = \frac{\ell x+jy+j\ell n-h\ell-jk}{j\ell m}, \\ Y = \frac{\ell x-jy+j\ell p-h\ell+jk}{j\ell o} \end{cases} {}^{78}.$$

7.5 Some 4D cases

We touch on the following:

Example 7.5.1. $\phi = 3w^2 - x^2 - y^2 - z^2 - 2w(x+y+z) = 0.$ (32)

Since

$$\frac{d\phi}{dw} = \frac{d}{dw}\{3w^2 - x^2 - y^2 - z^2 - 2w(x+y+z)\}$$
$$= 6w - 2x\frac{dx}{dw} - 2y\frac{dy}{dw} - 2z\frac{dz}{dw} - 2(x+y+z) - 2w(\frac{dx}{dw} + \frac{dy}{dw} + \frac{dz}{dw}),$$

we get the 1-form

$$\omega = d\phi = 6wdw - 2xdx - 2ydy - 2zdz - 2xdw - 2ydw - 2zdw - 2wdx$$
$$-2wdy - 2wdz$$
$$= (6w - 2x - 2y - 2z)dw - 2(w+x)dx - 2(w+y)dy - 2(w+z)dz.$$

So we have

$$\begin{cases} 6w - 2(x+y+z) = 0, & (33) \\ -2(w+x) = 0, & (34) \\ -2(w+y) = 0, & (35) \\ -2(w+z) = 0. & (36) \end{cases}$$

[78] *Cf.* (30).

It follows from (34) – (36) that $x = -w$, $y = -w$, and $z = -w$, which we plug into the LHS of (33) to get $12w = 0$. Hence, we have $w = 0$, from which it follows that $w = x = y = z = 0$. Thus, *SING* is the origin O (0, 0, 0, 0) [79] on (32), which is therefore classified into the category **ON** [80] .

Example 7.5.2. $\phi = 3w^2 + x^2 + y^2 + z^2 - 2w(x+y+z) = 0.$ (37)

Likewise,

$$\begin{aligned}\frac{d\phi}{dw} &= \frac{d}{dw}\{3w^2 + x^2 + y^2 + z^2 - 2w(x+y+z)\}\\ &= 6w + 2x\frac{dx}{dw} + 2y\frac{dy}{dw} + 2z\frac{dz}{dw} - 2(x+y+z) - 2w(\frac{dx}{dw} + \frac{dy}{dw} + \frac{dz}{dw}),\end{aligned}$$

and we get the 1-form

$$\begin{aligned}\omega = d\phi &= 6wdw + 2xdx + 2ydy + 2zdz - 2xdw - 2ydw - 2zdw - 2wdx\\ &\quad -2wdy - 2wdz\\ &= (6w - 2x - 2y - 2z)dw + 2(x-w)dx + 2(y-w)dy + 2(z-w)dz.\end{aligned}$$

So we have

$$\begin{cases} 6w - 2(x+y+z) = 0, & (38)\\ 2(x-w) = 0, & (39)\\ 2(y-w) = 0, & (40)\\ 2(z-w) = 0. & (41)\end{cases}$$

It follows from (39) – (41) that $w = x$, $w = y$, and $w = z$ [81] , and we have $w = x = y = z$. Thus, *SING* is the line $\frac{w}{1} = \frac{x}{1} = \frac{y}{1} = \frac{z}{1}$ [82] . Now we rewrite (37) as $(w-x)^2 + (w-y)^2 + (w-z)^2 = 0$, from which it follows that $w - x = w - y = w - z = 0$. Hence, we have $w = x = y = z$, which proves to be the same as the *SING*. (37) is therefore classified into the category **IT** [83] .

[79] But what if such a point comes from the intersection of lines ? *Cf. Question* 6.2 in **6**.

[80] See **3.5** and **4**.

[81] If we plug the RHS 's of these equation s into the LHS of (38), we get $6w - 2(w+w+w) = 0$, the trivial.

[82] But what if this line came from the intersection of planes ? See **6**.

[83] See **3.7** and **4**.