Positivity of the Fourier Transform of the Shortest Maximal Order Convolution Mask for Cardinal B-splines

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In [2] approximations of functions into manifolds were studied. For the transformation of function values to B-spline coefficients convolution masks were considered. Some of the proofs required that the convolution mask had a positive Fourier transform. This property was used to show that the inverse of the convolution exists and that the spatial dependency decays exponentially. For splines of degree \( m \) the existence of such convolution masks of length \( m^2/2 \) were constructively proven. It was posed as an open question if families with shorter sequences could satisfy this property. For \( m \leq 21 \) it was computationally verified that the shortest possible sequence that satisfies the polynomial reproduction property also has a positive Fourier transform. This sequence has length \( m \). It was conjectured that this holds true for all odd \( m > 0 \). In Section 1 of this work we will prove this fact. In Section 2 we describe how the convolution masks can be computed.

1 Theory

We start by defining cardinal B-splines

Definition 1. Cardinal B-splines can recursively be defined by

\[
B_0 = 1_{[-\frac{1}{2}, \frac{1}{2}]} \text{ and } B_m = B_{m-1} \ast B_0 \text{ for all } m \geq 1
\]

where \( 1_{[-\frac{1}{2}, \frac{1}{2}]} \) denotes the indicator function on the interval \( [-\frac{1}{2}, \frac{1}{2}] \) and \( \ast \) denotes the convolution.

Next we consider the polynomial reproduction property

\[
\sum_{i \in \mathbb{Z}} \sum_{j=-S}^{S} B_m(x - i) p(i - j) \lambda_j = p(x) \text{ for all } x \in \mathbb{R}
\] (1)
and for all polynomials $p$ of degree $\leq m$. As in [2] we will use the functions $N_m, \Lambda: \mathbb{C} \rightarrow \mathbb{C}$ defined by

$$N_m(z) = \sum_{j=-(m-1)/2}^{(m-1)/2} B_m(j)z^j, \quad \Lambda(z) = \sum_{j=-S}^{S} \lambda_j z^j. \quad (2)$$

We will now proof an equivalent formulation of the polynomial reproduction property.

**Lemma 1.** The polynomial reproduction property (1) is equivalent to

$$\left. \frac{d^l}{dz^l} (N_m(z)\Lambda(z)) \right|_{z=1} = \begin{cases} 1 & l = 0 \\ 0 & l \in \{1, \ldots, m\} \end{cases}. \quad (3)$$

**Proof.** We have

$$\left. \frac{d^l}{dz^l} (N_m(z)\Lambda(z)) \right|_{z=1} = \left. \frac{d^l}{dz^l} \left( \sum_{k=-(m-1)/2}^{(m-1)/2} B_m(k)z^k \right) \left( \sum_{j=-S}^{S} \lambda_j z^j \right) \right|_{z=1} \quad (4)$$

$$= \left. \frac{d^l}{dz^l} \left( \sum_{k=-(m-1)/2}^{S+(m-1)/2} \sum_{j=-S}^{S} B_m(k-j)\lambda_j z^{k} \right) \right|_{z=1} \quad (5)$$

$$= \left. \left( \sum_{k=-(m-1)/2}^{S+(m-1)/2} \sum_{j=-S}^{S} B_m(k-j)\lambda_j z^{k} \right) \right|_{z=1} \quad (6)$$

$$= \left. \left( \sum_{k=-S-(m-1)/2}^{S+(m-1)/2} \sum_{j=-S}^{S} B_m(k-j)\lambda_j p_l(-k) \right) \right|_{z=1} \quad (7)$$

where $p_0 = 1, p_l(x) = (-x)(-x-1)\ldots(-x-(l-1))$. If the polynomial reproduction property (1) holds Expression (8) is equal to

$$p_l(0) = \begin{cases} 1 & l = 0 \\ 0 & l \in \{1, \ldots, m\} \end{cases}. \quad (8)$$

On the other hand if (3) holds the polynomial reproduction property (1) holds for $x = 0$ and $p = p_l$ for all $l \in \{0, \ldots, m\}$. As the polynomials $p_0, \ldots, p_m$ build a basis for the space of polynomials of degree $\leq m$ the polynomial reproduction property (1) holds for all polynomials $p$ of degree $\leq m$ at $x = 0$. By replacing $x$ resp. $i$ by $x+1$ resp. $i+1$ it follows that the polynomial reproduction property holds at all integer points. Now consider the function

$$f(x) = \sum_{i \in \mathbb{Z}} \sum_{j=-S}^{S} B_m(x-i)p(i-j)\lambda_j.$$
Since \( B'_m(x) = B_{m-1}(x + \frac{1}{2}) - B_{m-1}(x - \frac{1}{2}) = \delta B_{m-1}(x - \frac{1}{2}) \) where \( \delta \) is the discrete difference operator \( \delta u(x) = u(x + 1) - u(x) \) it follows inductively that \( B^{(m)}_m(x) = \delta^m B_0(x - \frac{m}{2}) \). Since \( \delta \) commutes with the convolution it follows that

