ON THE FUNDAMENTALS OF COLLATZ CONJECTURE

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We differentiate even and odd numbers into various groups and subgroups. We provide the properties of the forms of numbers which fall into each groups and subgroups. We expound on the relationship of a special group of even numbers and the collatz conjecture, we also derive an accurate formula to calculate the steps involved when an even number of the group is the initial value of the collatz operation.

For each group and subgroup of odd and even numbers, we discuss the observed pattern of their sequences and also derive accurate formulas for each sequence. Throughout, b, d, k, N, n, x, m, and z all denote positive integers, with d, and N denoting odd numbers, x and z denoting even numbers, and b denoting special even-even numbers

The order of priority of the properties of each group is key in the differentiation of the numbers into their various groups and subgroups.

KEYWORDS:
Odd-even numbers, ordinary even-even numbers, special even-even numbers, odd-odd numbers, ordinary odd numbers, special odd numbers.

EVEN NUMBERS

All even numbers can be divided into two groups which include; odd-even numbers (OE), and even-even numbers (EE).

The odd-even numbers (OE) have the following properties;

1. They can be expressed as the product of 2 and any odd number d.
2. They are divisible by 2, but not by 4.
3. They are divisible by atleast one odd number other than 1

The even-even numbers (EE) have the following properties;

1. They can be expressed as the product of 2 and any even number x.
2. They are divisible by 2 and(or) 4.
3. Some even-even numbers are divisible by odd numbers, while some are not.

The even-even numbers can also be subdivided into two subgroups. These are; the ordinary even-even numbers (OEE), and the special even-even numbers (SEE).

The ordinary even-even numbers (OEE) have the following properties;

1. They cannot be expressed as 2^n.
2. They are divisible by atleast one odd number other than 1.
3. They are divisible by 2.

The properties of the special even-even numbers (SEE) include;

1. They can be expressed as 2^n
2. They are not divisible by odd numbers greater than 1
3. They are divisible by 2.

The table below shows the differentiation of even numbers from 2-98 into the group and subgroups mentioned above.
The rules of the differentiation is as follows;
i. If $x \div 2^n = d = 1$, n is a special even-even number .
ii. If $x \div 2 = d > 1$, n is an odd-even nimber.
iii. If $x \div 2^n = d > 1$, n is an ordinary even-even number.

x	X÷2^n (d)	2^n	X type
2	1	2	SEE/OE
4	1	4	SEE
6	3	2	OE
8	1	8	SEE
10	5	2	OE
12	3	4	OEE
14	7	2	OE
16	1	16	SEE
18	9	2	OE
20	5	4	OEE
22	11	2	OE
24	3	8	OEE
26	13	2	OE
28	7	4	OEE
30	15	2	OE
32	1	32	SEE
34	17	2	OE
36	9	4	OEE
38	19	2	OE
40	5	8	OEE
42	21	2	OE
44	11	4	OEE
46	23	2	OE
48	3	16	OEE
50	25	2	OE
52	13	4	OEE
54	27	2	OE
56	7	8	OEE
58	29	2	OE
60	15	4	OEE
62	31	2	OE
64	1	64	SEE
66	33	2	OE
68	17	4	OEE
70	35	2	OE
72	9	8	OEE
74	37	2	OE
76	19	4	OEE
78	39	2	OE
80	5	16	OEE
82	41	2	OE
84	21	4	OEE
86	43	2	OE
88	11	8	OEE
90	45	2	OE
92	23	4	OEE
94	47	2	OE
96	3	32	OEE
98	49	2	OE

We can now write down the even numbers according to the odd-even number group and even-even subgroups.

THE ODD-EVEN NUMBER SEQUENCE
The sequence of the odd-even numbers is as follows;

2,6,10,14,18,22,26,30,34,38,42,46,50,54,58,62,66,70,74,78,82,86,90,94,98...
The sequence has a first term and a common difference. Using the arithmetic progression formula, we can find an nth term of the sequence.
Let an nth term be G_n, then,

$$G_n=a+(n-1)d$$

where the first term a=2, and the common difference d=4

$$\therefore G_n=2+(n-1)4\_[1]$$

Therefore, every even number G_n is an odd-even number for every value of n.

