A rational cover

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Abstract

This letter demonstrates in an elegant way that Cantor's postulate that there's a bijection between any two countable infinite sets is flawed.

There's an interesting theorem in contemporary mathematics stating that one can cover all the rational numbers in the interval $[0, 1]$ by non-vanishing intervals whose union is of length less than 1. The proof goes as follows.

One can list all the rationals in interval $[0, 1]$ in an infinitely long list like this:

$$\begin{array}{llll} \frac{1}{2} & & & \\ \frac{1}{3} & \frac{2}{3} & & \\ \frac{1}{4} & \frac{3}{4} & & \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \\ \vdots & & & \end{array}$$

Now cover each of these rationals by an interval, each of length 3^{-n}, with n being naturals, and let numbers n count rationals in the order as written above. So, cover point $1/2$ on the number line by an interval of length $1/3$, cover $1/3$ by an interval of length $1/9$, cover $2/3$ by an interval of length $1/27$ and so on.

There are obviously countably many rationals in interval $[0, 1]$, since we've listed them all just now in the list above. There are also countably many naturals n. Hence, one can indeed cover each rational from the above list by an interval of length 3^{-n}. We notice that each of the covering intervals is of a non-vanishing length 3^{-n} for any n.

This way, the union of all the intervals of lengths 3^{-n} is at most of length

$$\sum_{n \in \mathbb{N}} 3^{-n} = \frac{1}{2} \tag{1}$$

This is a well known result.

Now, consider this.

There are infinitely many rationals in interval $[0, 1]$. Actually, there is a rational between any two distinct reals. Just consider any two distinct reals written in a decimal form. There's always a number with finite number of decimals that fits in-between any two reals. For instance, between 0.1 and 0.11 one can fit 0.105. This is obviously a trivial conclusion. So one can fit a rational between any two distinct reals.

Now, if the cover of all rationals in interval $[0, 1]$ is of length $1/2$, whilst the interval $[0, 1]$ is obviously of length 1, then there are plenty of pairs of reals not being covered. Since one can fit a rational between any two reals, it's obvious that this covering does not cover all the rationals in interval $[0, 1]$ then.

So what went wrong? This conclusion that one can cover rationals in interval $[0, 1]$ by a cover of arbitrary length is considered rigorous and true by contemporary mathematics. And it's obviously wrong. So what went wrong?

What went wrong is that there aren't as many rationals in interval $[0, 1]$ as there are naturals altogether. There are more rationals than naturals. Just consider the list above, listing some rationals in interval $[0, 1]$. The first column reads

$$\begin{array}{l} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \frac{1}{5} \\ \vdots \end{array}$$

So there's obviously a bijection between the first column above and all the naturals n.

But there are more rationals. By looking at the list, it seems that there are about $O(N^2)$ rationals for all sufficiently large naturals N. There are definitely more rationals in the list than there are naturals: just look at it.

Now, if we discard Cantor's postulate that there's a bijection between any two countable infinite sets, then we see where the problem is: the covering with covers of lengths 3^{-n} cannot cover all the rationals in interval $[0, 1]$. It only covers first N, as N grows without bounds. So some rationals are simply left uncovered, since there are infinitely more rationals in interval $[0, 1]$ than there are naturals overall.

This demonstrates in an elegant way that Cantor's postulate that there's a bijection between any two countable infinite sets is flawed.