Prove Grimm's Conjecture via Stepwise Forming Consecutive Composite Number's Points on the Original Number Axis

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Abstract

If regard positive integers which have a common prime factor as a kind, then the positive half line of the number axis consists of infinite many recurring segments which have same permutations of χ kinds of integer's points, where $\chi \geq 1$. In this article, the author shall prove Grimm's conjecture by the method that changes orderly symbols of each kind of composite number's points at the original number axis, so as to form consecutive composite number's points within limits that proven Bertrand's postulate restricts.

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1. Introduction

Grimm's conjecture named after Carl Albert Grimm. The conjecture states that if $n+1$, $n+2$, … $n+k$ are all composite numbers, then there are distinct primes p^{i}_{j} such that $p^{i}_{j} \mid (n+j)$, for $1 \leq j \leq k$, [1].

For example, for consecutive composite numbers between primes 523 and 541, take out one another's distinct prime factors from them as follows.

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Composite numbers:	524,	525,	526,	527,	528,	529,	530,
Decomposing prime factors:	$2^2\times131$,	$3\times5^2\times7$,	2×263,	17×31,	$2^4\times3\times11$,	23^2,	$2\times5\times53$,
Distinct prime factors:	131,	3,	263,	17,	11,	23,	53,

531,	532,	533,	534,	535,	536,	537,	538,	539,	540
$3^2\times59$,	$2^2\times7\times19$,	13×41,	$2\times3\times89$,	5×107,	$2^3\times67$,	3×179,	2×269,	$7^2\times11$,	$2^2\times3^3\times5$
59,	19,	41,	89,	107,	67,	179,	269,	7,	2

This conjecture was first published in *American Mathematical Monthly*, *76 (1969) 1126-1128*. Yet, it is still both unproved and un-negated a conjecture hitherto, [2].

2. Preparations before the Proof

In order to restrict the length of consecutive composite numbers, it is necessary to quote proven Bertrand's Postulate.

Bertrand's postulate, also called the Bertrand-Chebyshev theorem or Chebyshev's theorem, states that if $n>3$, there is always at least one prime p between n and $2n-2$. Equivalently, if $n>1$, then there is always at least one prime p such that $n<p<2n$, [3]. In addition to this, as well as may see also proven Legendre-Zhang's conjecture.

Legendre-Zhang's conjecture states that there is at least an odd prime between n^2 and $n(n+1)$, and there is at least an odd prime between $n(n+1)$ and $(n+1)^2$, where $n\geq2$, [4].

Undoubtedly, the length of consecutive composite numbers is shorter than the distance from n to $2n$.

It is well known that each and every integer's point at positive half line of the number axis expresses a positive integer, and that infinite many a distance between two adjacent integer's points equals one another.

So use the symbol • to denote a positive integer's point, then either such a symbol is in a formulation, or lies at positive half line of the number axis.

After a rightward-directional half line is marked with infinite many symbols of integer's point purely, it is regarded as positive half line of the original number axis, as listed below.

First Illustration

In addition, use too the symbol •s to denote at least two integer's points in the formulation. Also the positive half line of the number axis is called the half line for short, hereinafter. If the number axis is original, then the half line is called the original half line as well.

After regard smallest prime *2* as *№1* prime and regard prime P_χ as №χ prime where $\chi \geq 1$, prime *2* is written as P_1, and that the greater value of χ, show the greater prime P_χ. And then, regard positive integers which share prime factor P_χ as №χ kind of integers, moreover regard №χ kind of integers except for prime P_χ as №χ kind of composite numbers.

Actually, №χ kind of integers consists of infinitely many a product which multiplies each and every positive integer by P_χ, so there is a №χ kind's integer's point within consecutive P_χ integer's points at the half line. Excepting P_χ as a prime point, others are all composite number's points, and regard them as *№χ* kind of composite number's points within *№χ* kind of integer's points.

If a composite number contains at least two distinct prime factors, then the composite number belongs to at the least two kinds of composite numbers concurrently.

Since there are infinitely many primes, so there are infinitely many kinds of composite numbers as well. Correspondingly there are infinitely many kinds of composite number's points at the half line.

By now, what we need is to find a law that take out a set of distinct prime factors from a string of consecutive composite numbers, one for one.

