Newton's Second Law is Valid in Relativity for Proper Time

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Abstract

In Newtonian particle dynamics, time is invariant under inertial transformations, and speed has no upper bound. In special relativity, it is the observed particle's proper time, rather than the arbitrary observer's time, which is inertial-transformation invariant, and it is the particle's proper-time speed which has no upper bound. It is thus perhaps not surprising the Newton's Second Law is also valid in relativistic particle dynamics presented in terms of the particle's proper time. This follows from the special-relativistic dynamical principle that the time derivative of special-relativistic momentum is equal to the applied force, provided proper force is defined to have an additional factor of gamma. The four-vector fully Lorentz-covariant completion of proper force is obliged to have zero contraction with the particle's proper four-velocity. A special-relativistic particle in either a scalar or a four-vector (electromagnetic) potential is shown to adhere to the proper-time Newton's Second Law, and when it is consistently taken into account that a metric (gravitational) potential modifies the rate of change of a particle's proper time with observer time, that adherence is shown to hold for metric potentials as well.

Observed versus proper-length constant velocity of a special relativistic object

An object's constant velocity can be calculated *by dividing the vector segment of its trajectory which it instantaneously intersects by the time it requires to traverse that segment*, but *special-relativistic observed length contraction of that trajectory segment by the factor* γ^{-1} [1], where,

$$\gamma^{-1} \stackrel{\text{def}}{=} \left(1 - |\dot{\mathbf{r}}/c|^2\right)^{\frac{1}{2}} \leq 1, \tag{1a}$$

implies that the object's special-relativistic observed constant velocity $\dot{\mathbf{r}}$ *equals* γ^{-1} *times its proper-length constant velocity*, i.e., its *proper-length constant velocity* equals $\gamma\dot{\mathbf{r}}$, *whose magnitude* $|\gamma\dot{\mathbf{r}}| = \left(|\dot{\mathbf{r}}|^2/\left(1 - |\dot{\mathbf{r}}/c|^2\right)\right)^{\frac{1}{2}}$ *is unbounded, notwithstanding that* $|\dot{\mathbf{r}}| < c$. Moreover, $\gamma\dot{\mathbf{r}}$ is equal to $(d\mathbf{r}/d\tau)$, *the object's velocity calculated using its Lorentz-invariant differential proper time* $d\tau$, which of course is,

$$d\tau = \left((dt)^2 - |d\mathbf{r}/c|^2\right)^{\frac{1}{2}}. \tag{1b}$$

That $\gamma\dot{\mathbf{r}}$ is equal to $(d\mathbf{r}/d\tau)$ *is a general fact because*,

$$\begin{gathered}\gamma = \left(1/\left(1 - |\dot{\mathbf{r}}/c|^2\right)\right)^{\frac{1}{2}} = \left((dt)^2/\left((dt)^2 - |d\mathbf{r}/c|^2\right)\right)^{\frac{1}{2}} = \left((dt)^2/(d\tau)^2\right)^{\frac{1}{2}} = (dt/d\tau) \;\Rightarrow \\ \gamma\dot{\mathbf{r}} = (dt/d\tau)\dot{\mathbf{r}} = (dt/d\tau)(d\mathbf{r}/dt) = (d\mathbf{r}/d\tau).\end{gathered} \tag{2}$$

An object's Lorentz-transformation invariant differential proper time $d\tau$ *is somewhat analogous to the Galilean-transformation invariant time of Newtonian physics* and, since an object's proper-time speed $|d\mathbf{r}/d\tau| = |\gamma\dot{\mathbf{r}}|$ is unbounded, *that speed is somewhat analogous to the unbounded speed of Newtonian physics*.