THE SPECIAL EVEN-EVEN NUMBER SEQUENCE
The sequence of special even-even numbers are as follows;

2,4,8,16,32,64...

The sequence has a first term and a common ratio. Using the geometric progression formular, we can find any nth term of the sequence.
Let an nth term of the sequence be L_n, then;

$$L_n=ar^{n-1}$$

where the first term a=2, and the common ratio r=2, then;

$$L_n=2\times 2^{(n-1)}$$

$$=2^{(n-1)+1}$$

$$=2^{n-1+1}$$

$$\therefore L_n=2^n\_[2]$$

Thus, every even number $L_n$is a special even-even number.

THE ORDINARY EVEN-EVEN NUMBER SEQUENCE
The sequence of ordinary even-even numbers are as follows;

12,20,24,28,36,40,44,48,52,56,60,68,72,76,80,84,88,92,96...

From the properties of ordinary even-even numbers, it follows that zero is an ordinary even-even number, but since it clearly disobeys the rules of differentiation of even numbers, we do not include it in the above table, and in the sequence above.

The sequence has a first term, but does not have a common difference, or ratio, but it does follow a pattern. There are twin terms throughout the sequence where the difference between the two terms is twice the difference between other terms of the sequence.

The twin terms are as follows;

12 and 20, 28 and 36, 60 and 68…, the difference between these terms is 8.

The first term of the first twin is 12, and can be expressed as;

$2^4-4=12$, the next term of the twin is 20, and can be expressed as;

$2^4+4=20$. the other terms which follow are sums of 20 and n multiples of 4. The summation of 20 and m multiples of 4 stops with the term before the next twin terms. The sequence continues with the first term of the next twin term 28 which can be expressed as;

$2^5-4=28$, the next term of the twin is $2^5+4=36$, in a similar fashion, the terms that follow are sums of 36 and m multiples of 4 and ends with the term before the next twin term.

The first term of the next twin is; $2^6-4=60$, while the next term is; $2^6+4=68$, the terms which follow are sums of 60 and m multiples of 4

Following this pattern, we shall write the number sequence in one equation. Let Ω be the sequence of ordinary even-even numbers, so that;

$$\Omega=\{2^4-4,2^4+4,2^4+[m-(m-2)]\times4],\ldots,2^4+m(4),2^{4+n}-4,2^{4+n}+4,2^{4+n}+[m-(m-2)]\times4,\ldots,2^{4+n}+m(4),2^{4+(n+k)}-4,2^{4+(n+k)}+4,2^{4+(n+k)}+[m-(m-2)]\times4,\ldots,2^{4+(n+k)}+m(4)\ldots\}\_[3]$$

The term before a twin term is; $2^{4+n}+m(4)$, while the first term of the twin term is; $2^{5+n}-4$, the difference between the latter and the former is 4, thus, we can write;

$$2^{4+n}+m(4)=2^{5+n}-4-4$$

$$=2^{4+n}+m(4)=2^{5+n}-8$$

Subtracting 2^{4+n} from both sides, we get;

$$m(4)=2^{5+n}-8-2^{4+n}$$

$$=2^{5+n}-2^{4+n}-8$$

$$=2^{4+n}(2-1)-8$$

$$=(2^{4+n}-2^3)$$

$$\Rightarrow m=(2^{4+n}-2^3)\div2^2$$

$$=2^2(2^{2+n}-2)\div2^2$$

$$\therefore m=2^{2+n}-2\_[4]$$

Having found m, we can rewrite equation [3] as;

Ω={2^4-4,2^4+4,2^4+[2^{2+n}-2-(2^{2+n}-2-2)]×4],…,2^4+(2^{2+n}-2)4,2^{4+n}-4,2^{4+n}+4,2^{4+n}+[2^{2+n}-2-(2^{2+n}-2-2)] 4,…,2^{4+n}+(2^{2+n}-2)4,$2^{4+(n+k)}$-4,$2^{4+(n+k)}$+4,$2^{4+(n+k)}$+[2^{2+n}-2-(2^{2+n}-2-2)]×4,…,$2^{4+(n+k)}$+(2^{2+n}-2)4…}