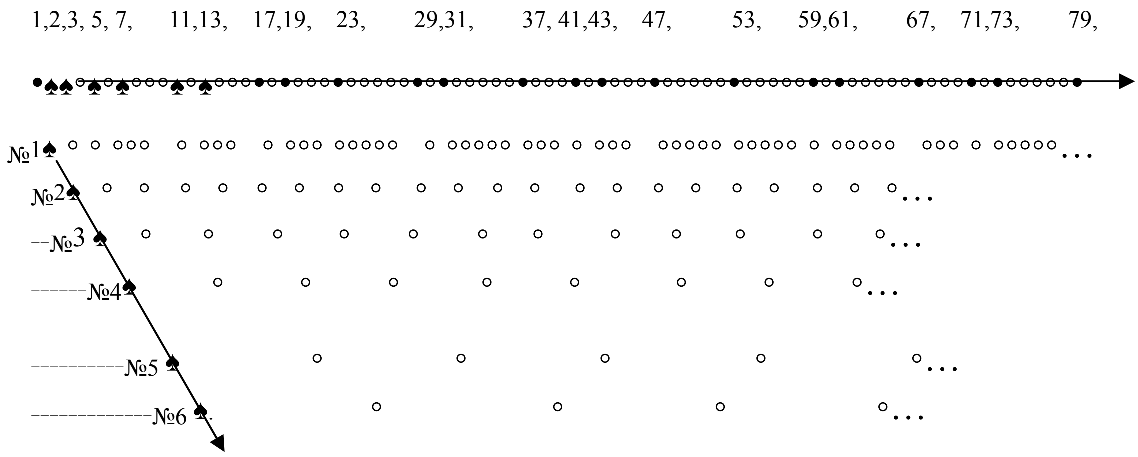
For this purpose, the author is planning to proceed from the formation of consecutive composite number's points at the half line. Accordingly first define symbols of prime point and composite number's point as follows.

If an integer's point • at the half line is defined as a composite number's point, then change the symbol ◦ for original symbol •. Withal, use the symbol ◦s to denote at least two composite number's points in the formulation.

If an integer's point • at the half line is defined as a prime point, then change the symbol ♠ for original symbol •.

Next, change ◦s for •s at places of *№χ* kind of composite number's points according to the order of χ from small to large, where $\chi \geq 1$.

Thus it can be seen, that one another's permutations of χ kinds of integer's points always assume infinite many recurrences on same pattern at the half line, irrespective of their prime/composite attribute and integer's points amongst them. Seriatim decompose $№\chi$ kind of integer's points at the half line according to the order of χ from small to large, please, see also the schematic illustration as follows:



Second Illustration

We regard one another's equivalent shortest line segments on same permutations of χ kinds of integer's points at the half line as recurring line segments of permutations of the χ kinds of integer's points. And that use the character string "$RLS_{№1\sim№\chi}$" to express a recurring line segment of $\sum№\chi$ kind of integer's points, where $\chi\geq1$. Besides, at least two such recurring line segments are expressed by $RLSS_{№1\sim№\chi}$, [5].

$№1$ $RLS_{№1\sim№\chi}$ begins with 1. Also, there are $\prod P_\chi$ integer's points per $RLS_{№1\sim№\chi}$, where $\chi\geq1$, and $\prod P_\chi=P_1P_2...P_\chi$. Thus a $RLS_{№1\sim№\chi}$ consists of consecutive P_χ $RLSS_{№1\sim№(\chi-1)}$, and that they link one by one.

As thus one another's permutations of χ kinds of integer's points at each of $RLSS_{№1\sim№\chi}$ is just the same, irrespective of integer's points amongst them.

Number the ordinals of integer's points at each of seriate $RLSS_{№1\sim№\chi}$ by consecutive natural numbers ≥*1*. Let from left to right each and every integer's point at each of seriate $RLSS_{№1\sim№\chi}$ be marked with from small to great a natural number ≥*1*.

Then, there is a $№\chi$ kind's integer's point within P_χ integer's points which share an ordinal at P_χ $RLSS_{№1\sim№(\chi-1)}$ of a $RLS_{№1\sim№\chi}$.

Of course, there is a $№\chi$ kind's composite number's point within P_χ integer's points which share an ordinal at P_χ $RLSS_{№1\sim№(\chi-1)}$ of each of seriate $RLSS_{№1\sim№\chi}$ on the right side of $№1$ $RLS_{№1\sim№\chi}$.