The Lorentz-covariant proper-time extension of Newton's Second Law

The usual presentation of single-particle special-relativistic dynamics is,

$$(d\mathbf{p}/dt) = \mathbf{f}, \tag{3a}$$

where $\mathbf{f}$ is the force and the relativistic single-particle momentum $\mathbf{p}$ is given by,

$$\mathbf{p} = m\gamma\dot{\mathbf{r}}, \tag{3b}$$

where m is the particle's rest mass. From Eq. (2) we see that Eq. (3b) can be rewritten,

$$\mathbf{p} = m(d\mathbf{r}/d\tau), \tag{3c}$$

so Eq. (3a) becomes,

$$m(d(d\mathbf{r}/d\tau)/dt) = \mathbf{f}. \tag{3d}$$

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We now multiply the left side of Eq. (3d) by $(dt/d\tau)$ and its right side by γ, as per Eq. (2), which yields,

$$m(d(d\mathbf{r}/d\tau)/dt)(dt/d\tau) = \gamma\mathbf{f}. \tag{3e}$$

We simplify the left side of Eq. (3e) and denote $\gamma\mathbf{f}$ on its right side *as the proper force* $\mathbf{F}$ to obtain,

$$m\big(d^2\mathbf{r}/d\tau^2\big) = \mathbf{F}, \tag{3f}$$

the relativistic extension of Newton's Second Law via proper time. An *example* of Eq. (3f) is the proper force exerted by an electromagnetic field on a particle of charge e, namely,

$$\mathbf{F} = e\gamma(\mathbf{E} + ((\dot{\mathbf{r}}/c)\times\mathbf{B}))\,. \tag{3g}$$

The *fully Lorentz-covariant four-vector completion of* Eq. (3f) *must of course read*,

$$m\big(d^2x^\mu/d\tau^2\big) = F^\mu, \tag{3h}$$

but the nature of proper time *ensures that only three of the four components of the proper force F^μ can be mutually independent*. We begin the demonstration of this fact by using Eq. (2) to show that,

$$(dx^\mu/d\tau)\,(dx_\mu/d\tau) = (\dot{x}^\mu\dot{x}_\mu)(dt/d\tau)^2 = (\dot{x}^\mu\dot{x}_\mu)\gamma^2 = \big(c^2 - |\dot{\mathbf{r}}|^2\big)/\big(1 - |\dot{\mathbf{r}}/c|^2\big) = c^2, \tag{3i}$$

which furthermore implies that,

$$\big(d^2x^\mu/d\tau^2\big)(dx_\mu/d\tau) = \tfrac{1}{2}(d((dx^\mu/d\tau)(dx_\mu/d\tau))/d\tau) = \tfrac{1}{2}\big(d\big(c^2\big)/d\tau\big) = 0. \tag{3j}$$

Eq. (3h) together with Eq. (3j) implies that,

$$F^\mu(dx_\mu/d\tau) = m\big(d^2x^\mu/d\tau^2\big)(dx_\mu/d\tau) = 0. \tag{3k}$$

Therefore only three of the four components of the proper force F^μ can be mutually independent. In greater detail, Eq. (3k) together with Eq. (2) yields that,

$$0 = F^\mu(dx_\mu/d\tau) = (F^\mu\dot{x}_\mu)(dt/d\tau) = \big(F^0c - \mathbf{F}\cdot\dot{\mathbf{r}}\big)/\big(1 - |\dot{\mathbf{r}}/c|^2\big)^{\frac{1}{2}}, \text{ which implies that } F^0 = \mathbf{F}\cdot(\dot{\mathbf{r}}/c). \tag{3l}$$

We thus see that F^0 *vanishes altogether in the nonrelativistic limit* $|\dot{\mathbf{r}}/c| \to 0$, for which it is *also* true that $(dt/d\tau) = \gamma = \big(1 - |\dot{\mathbf{r}}/c|^2\big)^{-\frac{1}{2}} \to 1$, so in the nonrelativistic limit Eq. (3h) *reduces to Newton's* $m\ddot{\mathbf{r}} = \mathbf{f}$.

Eq. (3f) shows that the concept of inertial mass, which is *the same* as rest mass, *is just as relevant to relativistic physics as it is to Newtonian physics*. Indeed, the development of Higgs field physics [2] has elaborated the inertial mass concept. An intriguing *inertia issue* is the *existence* of particles, e.g., *photons*, which have *zero inertial mass* (these are asserted *to not couple at all to the Higgs field*). According to Eq. (3c), a zero-inertial-mass particle which has nonzero momentum $|\mathbf{p}| > 0$ *has infinite proper-time speed* because $|d\mathbf{r}/d\tau| = \lim_{m\to 0}(|\mathbf{p}|/m) = \infty$. We now show that this corresponds *to observed speed* $|\dot{\mathbf{r}}|$ *being* c by *inverting* the Eq. (2) relation of proper-time velocity $(d\mathbf{r}/d\tau)$ to observed velocity $\dot{\mathbf{r}}$, namely,

$$(d\mathbf{r}/d\tau) = \gamma\dot{\mathbf{r}} = \big(1 - |\dot{\mathbf{r}}/c|^2\big)^{-\frac{1}{2}}\,\dot{\mathbf{r}}.$$