={2^4-4,2^4+4,2^4+[2^{2+n}-2-(2^{2+n}-4)]×4],…,2^4+(2^{2+n}-2)4,2^{4+n}-4,2^{4+n}+4,2^{4+n}+[2^{2+n}-2-(2^{2+n}-4)] ×4,…,2^{4+n}+(2^{2+n}-2)4,$2^{4+(n+k)}$-4,$2^{4+(n+k)}$+4,$2^{4+(n+k)}$+[2^{2+n}-2-(2^{2+n}-4)]×4,…,$2^{4+(n+k)}$+(2^{2+n}-2)4…}

={2^4-4,2^4+4,2^4+[2^{2+n}-2-2^{2+n}+4)]×4],…,2^4+(2^{2+n}-2)4,2^{4+n}-4,2^{4+n}+4,2^{4+n}+[2^{2+n}-2-2^{2+n}+4)] ×4,…,2^{4+n}+(2^{2+n}-2)4,$2^{4+(n+k)}$-4,$2^{4+(n+k)}$+4,$2^{4+(n+k)}$+[2^{2+n}-2-2^{2+n}+4)]×4,…,$2^{4+(n+k)}$+(2^{2+n}-2)4…}

={2^4-4,2^4+4,2^4+[2]×4],…,2^4+(2^{2+n}-2)4,2^{4+n}-4,2^{4+n}+4,2^{4+n}+[2] ×4,…,2^{4+n}+(2^{2+n}-2)4,$2^{4+(n+k)}$-4,$2^{4+(n+k)}$+4,$2^{4+(n+k)}$+[2]×4,…,$2^{4+(n+k)}$+(2^{2+n}-2)4…}

∴Ω={2^4-4,2^4+4,2^4+8,…,2^4+(2^{2+n}-2)4,2^{4+n}-4,2^{4+n}+4,2^{4+n}+8,…,2^{4+n}+(2^{2+n}-2)4,$2^{4+(n+k)}$-4,$2^{4+(n+k)}$+4,$2^{4+(n+k)}$ +8,…,$2^{4+(n+k)}$+(2^{2+n}-2)4…}_[5]

THE RELATIONSHIP OF SPECIAL EVEN-EVEN NUMBERS AND THE COLLATZ CONJECTURE

The table below shows an intricate relationship between the nth degree to which 2 is raised and the number of special even-even numbers from 2 to 2^n.

2^n	Special even-even numbers from 2 to 2^n	No. of Special even-even numbers from 2 to 2^n
2^1	2	1
2^2	2,4	2
2^3	2,4,8	3
2^4	2,4,8,16	4
2^5	2,4,8,16,32	5
	…	…
2^n	From $2^{n-(n-1)}$ to 2^n	m

From the above table, it follows that the number of special even-even numbers from 2 to another special even-even number 2^n is n.
we can find n by letting;

$$2^n=b$$

taking the natural logarithm of both sides, we have;

$$\ln(2^n)=\ln(b)$$

$$=n\ln(2)=\ln(b)$$

$$=> n=\ln(b)\div\ln(2)\_[6]$$

When we divide 2 by 2, we obtain 1, therefore, when 2^n is divided down to 1, n becomes;

$$n=\{\ln(b)\div\ln(2)\} +1\_[7]$$

The number of division of 2^n by 2 is the number of steps required to reach 1, when 2^n is the initial value of the collatz operation.
The table below shows a relationship between the nth degree of 2 and the steps z involved in the operation of 2^n as an initial value of the collatz operation.

2^n	No. of divisions of 2^n by 2	Steps (z)
2^1	1	1
2^2	2	2
2^3	3	3
2^4	4	4
2^5	5	5
...	...	...
2^n	m	z

The above table shows that the total number of steps z from 2^n to 1 is n, i.e,

$$z=n\_[8]$$

Substituting [6] into [8], we obtain;

$$z=\ln(b)\div\ln(2)\_[9], \text{ where } z\neq\infty.$$

Since an infinite z implies an infinite number of divisions, which also mean that b never reaches 1., a finite z imples that the collatz conjecture holds for any number of the form 2^n

ODD NUMBERS
All odd numbers can be differentiated into three groups, these are as follows; odd-odd numbers (O_2), ordinary odd numbers (OO), and special odd numbers (SO).

