ON THE IMPOSSIBILITY OF THE EXISTENCE OF CYCLES INCLUDING INTEGER MULTIPLES OF 3 IN A COLLATZ SEQUENCE

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ABSTRACT. As is well known, the Collatz sequence, which is also named as the hailstone sequence, follows the rule of Collatz conjecture. The rule requires us to divide any positive even integer by 2. We must multiply every positive odd number by 3 and then add 1 according to the rule. By investigating residues modulo 3, I will prove any integer multiple of 3 cannot appear more than one time in a Collatz sequence, which implies any multiple of 3 cannot be included in a possible cycle of the Collatz sequence.

As Lagarias [1] noted, the Collatz sequence seems to have diverse mathematical implications, and I focus on modulo arithmetic to handle this sequence. In Theorem 1 and 2, by showing that any multiple of 3 cannot appear more than once in a Collatz sequence, I will prove that any possible cycle of a Collatz sequence cannot have any multiple of 3. According to the premise of Collatz conjecture, every integer in this paper is supposed to be positive.

Theorem 1. Any odd multiple of 3 cannot be generated more than one time in a Collatz sequence, and any subsequent term after an odd multiple of 3 cannot be a multiple of 3.

Proof. Let x, which is an arbitrary odd multiple of 3, be the first term of a Collatz sequence. Since x is an odd integer, according to the odd number rule of Collatz conjecture, it moves into an even integer which can be written in the form $3x+1$. Since $3x+1$ cannot be divided by 3, it is not a multiple of 3. Thus, the next odd integer y, which is generated when we divide $3x+1$ by 2 one or more times according to the even number rule of Collatz conjecture, cannot be a multiple of 3. It is because when we suppose such an odd integer $3k$, it must be true that $(3x+1)/2^n = 3k$. Then $3x+1 = 3k \cdot 2^n$. This is contradictory since the left side of the equation is not divided by 3, but the right side must be a multiple of 3. Thus, we know y cannot be a multiple of 3. To sum up, when an odd integer x moves to an even integer $3x+1$, the next odd integer y cannot be a multiple of 3 because $3x+1$ is a multiple of y, but $3x+1$ is not a multiple of 3.
In addition, we know that this is the same with every possible even integer generated when $3x+1$ is divided by 2 one or more times, while moving toward y. Let us assume that s, an even multiple of 3, exists during the procedure. Then $s = 6t$ and $(3x+1)/2^m = s$ (the integer $m \geq 1$), so it leads to a contradiction that $3x+1 = 6t \cdot 2^m$. The left side of the equation is not a multiple of 3, but the right side is divisible by 3. Thus, any even number generated when $3x+1$ is divided by 2 one or more times cannot be a multiple of 3.
For sure, the very next term after the odd integer y must be $3y+1$, according to the odd number rule of Collatz conjecture. Since $3y+1$ is not a multiple of 3, it moves to the next odd integer z that is not a multiple of 3 for the same reason above. While $3y+1$ moves to z when divided by 2 one or more times, the every possible term between $3y+1$ and z must not be a multiple of 3 in the same way as above.
We find any odd or even integer which is generated when any even integer of the form $3k+1$ is divided by 2^n cannot be a multiple of 3 (Condition 1). And every odd integer that is not a multiple of 3 is always connected to the next even integer that can be represented in the form $3k^{'}+1$, which also cannot be divided by 3 (Condition 2). Thus, an even integer under Condition 2 turns into an odd integer under Condition 1 when we apply the even number rule of Collatz conjecture. Again, the odd integer under Condition 1 converts into an even integer under Condition 2 by following the odd number rule of the conjecture. Even though we repeat this task infinitely, the term must be under either Condition 1 or 2. Thus, once x, an arbitrary odd multiple of 3, moves into an even integer that can be expressed in the form $3x+1$ according to the odd number rule of Collatz conjecture, it is under Condition 1 at first and since then under either Condition 1 or 2. Therefore, there will be no odd multiple of 3 except the first term of the sequence, and there will be no other term that is a multiple of 3 after one odd multiple of 3.

Theorem 2. Any even multiple of 3 cannot be generated more than once in a Collatz sequence.

Proof. Suppose $6u$, an arbitrary even multiple of 3, reappears in a Collatz sequence. Then the very previous term before the reappearing $6u$ must not be an odd integer. This is because when we suppose such an odd integer v, it must be true that $3v + 1 = 6u$ according to the odd number rule of the sequence. However, this is a contradiction since the left side of the equation is not a multiple of 3, but the right side can be divided by 3. Thus, the very previous term before the reappearing $6u$ must be an even integer, so let w be an even integer that is the very previous term before the reappearing $6u$. Then it must be true that $w = 12u > 6u$. It is because any even integer, including w, must be divided by 2 so as to move into the next term according to the even number rule of Collatz conjecture. However, by Theorem 1, we know that any subsequent term following the odd multiple of 3 cannot be a multiple of 3. It implies that once every even multiple of 3, including $6u$, decreasingly moves to an odd multiple of 3 by being divided by 2 one or more times according to the even number rule of Collatz conjecture, there will be no other multiple of 3 following the odd multiple of 3. Therefore, if we let $6u$ be the first term of a Collatz sequence, any even multiple of 3 that is greater than $6u$ cannot exist during the sequence. Since $w = 12u > 6u$, w cannot be the term of the sequence. Thus, w cannot be the very previous term before the reappearing $6u$. It means that the very previous term before the reappearing $6u$ cannot be an even integer. Therefore, it leads to a contradiction that the very previous term before the reappearing $6u$ cannot be either an odd integer or an even integer though every term of a Collatz sequence must be an integer. Thus, we know that $6u$, an arbitrary even multiple of 3, cannot reappear in a Collatz sequence, so it is proven that any even integer multiple of 3 cannot be generated more than one time in a Collatz sequence.

By Theorem 1 and 2, we know that any integer multiple of 3 cannot appear more than once in a Collatz sequence. Therefore, we can conclude that any possible cycle of a Collatz sequence cannot include any multiple of 3. It is because any cycle in a sequence generates every number included in it an infinite number of times, but any Collatz sequence cannot generate a multiple of 3 more than one time.

REFERENCES

[1] Jeffrey C. Lagarias, *The Ultimate Challenge: the 3x+1 Problem*, American Mathematical Society, Providence, Rhode Island, 2010.

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