The *inverse* of this relation is,

$$\dot{\mathbf{r}} = (d\mathbf{r}/d\tau)\,\big(1 + |(d\mathbf{r}/d\tau)/c|^2\big)^{-\frac{1}{2}}\,, \tag{3m}$$

which has the asymptotic form,

$$\dot{\mathbf{r}} \sim c\,((d\mathbf{r}/d\tau)/|d\mathbf{r}/d\tau|) \;\text{ as } |(d\mathbf{r}/d\tau)/c| \to \infty. \tag{3n}$$

This result shows that zero-inertial-mass particles of nonzero momentum $|\mathbf{p}| > 0$, *which therefore have infinite proper-time speed* $|d\mathbf{r}/d\tau| = \lim_{m\to 0}(|\mathbf{p}|/m) = \infty$, consequently *have observed speed* $|\dot{\mathbf{r}}|$ *equal to* c.

We next work out the specific proper force F^μ exerted on a relativistic particle of mass m by (1) a scalar potential $\phi(x^\alpha)$, (2) a four-vector (electromagnetic) potential $A^\mu(x^\alpha)$ and (3) a dimensionless second-rank symmetric-tensor metric (gravitational) potential $g_{\mu\nu}(x^\alpha)$. To obtain the needed three equations of motion, we first must construct the three relativistic single-particle Lagrangians for these three potentials.

A rest-frame approach to constructing special-relativistic particle Lagrangians

A mass m, $\dot{\mathbf{r}} = \mathbf{0}$ special-relativistic particle has energy mc^2 *plus its* $\dot{\mathbf{r}} = \mathbf{0}$ *"rest" potential energy* V_{rest},

$$H_{\text{rest}} = mc^2 + V_{\text{rest}}, \tag{4a}$$

so since the usual Lagrangian term $\dot{\mathbf{r}} \cdot \mathbf{p}$ *vanishes entirely when* $\dot{\mathbf{r}} = \mathbf{0}$, the special-relativistic particle's "rest" Lagrangian L_{rest} and consequent "rest" action S_{rest} are,

$$L_{\text{rest}} = -H_{\text{rest}} = -\big(mc^2 + V_{\text{rest}}\big) \Rightarrow S_{\text{rest}} = -\int \big(mc^2 + V_{\text{rest}}\big)dt. \tag{4b}$$

The special-relativistic extension of S_{rest} *is required to be Lorentz invariant*, and therefore is of the form,

$$S_{\text{inv}} = -\int \big(mc^2 + V_{\text{inv}}\big)d\tau = -\int \big(mc^2 + V_{\text{inv}}\big)(d\tau/dt)dt, \tag{4c}$$

where $d\tau$ is the particle's Lorentz-invariant differential proper time, and V_{inv} *is its Lorentz-invariant potential energy, which must reduce to* V_{rest} *in the limit* $\dot{\mathbf{r}} \to \mathbf{0}$. Extending V_{rest} to *the Lorentz-invariant* V_{inv} is dealt with case-by-case. Eq. (4c) implies that the full special-relativistic Lagrangian L_{rel} is given by,

$$L_{\text{rel}} = -\big(mc^2 + V_{\text{inv}}\big)(d\tau/dt). \tag{4d}$$

The proper force exerted by a scalar potential

A relativistic particle of mass m which couples to a scalar potential $\phi(x^\alpha)$ with dimensionless coupling strength k has *both* V^{ϕ}_{rest} *and* V^{ϕ}_{inv} *equal to* $(k\phi)$, so from Eqs. (4d) and (2),

$$L_\phi = -\big(mc^2 + k\phi\big)(d\tau/dt) = -\big(mc^2 + k\phi\big)\gamma^{-1} = -\big(mc^2 + k\phi\big)\big(1 - |\dot{\mathbf{r}}/c|^2\big)^{\frac{1}{2}}. \tag{5a}$$