Prime points P_1, P_2 … $P_{\chi-1}$ and P_χ exist at $№1$ $RLS_{№1\sim№\chi}$. Yet, there are χ composite number's points at ordinals of P_1, P_2, P_3... and P_χ at each of seriate $RLSS_{№1\sim№\chi}$ on the right side of $№1$ $RLS_{№1\sim№\chi}$. Thus $№1$ $RLS_{№1\sim№\chi}$ is a special $RLS_{№1\sim№\chi}$, such as the permutations of many kinds of integer's points at $№1$ $RLS_{№1\sim№4}$ are as follows.

•♠♠○♠○♠○○○♠○♠○○○♠○♠○○○♠○○○○○♠○♠○○○○○♠○○○♠○♠○○○♠○○○○○♠○○○○○♠○♠○○○○○♠○○○♠

2,3, 5, 7, 11,13, 17,19, 23, 29,31, 37, 41,43, 47, 53, 59,61, 67, 71,

○♠○○○○○♠○○○♠○○○○○♠○○○○○○○♠○○○♠○♠○○○♠○♠○○♠○○○○○○○○○○○○○♠○○○♠○○○♠○♠○○○○○○○○○

73, 79, 83, 89, 97, 101,103,107,109,113, 127, 131, 137,139,

♠○♠○○○○○♠○○○○○♠○○○♠○○○○○♠○○○○○♠○♠○○○○○○○○○♠○○♠○○○♠○♠○○○○○○○○○○○⟶

149,151,157, 163, 167, 173, 179,181, 191,193, 197,199, 210

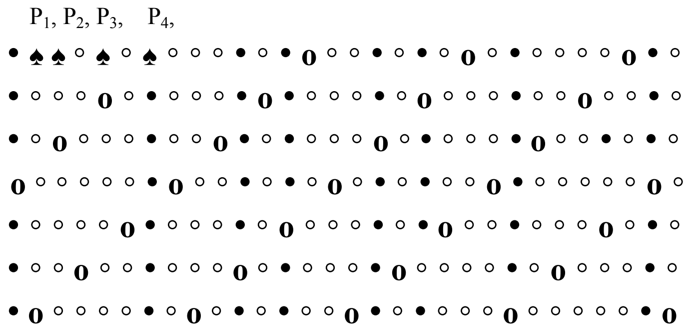
Third Illustration

$№1$ $RLS_{№1}$ ends with integer's point *2*; $№1$ $RLS_{№1\sim№2}$ ends with integer's point *6*; $№1$ $RLS_{№1\sim№3}$ ends with integer's point *30*; $№1$ $RLS_{№1\sim№4}$ ends with integer's point *210*.

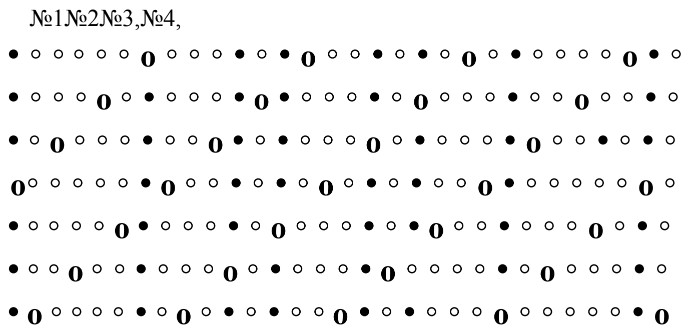
Unhappily, for such a seriation, whoever is difficult to distinguish the differentia of $№1$ $RLS_{№1\sim№\chi}$ in comparison with each of seriate

$RLSS_{№1\sim№\chi}$ on the right side of *№1* $RLS_{№1\sim№\chi}$, likewise difficult to find the same of two $RLSS_{№1\sim№\chi}$, therefore P_χ $RLSS_{№1\sim№(\chi-1)}$ of a $RLS_{№1\sim№\chi}$ can be folded at an illustration, [6].

For example, P_4 $RLSS_{№1\sim№3}$ of *№1* $RLS_{№1\sim№4}$, P_4 $RLSS_{№1\sim№3}$ of *№n* $RLS_{№1\sim№4}$ and *№1*, *№2*, *№3* and *№4* kinds of integer's points at two such $RLSS_{№1\sim№4}$, as listed below, where *n≥2*.



№1 $RLS_{№1\sim№4}$



№n $RLS_{№1\sim№4}$ where n≥2

Fourth Illustration

Annotation: each of *№1, №2* and *№3* kind's composite number's points is expressed by ∘, besides each of *№4* kind's composite number's points is expressed by **o**.

