The classical spin-rotation coupling and the kinematic origin of inertia

Louai Hassan Elzein Bashier*

Abstract

This paper is prepared to show that a rigid body which accelerates curvilinearly from its center of mass relative to a fixed point must simultaneously accelerate angularly relative to its center of mass. Formulae which coupling of the angular momentum and kinetic energy due to induced spin motion in the rigid body to the angular momentum and kinetic energy due to rotational motion of the same spinning rigid body have been derived. The paper also bringing to light the nature of the forces which cause induction of spin motion in that rigid body and a formula which coupling of this highlighted forces to the force which causes rotation of the rigid body has been also derived.

Keywords : Rigid body and gyroscope motion; origin of inertia; Spin-rotation coupling; Mach's principle; mass fluctuations.

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*Khartoum, Sudan. Postcode: 11123 Email: *louaielzein@gmail.com*

Nomenclature

m	=	mass of the rigid body
m_i	=	mass element in the rigid body
$\mathbf{r}_{CM}$	=	vector position of the center of mass of the rigid body relative to the axis of rotation O
$\mathbf{r}_i$	=	vector position of mass element m_i relative to the axis of rotation O of the rigid body, also position vector of the point i in the space-fixed reference system S
$\boldsymbol{\rho}'_i$	=	vector position of mass element m_i relative to the center of mass of the rigid body
$\mathbf{p}$	=	linear momentum of the rigid body
$\boldsymbol{\omega}$	=	rotational angular velocity of the rigid body relative to the axis of rotation O
$\boldsymbol{\Omega}$	=	spin angular velocity relative to the center of mass of the rigid body
I_O	=	moment of inertia of the rigid body relative to the axis of rotation O
I_{CM}	=	moment of inertia of the rigid body relative to its center of mass
$\mathbf{L}$	=	total angular momentum of the rigid body
$\mathbf{L}_R$	=	rotational angular momentum of the rigid body relative to the axis of rotation O
$\mathbf{L}_S$	=	coupled spin angular momentum of the rigid body relative to its center of mass
T	=	total kinetic energy of the rigid body
T_R	=	rotational kinetic energy of the rigid body relative to the axis of rotation O
T_S	=	coupled spin kinetic energy of the rigid body relative to its center of mass
$\boldsymbol{a}$	=	rectilinear acceleration of the rigid body
$\mathbf{F}$	=	external rectilinear force acts on the center of mass of the rigid body
$\dot{\boldsymbol{\omega}}$	=	angular acceleration of the rigid body relative to the axis of rotation O
$\boldsymbol{\tau}$	=	total torque over the rigid body
$\boldsymbol{\tau}_R$	=	rotational torque of the rigid body relative to the axis of rotation O (or the *active* torque)
$\boldsymbol{\tau}_S$	=	coupled spin torque of the rigid body relative to its center of mass (or the *inertial* torque)
$\mathbf{r}_Q$	=	position vector of the reference point Q in the space-fixed reference system S
$\mathbf{r}'_i$	=	position vector of the point i in the body-fixed reference system S'
$\boldsymbol{v}_Q$	=	velocity of the origin of S'-frame relative to the origin of S-frame
$\boldsymbol{v}_i$	=	velocity of the point i relative to the origin of S-frame
$\boldsymbol{v}'_i$	=	velocity of the point i relative to the origin of S'-frame
$\boldsymbol{a}_Q$	=	rectilinear acceleration of the origin of S'-frame relative to the origin of S-frame
$\boldsymbol{a}_i$	=	rectilinear acceleration of the point i relative to the origin of S-frame
$\boldsymbol{a}'_i$	=	rectilinear acceleration of the point i relative to the origin of S'-frame
$\boldsymbol{\tau}_{Euler}$	=	inertial torque occurs due to Euler force
$\boldsymbol{\tau}_{\text{Coriolis}}$	=	inertial torque occurs due to Coriolis force

1 Introduction

One can define the problem by the following statement:"division of the total angular momentum into its orbital and spin parts is especially useful because it is often true (at least to a good approximation) that the two parts are *separately* conserved."[*see* 1, p. 370]. The statement briefs the common understanding within scientific community about spin-rotation relation for a rigid body in rotational motion. Nevertheless, we are going to prove that the negation of this statement is what is true.