Since the generic equation of motion implied by any single-particle Lagrangian L is,

$$d(\partial L/\partial \dot{x}^i)/dt = (\partial L/\partial x^i), \text{ where } i = 1, 2, 3, \tag{5b}$$

it is very useful in the case of the Lagrangian L_ϕ of Eq. (5a) to note that,

$$\big(\partial\left(\gamma^{-1}\right)/\partial \dot{x}^i\big) = -c^{-2}\gamma\dot{x}^i = -c^{-2}(dt/d\tau)\dot{x}^i = -c^{-2}(dx^i/d\tau). \tag{5c}$$

From Eqs. (5a)–(5c) we obtain that,

$$d\big(\big(m + \big(k\phi/c^2\big)\big)(dx^i/d\tau)\big)/dt = -k(\partial\phi/\partial x^i)(d\tau/dt). \tag{5d}$$

Upon multiplying both sides of Eq. (5d) by $(dt/d\tau)$ and noting that $x^i = -x_i$, it becomes,

$$d\big(\big(m + \big(k\phi/c^2\big)\big)(dx^i/d\tau)\big)/d\tau = k(\partial\phi/\partial x_i), \tag{5e}$$

whose fully Lorentz-covariant four-vector completion clearly is,

$$d\big(\big(m + \big(k\phi/c^2\big)\big)(dx^\mu/d\tau)\big)/d\tau = k(\partial\phi/\partial x_\mu). \tag{5f}$$

Noting that $(d\phi/d\tau) = (\partial\phi/\partial x_\nu)(dx_\nu/d\tau)$, we carry out the outer $d/d\tau$ differentiation on the left side of Eq. (5f) and then move all terms except $m\big(d^2x^\mu/d\tau^2\big)$ to its right side to obtain,

$$m\big(d^2x^\mu/d\tau^2\big) = k\big[(\partial\phi/\partial x_\mu) - \big(1/c^2\big)\big[(dx^\mu/d\tau)(\partial\phi/\partial x_\nu)(dx_\nu/d\tau) + \phi\big(d^2x^\mu/d\tau^2\big)\big]\big] = F^\mu, \tag{5g}$$

where F^μ *is the proper force* $\phi(x^\alpha)$ *exerts on a mass m particle of coupling strength k.*

By applying the identities given by Eqs. (3i) and (3j), namely that,

$$(dx^\mu/d\tau)(dx_\mu/d\tau) = c^2 \text{ and } \big(d^2x^\mu/d\tau^2\big)(dx_\mu/d\tau) = 0, \tag{5h}$$

we verify that the Eq. (5g) proper force F^μ *satisfies the requirement* $F^\mu(dx_\mu/d\tau) = 0$ *of* Eq. (3k).

We also note that if the scalar potential $\phi(x^\alpha)$ is *constant* in x^α, Eq. (5g) implies that,

$$\big(m + \big(k\phi/c^2\big)\big)\big(d^2x^\mu/d\tau^2\big) = 0, \tag{5i}$$

i.e., the particle's *mass* m is effectively *modified by the addition to it of the constant term* $\big(k\phi/c^2\big)$. The Higgs field is thought of as such a constant scalar potential which is able to give an effective mass to otherwise zero-mass particles if they have nonzero dimensionless coupling strength k with that scalar potential [2].

The proper force exerted by a four-vector (electromagnetic) potential

A particle of mass m and charge e at rest in a four-vector electromagnetic potential $A^\mu(x^\alpha)$ has potential energy $V^{A^\mu}_{\text{rest}} = eA^0$, with Lorentz-invariant extension $V^{A^\mu}_{\text{inv}} = (e/c)(dx_\nu/d\tau)A^\nu$. Thus from Eqs. (4d) and (2),

$$L_{A^\mu} = -\big(mc^2 + (e/c)(dx_\nu/d\tau)A^\nu\big)(d\tau/dt) = -mc^2\gamma^{-1} - (e/c)\dot{x}_\nu A^\nu = -mc^2\gamma^{-1} - eA^0 + (e/c)\dot{\mathbf{r}}\cdot\mathbf{A}. \tag{6a}$$