The odd-odd numbers (O_2) have the following properties;
1. When the "3N+1" rule is applied to an odd-odd number, an odd-even number is obtained.
2. Dividing the odd-even number by 2 gives an odd number.

The ordinary odd numbers (OO) have the following properties;
1. when the "3N+1" rule is applied to an ordinary odd number, an ordinary even-even number is obtained.
2. Dividing the ordinary even-even number by 2^n gives an odd number, provided that 2^n is less than the ordinary even-even number.

The special odd numbers (SO) have the following properties;
1. When the "3N+1" rule is applied to a special odd number, a special even-even number is obtained.
2. The special even-even number $b=2^n$ has exactly n steps to 1.

The table below shows the differentiation of odd numbers N from 1-99 into the three groups mentioned above.

The rules of the differentiation are as follow;

i. If (3N+1)÷2=d>1, then, N is an odd-odd number.

ii. If (3N+1)÷2^n=d>1, then N is an ordinary odd number for n>1.

iii. if (3N+1)÷2^n=d=1, then N is a special odd number.

N	3N+1	d	2^n	N type
1	4	1	4	SO
3	10	5	2	O_2
5	16	1	16	SO
7	22	11	2	O_2
9	28	7	4	OO
11	34	17	2	O_2
13	40	5	8	OO
15	46	23	2	O_2
17	52	13	4	OO
19	58	29	2	O_2
21	64	1	64	SO
23	70	35	2	O_2
25	76	19	4	OO
27	82	41	2	O_2
29	88	11	8	OO
31	94	47	2	O_2
33	100	25	4	OO
35	106	53	2	O_2
37	112	7	16	OO
39	118	59	2	O_2
41	124	31	4	OO
43	130	65	2	O_2
45	136	17	8	OO
47	142	71	2	O_2
49	148	37	4	OO
51	154	77	2	O_2
53	160	5	32	OO
55	166	83	2	O_2
57	172	43	4	OO
59	178	89	2	O_2
61	184	23	8	OO
63	190	95	2	O_2
65	196	49	4	OO
67	202	101	2	O_2
69	208	13	16	OO
71	214	107	2	O_2
73	220	55	4	OO
75	226	113	2	O_2
77	232	29	8	OO
79	238	119	2	O_2
81	244	61	4	OO
83	250	125	2	O_2
85	256	1	256	SO
87	262	131	2	O_2
89	268	67	4	OO
91	274	137	2	O_2
93	280	35	8	OO
95	286	143	2	O_2
97	292	73	4	OO
99	298	149	2	O_2

We can now write down the odd numbers according to their groups.

THE ODD-ODD NUMBER SEQUENCE

For odd-odd numbers, the sequence is as follows;

3,7,11,15,19,23,27,31,35,39,43,47,51,55,59,63,67,71,75,79,83,87,91,95,99...

The sequence has a first term and a common difference, hence, we can use the Arithmetic progression formula to find an nth term.

Let the nth term be D_n, so that;

$D_n=a+(n-1)d$, where the first term a=3, the common difference d=4, therefore;

$D_n=3+(n-1)4$ _[10]

Thus, every odd number D_n is an odd-odd number.

THE SPECIAL ODD NUMBER SEQUENCE

The special odd number sequence is as follows;

1,5,21,85...

Let the nth term be E_n. The difference between an nth term and an (n-1)th term in the above sequence is as follows;

$$E_2-E_1=2^2$$

$$\Rightarrow E_2=E_1+2^2,$$

$$E_3-E_2=2^4$$

$$\Rightarrow E_3=E_2+2^4$$

also,

$$E_4-E_3=2^6$$

$$\Rightarrow E_4=E_3+2^6$$

It thus follows that;

$E_n=E(n-1)+4^{(n-1)}$ _[11],

For every (n>1)th term,

$E(n-1)=4^{(n-n)}+4^{(n-(n-1))}+4^{(n-(n-2))}+...+4^{(n-(n-k))}$, where k=2,3,4...(n-2).