\[
 f^{(m)}(x) = \sum_{i=0}^{S} \sum_{j=-S}^{S} B^{(m)}_m(x - i)p(i - j)\lambda_j 
\]

\[
 = \sum_{i=0}^{S} \sum_{j=-S}^{S} \delta^m B_0 \left( x - i - \frac{m}{2} \right) p(i - j)\lambda_j 
\]

\[
 = \sum_{i=0}^{S} \sum_{j=-S}^{S} B_0 \left( x - i - \frac{m}{2} \right) \delta^m p(i - j)\lambda_j 
\]

Note that every time \( \delta \) is applied the polynomial degree decreases by 1. Hence \( \delta^m p(i - j) \) and therefore \( f^{(m)} \) is constant. It follows that \( f \) is a polynomial and since \( p \) coincides with \( f \) on all integers that \( f = p \). Hence the polynomial reproduction property holds for all polynomials of degree \( \leq m \) and all \( x \in \mathbb{R} \). \( \square \)

We can now formulate and proof our theorem.

**Theorem 1.** Let \( k \geq 0 \) be an nonnegative integer. Then there exist a unique symmetric (i.e. \( \lambda_{-i} = \lambda_i \)) sequence \( (\lambda_j)_{j=-k}^{k} \subset \mathbb{R} \) that satisfies the polynomial reproduction property (1) for \( m = 2k + 1 \). Furthermore we have

\[
 \sum_{j=-k}^{k} \lambda_{j} e^{2\pi i j \omega} \geq 1 > 0 \text{ for all } \omega \in \mathbb{R}.
\]

**Proof.** Since \( \lambda \) and \( B_m \) are symmetric, i.e. \( \lambda_{-i} = \lambda_i \) and \( B_m(-x) = B_m(x) \), both \( N_m(z) \) and \( \Lambda(z) \) can be written as polynomials of degree \( k \) in

\[
x = z + z^{-1} = 2 \left( \frac{z - 1}{z} \right), \text{ i.e. } N_m(z) = p \left( \frac{(z - 1)^2}{z} \right), \quad \Lambda(z) = q \left( \frac{(z - 1)^2}{z} \right)
\]

for polynomials \( p, q \) of degree \( k \). Condition (3) is by Lemma 1 equivalent to

\[
p(x)q(x) = 1 + x^{k+1}(\ldots),
\]

i.e. that the constant coefficient of \( p(x)q(x) \) is one and coefficients of order 1 up to order \( k \) are zero. We first prove uniqueness of \( q \) and therefore of \( (\lambda_j)_{j=-k}^{k} \subset \mathbb{R} \). Let \( q_1, q_2 \) be two polynomials of degree \( k \) satisfying (12). Then it follows that

\[
p(x)(q_1(x) - q_2(x)) = x^{k+1}(\ldots)
\]

and since \( p(0) = 1 \neq 0 \) that \( q_1(x) - q_2(x) = x^{k+1}(\ldots) \) and since \( q_1 \) and \( q_2 \) are polynomials of degree \( k \) that \( q_1 = q_2 \).

The polynomial \( N_m(z)z^{(m-1)/2m} \) is known as the Eulerian polynomial. By [1] the Eulerian polynomial and therefore \( N_m \) has only negative and simple
real roots. If \( z_1 \) is a root of \( N_m \) then \( x_i = z_i + z_1^{-1} - 2 \leq -4 \) is a root of \( p \). Furthermore all \( k \) roots \( x_1, \ldots, x_k \) of \( p \) can be constructed in this way. Therefore the roots of \( p \) are all smaller or equal to \(-4\). Note that for \( |x| < 4 \) we have

\[
\frac{1}{p(x)} = \frac{1}{\prod_{i=1}^{k} \left(1 - \frac{x}{z_i}\right)} = \prod_{i=1}^{k} \sum_{j=0}^{\infty} \frac{x^j}{x_i^j}.
\]

Define \( q \) by the truncating this power series at order \( k + 1 \), i.e. such that

\[
\frac{1}{p(x)} = q(x) + x^{k+1}(\ldots)
\]

Then we have

\[
p(x)q(x) = p(x) \left( \frac{1}{p(x)} + x^{k+1}(\ldots) \right) = 1 + x^{k+1}(\ldots),
\]

which shows that (12) is satisfied, i.e. the sequence \((\lambda_j)_{j=-k}^{k} \) corresponding to \( q \) satisfies the polynomial reproduction property.