Applying Eqs. (5b) and (5c) to the Eq. (6a) Lagrangian L_{A^μ} yields,

$$d\big(m(dx^i/d\tau) + (e/c)A^i\big)/dt = -(e/c)\dot{x}_\nu\big(\partial A^\nu/\partial x^i\big). \tag{6b}$$

Multiplying both sides of Eq. (6b) by $(dt/d\tau)$ and noting that $x^i = -x_i$ produces,

$$d\big(m(dx^i/d\tau) + (e/c)A^i\big)/d\tau = (e/c)(dx_\nu/d\tau)(\partial A^\nu/\partial x_i), \tag{6c}$$

which we reexpress as,

$$m\big(d^2x^i/d\tau^2\big) = (e/c)\big[(dx_\nu/d\tau)(\partial A^\nu/\partial x_i) - \big(dA^i/d\tau\big)\big]. \tag{6d}$$

Since $\big(dA^i/d\tau\big) = \big(\partial A^i/\partial x_\nu\big)(dx_\nu/d\tau)$, we can rewrite Eq. (6d) as,

$$m\big(d^2x^i/d\tau^2\big) = (e/c)(dx_\nu/d\tau)\big[(\partial A^\nu/\partial x_i) - \big(\partial A^i/\partial x_\nu\big)\big], \tag{6e}$$

whose fully Lorentz-covariant four-vector completion clearly is,

$$m\big(d^2x^\mu/d\tau^2\big) = (e/c)(dx_\nu/d\tau)[(\partial A^\nu/\partial x_\mu) - (\partial A^\mu/\partial x_\nu)] = F^\mu, \tag{6f}$$

where F^μ *is the proper force* $A^\mu(x^\alpha)$ *exerts on a mass* m *particle of charge* e. The proper force F^μ *satisfies the requirement* $F^\mu(dx_\mu/d\tau) = 0$ *of* Eq. (3k) because $(dx_\nu/d\tau)(dx_\mu/d\tau)$ is *symmetric* under interchange of ν and μ, whereas $[(\partial A^\nu/\partial x_\mu) - (\partial A^\mu/\partial x_\nu)]$ is *antisymmetric* under that interchange. Eq. (6f) also implies Eq. (3g), since for $\mu = i = 1$, 2, or 3,

$$\begin{gathered} F^i = (e/c)(\gamma\dot{x}_\nu)\big[(\partial A^\nu/\partial x_i) - \big(\partial A^i/\partial x_\nu\big)\big] = \\ e\gamma\big[-\big(\partial A^0/\partial x^i\big) - (1/c)\dot{A}^i\big] + (e/c)\gamma\sum_{j=1}^{3}\big(\dot{x}^j\big)\big[\big(\partial A^j/\partial x^i\big) - \big(\partial A^i/\partial x^j\big)\big] = \\ e\gamma\big(-\big(\nabla_{\mathbf{r}}A^0\big) - (1/c)\dot{\mathbf{A}}\big)^i + (e/c)\gamma\left((\nabla_{\mathbf{r}}(\dot{\mathbf{r}}\cdot\mathbf{A})) - ((\dot{\mathbf{r}}\cdot\nabla_{\mathbf{r}})\,\mathbf{A})\right)^i = \\ e\gamma\left(\mathbf{E} + ((\dot{\mathbf{r}}/c)\times(\nabla_{\mathbf{r}}\times\mathbf{A}))\right)^i = e\gamma\left(\mathbf{E} + ((\dot{\mathbf{r}}/c)\times\mathbf{B})\right)^i. \end{gathered} \tag{6g}$$