Rewriting equation [11], we have;

$E_n=4^{(n-n)}+4^{(n-(n-1))}+4^{(n-(n-2))}+...+4^{(n-(n-k))}+4^{(n-1)}$

$=4^0+4^1+4^2+...+4^{(n-(n-k))}+4^{(n-1)}$

$E_n=1+4(1+4+...+4^{(n-(n-k))-1}+4^{(n-1)-1})$_[12]

The terms in the arc bracket have a common ratio i.e 4, hence it is a geometric progression. It can therefore be expressed as;

ar^{n-1},

where the first term, a=1, the common ratio, r=4, we then have;

$(1)4^{n-1}$

$=4^{n-1}$

The sum of a geometric progression when the common ratio r>1 is given by;

$Sn=a(r^n-1)\div(r-1)$

when a=1, and r=4, then;

$=1(4^n-1)\div(4-1)$

$Sn=(4^n-1)\div3$_[13]

Substituting the terms in the arc bracket in equation [12] with equation[13], we obtain:

$E_n=1+4\{(4^n-1)\div3\}$

To obtain an nth term, we subtract 1 from the power to which 4 is raised, i.e

$E_n=1+\{4(4^{(n-1)}-1)\div3\}$

$=1+(4^{(n-1)+1}-4)\div3$

$=(3+4^n-4)\div3$

$E_n=(4^n-1)\div3$__[14]

Equation [14] is the general equation of the special odd numbers sequence.

The number of steps from E_n to a special even-even number 3En+1 is 1. Therefore, when a special odd number is the initial value of the collatz operation, the number of steps $z(E_n)$ is given by;

$z(E_n)=z+1$_[15]

substituting [9] into [15], we have;

$$z(E_n)=\{\ln(b)\div\ln(2)\}+1\_[16], \text{ where } z(E_n)\neq\infty,$$

hence, the collatz conjecture holds for every special odd numbers.

THE ORDINARY ODD NUMBER SEQUENCE

The sequence of ordinary odd numbers is as follows;

9,13,17,25,29,33,37,41,45,49,53,57,61,65,69,73,77,81,89,93,97...

The sequence has a first term (9) and a difference (4), but there are gaps, where the difference between two terms is twice the difference (4). The gaps appear at the {4(1)-1}th and 4(1)th terms, and between the {4(5)-2}th and {4(5)-1}th terms (notice that the numbers in the arc brackets are numbers of the special odd numbers sequence).

The gaps appear due to the absence of numbers which are originally sums of the common difference and the {4(1)-1}th and {4(5)-2}th terms respectively.These numbers are 17+4=21, and 81+4=85, these are numbers of the special odd numbers sequence.

Following the pattern of the gaps, we see that the gaps appear between;
$\{4(E_n)-n\}$th and $\{4(E_n)-(n-1)$th terms.
Let B_m be an nth term, Using the A.P formula, we have that;

$$B_m=a+(m-1)d, \text{ where } a=9, \text{ and } d=4, \text{ then;}$$

$$B_m=9+(m-1)4\_[17]$$

[17] holds only for terms less than 4(1), this is due to the difference between the terms before and after the gap. Hence, to find the term after the gap, we add 4 to the equation of the term before the gap, to compensate for the difference. However, for terms other than the first term after a gap, this is not necessary.

It's also worthy to note that due to the gaps, the first term changes from an initial first term to the term after the gap. This is because an initial first term fails after each gap.
∴for $m>\{4(E_n)-(n-1)\}$th term, we have that;

$$B_m=a+(m-1)4,$$

where the first term after a gap $a=B\{4(E_n)-(n-1)\}$, then;

$$B_m=B\{4(E_n)-(n-1)\}+(m-1)4\_[18]$$

For $m=\{4(E_n)-(n-1)\}$th term,

$a=B\{4[E(n-1)]-(n-1)\}$ (which is the initial first term after a previous gap before m).
Thus;

$$B_m=B\{4[E(n-1)]-(n-1)\}+(m-1)4+4[17]$$

CONCLUSION

The idea of differentiating odd numbers and even numbers into the various groups discussed above is central to the collatz conjecture. with this idea and other established facts about the conjecture, the author believes that a general proof of the conjecture for all whole numbers is imminent.