The curvature and dimension of a closed surface

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September 26, 2019

Abstract

The curvature of a closed surface can lead to fractional dimension. In this paper, the properties of the 2-sphere surface of a three-dimensional ball and the \( 2.\times \)-dimensional surface (2.\times -surface) of a three-dimensional fractal set are considered. Tessellation is used to approximate each surface, primarily because the 2.\times -surface of a three-dimensional fractal set is otherwise non-differentiable (having no well-defined surface normals).

1 Tessellation of closed surfaces

Approximating the 2.\times -surface of a three-dimensional shape via triangular tessellation (a mesh) allows us to calculate the 2.\times -surface’s dimension \( D \in (2.0, 3.0) \).

First we calculate, for each triangle, the average dot product of the triangle’s normal \( \hat{n}_i \) and its three neighbouring triangles’ normals \( \hat{o}_1, \hat{o}_2, \hat{o}_3 \):

\[
d_i = \frac{\hat{n}_i \cdot \hat{o}_1 + \hat{n}_i \cdot \hat{o}_2 + \hat{n}_i \cdot \hat{o}_3}{3}.
\]  
(1)

Because we assume that there are three neighbours per triangle, the mesh must be closed (no cracks or holes, precisely two triangles per edge).

Then we calculate the normalized measure:

\[
m_i = \frac{1 - d_i}{2}.
\]  
(2)

Once \( m_i \) has been calculated for all triangles, we can then calculate the average normalized measure \( \lambda \), where \( t \) is the number of triangles:

\[
\lambda = \frac{\sum_{i=1}^{t} m_i}{t}.
\]  
(3)

The dimension of the closed surface is:

\[
D = 2 + \lambda.
\]  
(4)

In this paper, Marching Cubes [1] is used to generate the 2.\times -dimensional triangle meshes. The full C++ code for this paper can be found at [2].

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2 Conclusions

For a 2-sphere, the local curvature all but vanishes as the maximum triangle edge length $L$ decreases:

$$\lim_{L \to 0} \lambda(L) = 0. \quad (5)$$

To decrease $L$, one must increase the sampling resolution $r$ (an integer, greater than or equal to 2), where $g_{\text{max}}$ is the sampling grid maximum and $g_{\text{min}}$ is the sampling grid minimum:

$$L = \sqrt{3} \left( \frac{g_{\text{max}} - g_{\text{min}}}{r - 1} \right). \quad (6)$$

This results in a dimension of practically (but never quite) 2.0, which is to be expected from a non-fractal surface. See Figures 1 - 3.

On the other hand, for the 2.x-surface of a three-dimensional fractal set, the local curvature does not vanish:

$$\lim_{L \to 0} \lambda(L) \neq 0. \quad (7)$$

This results in a dimension considerably greater than 2.0, but not equal to or greater than 3.0, which is to be expected from a fractal surface. See Figures 4 - 7.

As far as we know, this method of calculating the dimension of a closed surface is novel.

References


Figure 1: Low resolution ($r = 10$) surface for the iterative equation is $Z = Z^2$. The surface’s dimension is $2.02$.

Figure 2: Medium resolution ($r = 100$) surface for the iterative equation is $Z = Z^2$. The surface’s dimension is $2.06$.

Figure 3: High resolution ($r = 1000$) surface for the iterative equation is $Z = Z^2$. The surface’s dimension is practically $2.0$. 
Figure 4: Low resolution \((r = 10)\) surface for the iterative equation is \(Z = Z \cos(Z)\). The surface’s dimension is 2.05.

Figure 5: Medium resolution \((r = 100)\) surface for the iterative equation is \(Z = Z \cos(Z)\). The surface’s dimension is 2.11.

Figure 6: High resolution \((r = 1000)\) surface for the iterative equation is \(Z = Z \cos(Z)\). The surface’s dimension is 2.08.
Figure 7: A two-dimensional slice of $Z = Z \cos(Z)$, showing the fractal nature of the set.