The proper force exerted by a metric (gravitational) potential

A sufficiently simple special-relativistic dynamical system is coupled to a dimensionless symmetric-tensor metric potential $g_{\mu\nu}(x^\alpha)$ *by substituting* $g_{\mu\nu}(x^\alpha)$ *for occurrences of the Minkowski metric tensor* $\eta_{\mu\nu}$ *in the system's Lorentz-invariant action*. To study a single particle's interaction with $g_{\mu\nu}(x^\alpha)$ *only*, the Lorentz-invariant action in which occurrences of $\eta_{\mu\nu}$ are replaced by $g_{\mu\nu}(x^\alpha)$ *must be that of the free particle, i.e.*,

$$S_{\text{free}} = -\textstyle\int mc^2 d\tau, \tag{7a}$$

whose Lorentz-invariant proper differential time $d\tau$ is given by Eq. (1b),

$$d\tau = \big((dt)^2 - |d\mathbf{r}/c|^2\big)^{\frac{1}{2}} = (dx^\mu dx_\mu)^{\frac{1}{2}}/c = (dx^\mu\eta_{\mu\nu}dx^\nu)^{\frac{1}{2}}/c. \tag{7b}$$

Having expressed $d\tau$ in terms of $\eta_{\mu\nu}$, we replace $\eta_{\mu\nu}$ by $g_{\mu\nu}(x^\alpha)$, which changes the Eq. (7a) free-particle action $S_{\rm free}$ to the following action $S_{g_{\mu\nu}}$ for the interaction of the particle with the metric potential $g_{\mu\nu}(x^\alpha)$,

$$S_{g_{\mu\nu}} = -\int mc(dx^\mu g_{\mu\nu} dx^\nu)^{\frac{1}{2}} = -\int mc(\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu)^{\frac{1}{2}}dt,$$

from which the Lagrangian $L_{g_{\mu\nu}}$ for the interaction of the particle with $g_{\mu\nu}(x^\alpha)$ follows,

$$L_{g_{\mu\nu}} = -mc(\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu)^{\frac{1}{2}} = -mc\Big(\dot{x}^0 g_{00}\dot{x}^0 + 2\dot{x}^0\textstyle\sum_{j=1}^3 g_{0j}\dot{x}^j + \sum_{j=1}^3\sum_{k=1}^3 \dot{x}^j g_{jk}\dot{x}^k\Big)^{\frac{1}{2}}, \text{ where } \dot{x}^0 = c. \quad (7c)$$

Since $L_{g_{\mu\nu}}$ is given in the observer's time t, presenting its equation of motion in the particle's proper time τ requires $(d\tau/dt)$—we saw above that the particle's coupling to $g_{\mu\nu}(x^\alpha)$ *changed* $d\tau$ *from* $(dx^\mu\eta_{\mu\nu}dx^\nu)^{\frac{1}{2}}/c$ *to* $(dx^\mu g_{\mu\nu}dx^\nu)^{\frac{1}{2}}/c$, which implies that that coupling *correspondingly changed* $(d\tau/dt)$ *from* $(\dot{x}^\mu\eta_{\mu\nu}\dot{x}^\nu)^{\frac{1}{2}}/c$ *to*,

$$(d\tau/dt)_{g_{\mu\nu}} \stackrel{\rm def}{=} (\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu)^{\frac{1}{2}}/c. \quad (7d)$$

A crucial physics-related restriction on $g_{\mu\nu}(x^\alpha)$ is that for all x^α, *its four matrix eigenvalues are required have the same signs as the matrix eigenvalues of* $\eta_{\mu\nu}$, *namely* $\{+,-,-,-\}$ [3]. Therefore for all x^α, $g_{\mu\nu}(x^\alpha)$ *has a matrix inverse*, which is conventionally denoted $g^{\lambda\kappa}(x^\alpha)$. Thus, for example,

$$g^{\lambda\kappa}(x^\alpha)g_{\kappa\nu}(x^\alpha) = \delta^\lambda_\nu. \quad (7e)$$

Before we work out the equation of motion implied by the Eq. (7c) Lagrangian $L_{g_{\mu\nu}}$, we note *the generalizations of the proper-velocity and proper-acceleration identities given by* Eqs. (3i) *and* (3j) *which ensue in the presence of a metric potential* $g_{\mu\nu}(x^\alpha)$. The Eq. (3i) identity's generalization is easy to surmise, i.e.,

$$(dx^\mu/d\tau)g_{\mu\nu}(dx^\nu/d\tau) = (\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu)/((d\tau/dt)_{g_{\mu\nu}})^2 = c^2, \quad (7f)$$

where the last equality follows from Eq. (7d). The Eq. (3j) identity's generalization is then obtained via differentiation with respect to τ of the Eq. (7f) identity,