3. Proving Grimm's Conjecture

Pursuant to basic conceptions and grounds of argument concerned within the preceding chapter, by now let us set about proving Grimm's conjecture via changes of symbols at places of *№*χ kind's composite number's points at the original half line to form consecutive composite number's points, where $\chi \geq 1$, ut infra.

(1). Change ◦s for •s at places of *№1* kind of composite number's points, then there is one ◦ at each of $RLSS_{№1}$ on the right side of *№1* $RLS_{№1}$. That is to say, there is n_1 ◦ between adjacent two •s, where $n_1=0, 1$.

Evidently, smallest prime factor of *№1* kind's composite numbers is p_1.

(2). Change ◦s for •s at places of *№2* kind of composite number's points successively. Since there is a *№2* kind's composite number's point within P_2 integer's points which share an ordinal at P_2 $RLSS_{№1}$ of a $RLS_{№1\sim№2}$, then *№2* kind's composite number's points coincide with two •s and one ◦ of *№1* kind's composite number's point at each and every $RLS_{№1\sim№2}$ monogamously. Therein, *№2* kind's composite number's points which coincide with ◦s are all repetitive composite number's points, yet each *№2* kind's composite number's point which coincides with one • starts to constitute consecutive composite number's points or turned into a separate *№2* kind's composite number's point.

As a consequence, there are n_2 ◦s between two adjacent •s at the half line, where $n_2=0, 1, 2$ and 3. Thus, except repetitive composite numbers, smallest prime factor of newly-added *№2* kind's composite numbers is p_2.

(3). Change ◦s for •s at places of *№3* kind of composite number's points successively. Since there is a *№3* kind's composite number's point within

P_3 integer's points which share an ordinal at P_3 $RLSS_{№1\sim№2}$ of a $RLS_{№1\sim№3}$, then *№3* kind's composite number's points coincide with one •, two pure *№1* kind's composite number's points, a pure *№2* kind's composite number's point, and a *№1* plus *№2* kind's composite number's point at a $RLS_{№1\sim№3}$ monogamously. Therein, *№3* kind's composite number's points which coincide with ◦s are all repetitive composite number's points, yet each *№3* kind's composite number's point which coincides with one • to constitute consecutive composite number's points or turned a separate *№3* kind's composite number's point.

As a consequence, there are n_3 ◦s between two adjacent •s at the half line where $n_3=0, 1, 2, 3, 4$ and 5. So, except repetitive composite numbers, smallest prime factor of newly-added *№3* kind's composite numbers is p_3.

By parity of reasoning, successively change ◦s for •s at places of *№*χ kind of composite number's points where $\chi>3$, on and on, and that except repetitive composite numbers, smallest prime factor of newly-added *№*χ kind's composite numbers is p_χ.

On balance, after successively change ◦s for •s at places of *№*χ kind of composite number's points where $\chi\geq 1$, excepting repetitive composite number's points which coincide with *№1*, *№2*… *№*$(\chi-1)$ kind's composite number's points monogamously, each of newly-added *№*χ kind's composite number's points is to constitute consecutive composite number's points or turned a separate *№*χ kind's composite number's point.

After change ◦s for •s at places of $\sum№X$ kinds of composite number's points where $X=1, 2, 3\ldots\chi$, every string of consecutive composite number's points on the left side of P_χ^2 is fixed. Namely every such string of consecutive composite number's points can't add a composite number's point any longer.

By this token, pursuant to the order of χ from small to large to determine each kind of composite number's points, if every prime factor of a $№\chi$ kind's composite number is not less than P_χ, then the $№\chi$ kind's composite number's point is exactly a newly-added $№\chi$ kind's composite number's point.

Thus, for any kind of composite numbers, it can not take out two identical greatest prime factors in a string of consecutive composite number's points according to the proven Bertrand's postulate. Yet, for two kinds of composite numbers in a string of consecutive composite number's points, that is not necessarily.

Pursuant to the order of χ from small to large to determine each kind of composite number's points, for taking out prime factors from a certain string of consecutive composite numbers as distinct primes, in reality, equivalently, first take out a greatest prime factor from each of the string of consecutive composite numbers, then check prime factors which are taken out whether there are identical two.