The statement for the Fourier transform is equivalent to

\[
\Lambda(z) \geq 1 \text{ for all } |z|=1
\]

which is equivalent to

\[
q(x) \geq 1 \text{ for all } -4 \leq x \leq 0.
\]

Since all roots \( x_1, \ldots, x_k \) of \( p \) are negative all terms \( \frac{x^j}{x_i^j} \) in the power series of \( \frac{1}{p(x)} \) are positive for \( x \in (-4,0) \). Therefore also all terms of the power series of \( \frac{1}{p(x)} \) and therefore also all polynomial terms of \( q(x) \) are positive and since the zero-order term is one we have \( q(x) \geq 1 \). The special cases \( x = -4 \) and \( x = 0 \) follow from continuity.

\[
\square
\]

2 Construction

In this section, we describe how to construct the coefficients. It is tedious even for small \( m \) to compute the roots of the polynomial \( p \) and the power series of \( 1/p \) used in the proof in the previous section. To get the coefficients it is easier to determine the polynomial coefficients the polynomial \( q \) recursively. To compute the coefficients of the \( N_m \), i.e. the values of the B-splines at integers de Boors 3-point recursion can be used which for our case of uniform grids reads

\[
B_m(x) = \frac{(m+1)}{m} B_{m-1}(x + \frac{1}{2}) + \frac{(m+1)}{m} B_{m-1}(x - \frac{1}{2}).
\]

(13)
2.1 The cases $m = 3$ and $m = 5$

For $m = 3$ we have

\[
N_m(z) = \frac{1}{6} z^{-1} + \frac{4}{6} + \frac{1}{6} z^1 = 1 + \frac{1}{6} (z + z^{-1} - 2) \tag{14}
\]

\[
p(x) = 1 + \frac{x}{6} \tag{15}
\]

\[
\frac{1}{p(x)} = 1 - \frac{x}{6} + \left(\frac{x}{6}\right)^2 - \cdots \tag{16}
\]

\[
q(x) = 1 - \frac{1}{6} x \tag{17}
\]

\[
\Lambda(z) = 1 - \frac{1}{6} (z + z^{-1} - 2) = -\frac{1}{6} z^{-1} + \frac{4}{3} - \frac{1}{6} z^1 \tag{18}
\]

\[
(\lambda_j)^k = \left( -\frac{1}{6}, \frac{4}{3}, \frac{1}{6} \right) \tag{19}
\]

For $m = 5$ we have

\[
N_m(z) = \frac{1}{120} z^{-2} + \frac{26}{120} z^{-1} + \frac{66}{120} + \frac{26}{120} z^1 + \frac{1}{120} z^2 \tag{20}
\]

\[
= 1 + \frac{1}{4} (z + z^{-1} - 2) + \frac{1}{120} (z + z^{-1} - 2)^2 \tag{21}
\]

\[
p(x) = 1 + \frac{x}{4} + \frac{x^2}{120} \tag{22}
\]

\[
q(x) = 1 - \frac{x}{4} + \frac{13 x^2}{240} \tag{23}
\]

\[
\Lambda(z) = 1 - \frac{1}{4} (z + z^{-1} - 2) + \frac{13}{240} (z + z^{-1} - 2)^2 \tag{24}
\]

\[
= \frac{13}{240} z^{-2} - \frac{7}{15} z^{-1} + \frac{73}{40} - \frac{7}{15} z^1 + \frac{13}{240} z^2 \tag{25}
\]

\[
(\lambda_j)^k = \left( \frac{13}{240}, \frac{7}{15}, \frac{73}{40}, \frac{7}{15}, \frac{13}{240} \right) \tag{26}
\]

2.2 Code to construct convolution mask for arbitrary $m$

Below an octave/matlab code for computing the convolution mask.

```matlab
function lambda = conv_mask(m)
    assert(mod(m,2)==1,'m must be an odd positive integer')
    k=(m-1)/2;
    %compute B-spline at integer by de Boor three point recursion
    b=1;
    for j=2:m
        x=(j-1)/2*linspace(-1,1,j);
        %
```
b = (((j+1)/2+x).*[b 0]+((j+1)/2-x).*[0 b])/j;
end

%coefficient wrt z of powers of x=1/z+z-2
powx=cell(1,k);
powx{1}=[1 -2 1];
for j=2:k
    powx{j}=conv(powx{1},powx{j-1});
end

%determine recursively coefficients of p(x)
cp=zeros(1,k+1);
for j=1:k
    cp(j)=b(1)/powx{k+1-j}(end);
    b=b-b(1)*powx{k+1-j};
    %remove first and last coefficient which is zero
    b([1 end])=[];
end

%determine coefficients of q
cq=zeros(1,k+1);
cq(k+1)=1;
for j=k:-1:1
    cq(j)=-sum(cq(j+1:end).*cp(end-1:-1:j));
end

%determine coefficients of lambda(z)
lambda=cq(end);
for j=1:k
    lambda=[0 lambda 0]+cq(end-j)*powx{j};
end
end

References