$$\begin{gathered} 0 = d\left(c^2\right)/d\tau = d((dx^\mu/d\tau)g_{\mu\nu}(dx^\nu/d\tau))/d\tau = \\ 2\big(d^2x^\mu/d\tau^2\big)g_{\mu\nu}(dx^\nu/d\tau) + (dx^\mu/d\tau)(\partial g_{\mu\nu}/\partial x^\kappa)(dx^\kappa/d\tau)(dx^\nu/d\tau), \end{gathered} \quad (7g)$$

which implies the following generalization of the Eq. (3j) identity,

$$\big(d^2x^\lambda/d\tau^2\big)g_{\lambda\gamma}(dx^\gamma/d\tau) = -\tfrac{1}{2}(dx^\mu/d\tau)(\partial g_{\mu\nu}/\partial x^\kappa)(dx^\kappa/d\tau)(dx^\nu/d\tau). \quad (7h)$$

Given this identity, a purported proper force F^λ on a particle of mass m that is claimed to adhere to,

$$m\big(d^2x^\lambda/d\tau^2\big) = F^\lambda, \quad (7i)$$

must be such that it satisfies the consistency requirement,

$$F^\lambda g_{\lambda\gamma}(dx^\gamma/d\tau) = -\tfrac{1}{2}m(dx^\mu/d\tau)(\partial g_{\mu\nu}/\partial x^\kappa)(dx^\kappa/d\tau)(dx^\nu/d\tau). \quad (7j)$$

We now work out the equation of motion implied by the Eq. (7c) Lagrangian $L_{g_{\mu\nu}} = -mc(\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu)^{\frac{1}{2}}$,

$$\begin{gathered} (\partial L_{g_{\mu\nu}}/\partial\dot{x}^i) = -\tfrac{1}{2}mc\Big(2g_{i0}\dot{x}^0 + 2\textstyle\sum_{j=1}^3 g_{ij}\dot{x}^j\Big)/(\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu)^{\frac{1}{2}} = \\ -mc(g_{i\nu}\dot{x}^\nu)/(c(d\tau/dt)_{g_{\mu\nu}}) = -mg_{i\nu}(dx^\nu/d\tau), \end{gathered} \quad (7k)$$

and,

$$\begin{gathered} (\partial L_{g_{\mu\nu}}/\partial x^i) = -\tfrac{1}{2}mc(\dot{x}^\mu(\partial g_{\mu\nu}/\partial x^i)\dot{x}^\nu)/(\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu)^{\frac{1}{2}} = \\ -\tfrac{1}{2}mc(\dot{x}^\mu(\partial g_{\mu\nu}/\partial x^i)\dot{x}^\nu)/(c(d\tau/dt)_{g_{\mu\nu}}) = -\tfrac{1}{2}m(\dot{x}^\mu(\partial g_{\mu\nu}/\partial x^i)(dx^\nu/d\tau)). \end{gathered} \quad (7l)$$

Inserting the Eq. (7k) and (7l) results into the generic Eq. (5b) equation of motion produces,

$$-m(d(g_{i\nu}(dx^\nu/d\tau))/dt) = -\tfrac{1}{2}m(\dot{x}^\mu(\partial g_{\mu\nu}/\partial x^i)(dx^\nu/d\tau)). \quad (7m)$$

After dividing both sides of Eq. (7m) by $(d\tau/dt)_{g_{\mu\nu}}$, this equation of motion can be reexpressed as,

$$-m(d(g_{i\nu}(dx^\nu/d\tau))/d\tau) = -\tfrac{1}{2}m((dx^\mu/d\tau)(\partial g_{\mu\nu}/\partial x^i)(dx^\nu/d\tau)). \tag{7n}$$

The four-vector completion of Eq. (7n) clearly is,

$$-m(d(g_{\kappa\nu}(dx^\nu/d\tau))/d\tau) = -\tfrac{1}{2}m((dx^\mu/d\tau)(\partial g_{\mu\nu}/\partial x^\kappa)(dx^\nu/d\tau)). \tag{7o}$$