We begin with the distinction between rotational (also circular or orbital) and spin motion of a rigid body. Hence, we define *rotational* motion as the angular motion of center of mass of a rigid body relative to a fixed point whereas the distance between the center of mass of the rigid body and the axis of rotation remains fixed. The *spin* motion is the angular motion of a rigid body relative to its center of mass. Another thing is that; the analysis is going to be on 3-dimensional Euclidean space and with a planar rigid body undergoes planar motion.

The following three subsections (2.1), (2.2) and (2.3) can be considered as the observation of coupling phenomena and which has been obtained from mathematical analysis of rotational motion of a rigid body. The last subsection (Subsection (2.4)) gives theoretical explanation to this phenomena.

2 Analysis

2.1 Coupling of spin and rotational angular momentums

Referring to Figure1, the rigid body "*A*" of mass m is free to rotate relative to its center of mass CM as it is also simultaneously free to rotate relative to the fixed point O (inertial frame). Thus, it is pivoted at these two points. It is known that the total angular momentum $\mathbf{L}$ of such rigid body in rotational motion is given by[*see* 1, p. 369]:

$$\mathbf{L} = \mathbf{r}_{CM} \times \mathbf{p} \; + \; \sum_i \boldsymbol{\rho}'_i \times \dot{\boldsymbol{\rho}}'_i m_i \tag{1}$$

The first term is the angular momentum (relative to O) of the motion of the center of mass. The second is the angular momentum of the motion relative to the center of mass. Thus, we can re-express Equation (1) to say[*see* 1, p. 369]

$$\mathbf{L} = \mathbf{L}_{\text{motion of CM}} + \mathbf{L}_{\text{motion relative to CM}} \tag{2}$$

Since the mass is constrained to a circle then the tangential velocity of the mass of the rigid body is $\boldsymbol{\omega} \times \mathbf{r}_{CM}$ and its linear momentum is $\mathbf{p} = m(\boldsymbol{\omega} \times \mathbf{r}_{CM})$. Therefore, the total angular momentum equation (Equation (1)) becomes (assuming the motion is planar, thus both axises of rotation O and CM are parallel):

$$\mathbf{L} = \mathbf{r}_{CM} \times m(\boldsymbol{\omega} \times \mathbf{r}_{CM}) + \sum_i \boldsymbol{\rho}'_i \times m_i(\boldsymbol{\Omega} \times \boldsymbol{\rho}'_i) \tag{3}$$

Taking the first term in the RHS and using the position vector equation

$$\mathbf{r}_i = \mathbf{r}_{CM} + \boldsymbol{\rho}'_i\ , \tag{4}$$

one finds (see appendix A, I)

$$\mathbf{L} = \mathbf{L}_R + \mathbf{L}_S \tag{5}$$

Taking the dot product of Equation (5) with itself, we get

$$\mathbf{L}^2 = \mathbf{L}_R^2 + \mathbf{L}_S^2 + 2\,\mathbf{L}_R \cdot \mathbf{L}_S\ , \tag{6}$$

(since $\mathbf{L}_R$ and $\mathbf{L}_S$ commute), and therefore

$$\mathbf{L}_R \cdot \mathbf{L}_S = \tfrac{1}{2}\left(\mathbf{L}^2 - \mathbf{L}_R^2 - \mathbf{L}_S^2\right)\ . \tag{7}$$

Since the total angular momentum (Equation (5)) is conserved, it is implies that the rotational and coupled spin angular momentum are mutually exchange and that in order to conserve the total angular momentum, that is

$$\mathbf{L} = \downarrow\uparrow \mathbf{L}_R + \uparrow\downarrow \mathbf{L}_S \tag{8}$$

Equation (7) and (8) negate the statement of uncoupling of rotational and spin angular momentums with which we had began the argument since $\mathbf{L}_R \cdot \mathbf{L}_S \neq 0$.