Consequently, for any string of consecutive composite numbers, first take out P_χ from P_χ^2, if it has P_χ^2; after that, take out a greatest prime factor from each of others. If in these greatest prime factors, a greatest prime factor of some composite number is pre-existing P_χ, this indicates that the composite number first belongs to $№(\chi\text{-}k)$ kind where $\chi>k\geq 1$ surely, so again take out $P_{\chi\text{-}k}$ from the composite number to replace the greatest prime factor P_χ. This is the very the law we need to look for.

For example, the string of consecutive composite numbers: 24, 25, 26, 27 and 28, i.e. $2^3\times3$, 5^2, 2×13, 3^3 and $2^2\times7$, first take out 3 from 3^3; after that, orderly take out a greatest prime factor from each of others: 3, 5, 13 and 7; again take out 2 from $2^3\times3$ to replace 3. Then 2, 3, 5, 7 and 13 are exactly a set of satisfactory distinct primes for the string of consecutive composite numbers.

Well then, all such prime factors which are taken out from each string of consecutive composite numbers are exactly satisfactory distinct primes.

By now, need us to determine two of P_χ which takes out from P_χ^α and $P_\chi^{\alpha+1}$ whether they exist in a string of consecutive composite numbers.

As stated, known that the length of consecutive composite numbers is shorter than the distance from n to $2n$, then let $n=P_\chi^\alpha$, so there is a prime between P_χ^α and $2P_\chi^\alpha$.

Since it has $2P_\chi^\alpha< P_\chi^{\alpha+1}$except for $P_\chi=2$ and $\alpha=1$, then there is a prime between P_χ^α and $P_\chi^{\alpha+1}$.

That is to say, the length of consecutive composite numbers is shorter than the distance from P_χ^{α} to $P_\chi^{\alpha+1}$.

On balance, if can take out two identical greatest prime factors from two composite numbers monogamously, then the two composite numbers exist not in a string of consecutive composite numbers definitely. Accordingly, two such identical greatest prime factors derived from two strings of consecutive composite numbers are not in a set of distinct primes.

To sum up, the conjecture is proved by the author as the true. Yet, in order to impress upon readers the aforesaid way of doing things, might as well again give a concrete example. Namely monogamously find out distinct prime factors from the string of consecutive composite numbers between prime 1327 and prime 1361, ut infra.

First decompose these composite numbers into prime factors, they are: $1328=2^4\times83$, $1329=3\times443$, $1330=2\times5\times7\times19$, $1331=11^3$, $1332=2^2\times3^2\times37$, $1333=31\times43$, $1334=2\times23\times29$, $1335=3\times5\times89$, $1336=2^3\times167$, $1337=7\times191$, $1338=2\times3\times223$, $1339=13\times103$, $1340=2^2\times5\times67$, $1341=3^2\times149$, $1342=2\times11\times61$, $1343=17\times79$, $1344=2^6\times3\times7$, $1345=5\times269$, $1346=2\times673$, $1347=3\times449$, $1348=2^2\times337$, $1349=19\times71$, $1350=2\times3^3\times5^2$, $1351=7\times193$, $1352=2^3\times13^2$, $1353=3\times11\times41$, $1354=2\times677$, $1355=5\times271$, $1356=2^2\times3\times113$, $1357=23\times59$, $1358=2\times7\times97$, $1359=3^2\times151$ and $1360=2^4\times5\times17$ according to the order of the composite numbers from small to large.

Then, greatest prime factors which are taken out from the string of consecutive composite numbers are: 83, 443, 19, 11, 37, 43, 29, 89, 167, 191, 223, 103, 67, 149, 61, 79, 7, 269, 673, 449, 337, 71, 5, 193, 13, 41, 677, 271, 113, 59, 97, 151 and 17, according to the order of the string of consecutive composite numbers.

In order to watch these distinct primes conveniently, it is necessary to renewedly arrange them into: 5, 7, 11, 13, 17, 19, 29, 37, 41, 43, 59, 61, 67, 71, 79, 83, 89, 97, 103, 113, 149, 151, 167, 191, 193, 223, 269, 271, 337, 443, 449, 673 and 677, according to the order from small to large.

It is obvious that every two primes in the string of primes are not alike.

This shows that the way of doing things is able to take out one another's-distinct prime factors from consecutive composite numbers as satisfactory primes monogamously.

In other words, consecutive composite numbers are able to be divided exactly by one another's- distinct primes respectively.

The proof was thus brought to a close. As a consequence, Grimm's conjecture is tenable.

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