Carrying out the outer $d/d\tau$ differentiation on the left side of Eq. (7o) yields two terms,

$$-mg_{\kappa\nu}\big(d^2x^\nu/d\tau^2\big) - m((dx^\mu/d\tau)(\partial g_{\kappa\nu}/\partial x^\mu)(dx^\nu/d\tau)) = -\tfrac{1}{2}m((dx^\mu/d\tau)(\partial g_{\mu\nu}/\partial x^\kappa)(dx^\nu/d\tau)). \tag{7p}$$

Because $((dx^\mu/d\tau)(dx^\nu/d\tau))$ is *symmetric* under interchange of μ and ν, Eq. (7p) can be rewritten as,

$$-mg_{\kappa\nu}\big(d^2x^\nu/d\tau^2\big) = \tfrac{1}{2}m((dx^\mu/d\tau)[(\partial g_{\kappa\nu}/\partial x^\mu) + (\partial g_{\kappa\mu}/\partial x^\nu) - (\partial g_{\mu\nu}/\partial x^\kappa)](dx^\nu/d\tau)). \tag{7q}$$

Making use of Eq. (7e), we multiply both sides of Eq. (7q) by $-g^{\lambda\kappa}$ and sum over the index κ to obtain,

$$m\big(d^2x^\lambda/d\tau^2\big) = -\tfrac{1}{2}mg^{\lambda\kappa}(dx^\mu/d\tau)[(\partial g_{\kappa\nu}/\partial x^\mu) + (\partial g_{\kappa\mu}/\partial x^\nu) - (\partial g_{\mu\nu}/\partial x^\kappa)](dx^\nu/d\tau) = F^\lambda, \tag{7r}$$

where F^λ *is the proper force* $g_{\mu\nu}(x^\alpha)$ *exerts on a mass* m *particle*.

To check that F^λ satisfies the consistency requirement of Eq. (7j), we use $g^{\lambda\kappa} = g^{\kappa\lambda}$ to rewrite F^λ as,

$$F^\lambda = -\tfrac{1}{2}m(dx^\mu/d\tau)[(\partial g_{\kappa\nu}/\partial x^\mu) + (\partial g_{\kappa\mu}/\partial x^\nu) - (\partial g_{\mu\nu}/\partial x^\kappa)](dx^\nu/d\tau)g^{\kappa\lambda}, \tag{7s}$$

which, since $g^{\kappa\lambda}g_{\lambda\gamma} = \delta^\kappa_\gamma$, yields that,

$$\begin{gathered} F^\lambda g_{\lambda\gamma}(dx^\gamma/d\tau) = -\tfrac{1}{2}m(dx^\mu/d\tau)[(\partial g_{\kappa\nu}/\partial x^\mu) + (\partial g_{\kappa\mu}/\partial x^\nu) - (\partial g_{\mu\nu}/\partial x^\kappa)](dx^\nu/d\tau)(dx^\kappa/d\tau) = \\ -\tfrac{1}{2}m(dx^\mu/d\tau)(\partial g_{\mu\nu}/\partial x^\kappa)(dx^\kappa/d\tau)(dx^\nu/d\tau), \end{gathered} \tag{7t}$$

as required by Eq. (7j), where the last equality ensues after appropriately renaming contracted indices.

It is to be noted that Eq. (7r) *is conventionally written using the Christoffel symbol* $\Gamma^\lambda_{\mu\nu}$, i.e. [4],

$$\begin{gathered} \big(d^2x^\lambda/d\tau^2\big) + (dx^\mu/d\tau)\Gamma^\lambda_{\mu\nu}(dx^\nu/d\tau) = 0 \ \text{ where,} \\ \Gamma^\lambda_{\mu\nu} = \tfrac{1}{2}g^{\lambda\kappa}[(\partial g_{\kappa\nu}/\partial x^\mu) + (\partial g_{\kappa\mu}/\partial x^\nu) - (\partial g_{\mu\nu}/\partial x^\kappa)]. \end{gathered} \tag{7u}$$

References

[1] Length contraction–Wikipedia, https://en.wikipedia.org/wiki/Length_contraction.

[2] Higgs field–Simple Wikipedia, https://simple.wikipedia.org/wiki/Higgs_field.

[3] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, New York, 1972), Section 3.6, pp. 85–86.

[4] S. Weinberg, op. cit., Eq. (3.2.3), p. 71 and Eq. (3.3.7), p. 75.