Another thing we can notice is that if we fully do the dot product of Equation (6) and then rearrange it, we obtain the *parallel axis theorem.* (see appendix A, II)

2.2 Coupling of spin and rotational kinetic energies

It is known that the kinetic energy T of the rigid body "A" in its rotational motion relative to the axis of rotation O is given by[*see* 2, p. 206]:

$$T = \tfrac{1}{2}m\,(\omega \times \mathbf{r}_{CM} \cdot \omega \times \mathbf{r}_{CM}) + \tfrac{1}{2}I_{CM}\,(\mathbf{\Omega} \cdot \mathbf{\Omega}) \tag{9}$$

where $\omega \times \mathbf{r}_{CM}$ is the tangential velocity of the center of mass of the rigid body relative to the axis of rotation O and $\mathbf{\Omega}$ is the spin velocity relative to the center of mass of the rigid body. Taking the first term in the RHS of Equation (9), one finds (see appendix A, III)

$$T = T_R + T_S \tag{10}$$

If we fully do the dot products of Equation (10) and then rearrange it, we again will obtain the parallel axis theorem. (see appendix A, IV)

2.3 Coupling of the acting forces

At this section we will explore the coupling between the force causes the rotation of the rigid body "A" and the force causes its spin. Referring to Figure1, if an external force $\mathbf{F}$ acts on the center of mass of the rigid body, and since the mass m is constrained to a circle, then the tangential acceleration of the rigid body is $\dot{\omega} \times \mathbf{r}_{CM}$, and since $\mathbf{F} = m\boldsymbol{a}$, the total torque $\boldsymbol{\tau}$ is given by:

$$\boldsymbol{\tau} = \mathbf{r}_{CM} \times \mathbf{F} = \mathbf{r}_{CM} \times m(\dot{\omega} \times \mathbf{r}_{CM}) \tag{11}$$

Substituting Equation (4) into (11), one finds (see appendix A, V)

$$\boldsymbol{\tau} = \boldsymbol{\tau}_R + \boldsymbol{\tau}_S \tag{12}$$

Taking the dot product of Equation (12) with itself, we obtain

$$\boldsymbol{\tau}^2 = \boldsymbol{\tau}_R^2 + \boldsymbol{\tau}_S^2 + 2\,\boldsymbol{\tau}_R \cdot \boldsymbol{\tau}_S \;, \tag{13}$$

(since $\boldsymbol{\tau}_R$ and $\boldsymbol{\tau}_S$ commute), and therefore

$$\boldsymbol{\tau}_R \cdot \boldsymbol{\tau}_S = \tfrac{1}{2}\left(\boldsymbol{\tau}^2 - \boldsymbol{\tau}_R^2 - \boldsymbol{\tau}_S^2\right) \;. \tag{14}$$

If we fully do the dot products of Equation (13) and then rearrange it, we again will obtain the parallel axis theorem. (see appendix A, VI)

2.4 The nature of the forces which cause the spin torque

To find out the nature of the force behinds the spin torque $\boldsymbol{\tau}_S$ we are going to take the kinetic approach to find the same term that assigned to it and which appears in Equation (12), that is, $I_{CM}(-\dot{\boldsymbol{\omega}})$. Referring to Figure2, we have a space-fixed coordinate system S which is a coordinate system with the origin fixed in space at point O, and with space-fixed directions for the axes. We have also a body-fixed coordinate system S' with an arbitrary point Q (reference point) on the rigid body is selected as the coordinate origin. Therefore, the quantities in the reference systems S and S' are related as follows[*see* 3, p. 96-97]:

$$\mathbf{r}_i = \mathbf{r}_Q + \mathbf{r}'_i \tag{15}$$

Taking the first change in position vector (Equation (15)) with respect to time, yields (see appendix B, VII)

Rectilinear velocity[*see* 4, p. 17]:

$$\boldsymbol{v}_Q = \boldsymbol{v}_i - \boldsymbol{v}'_i\ , \quad \text{and} \tag{16}$$

Azimuthal velocity:

$$\boldsymbol{\omega} \times \mathbf{r}_Q = \boldsymbol{\omega} \times \mathbf{r}_i - \boldsymbol{\omega} \times \mathbf{r}'_i\ . \tag{17}$$

If the rigid body "A" is *rotationally* moves, then by letting the coordinate origin (point Q) of the S'-frame to be the center of mass of the rigid body then that yields identically Equation (5), the angular momentums coupling formula which derived earlier in Subsection (2.1). (see appendix B, VIII)

Taking the second change in position vector (Equation (15)) with respect to time, we obtain (see appendix B, IX)

Azimuthal acceleration:

$$\dot{\boldsymbol{\omega}} \times \mathbf{r}_Q = \dot{\boldsymbol{\omega}} \times \mathbf{r}_i + (-\dot{\boldsymbol{\omega}} \times \mathbf{r}'_i) \tag{18}$$

Coriolis acceleration:

$$2\boldsymbol{\omega} \times \boldsymbol{v}_Q = 2\boldsymbol{\omega} \times \boldsymbol{v}_i + (-2\boldsymbol{\omega} \times \boldsymbol{v}'_i) \tag{19}$$

Centripetal acceleration:

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_Q) = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i) + (-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}_i)) \tag{20}$$

Rectilinear acceleration:

$$\boldsymbol{a}_Q = \boldsymbol{a}_i + (-\boldsymbol{a'}_i) \tag{21}$$

The combination of these accelerations gives the total acceleration (accelerations relative to S, S'-frames simultaneously):

$$\begin{aligned} &\boldsymbol{a}_Q + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_Q) + \dot{\boldsymbol{\omega}} \times \mathbf{r}_Q + 2\boldsymbol{\omega} \times \boldsymbol{v}_Q \\ &= \overbrace{\boldsymbol{a}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i) + \dot{\boldsymbol{\omega}} \times \mathbf{r}_i + 2\boldsymbol{\omega} \times \boldsymbol{v}_i}^{\text{Accelerations relative to } S\text{-frame (inertial frame)}} \\ &+ \\ &\overbrace{(-\boldsymbol{a'}_i) + (-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}_i)) + (-\dot{\boldsymbol{\omega}} \times \mathbf{r'}_i) + (-2\boldsymbol{\omega} \times \boldsymbol{v'}_i)}^{\text{Accelerations relative to } S'\text{-frame (active frame)}} \end{aligned} \tag{22}$$

From equations (18) to (21) and by multiplying by mass m, remembering that $m = \sum_i m_i$, we obtain the forces acting over the rigid body, that is

Azimuthal force:

$$m\,(\dot{\boldsymbol{\omega}} \times \mathbf{r}_Q) = \sum_i m_i\,(\dot{\boldsymbol{\omega}} \times \mathbf{r}_i) + \sum_i m_i\,(-\dot{\boldsymbol{\omega}} \times \mathbf{r'}_i) \tag{23}$$

Coriolis force:

$$m\,(2\boldsymbol{\omega} \times \boldsymbol{v}_Q) = \sum_i m_i\,(2\boldsymbol{\omega} \times \boldsymbol{v}_i) + \sum_i m_i\,(-2\boldsymbol{\omega} \times \boldsymbol{v'}_i) \tag{24}$$

Centripetal force:

$$m\,(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_Q)) = \sum_i m_i\,(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i)) + \sum_i m_i\,(-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}_i)) \tag{25}$$

Rectilinear force:

$$m\boldsymbol{a}_Q = \sum_i m_i \boldsymbol{a}_i + \sum_i m_i\,(-\boldsymbol{a'}_i) \tag{26}$$

The combination of these forces gives the total force:

$$\overbrace{m[\boldsymbol{a}_Q + \omega \times (\omega \times \mathbf{r}_Q) + \dot{\omega} \times \mathbf{r}_Q + 2\omega \times \boldsymbol{v}_Q]}^{\text{Total force (forces relative to } S, S'\text{-frames simultaneously)}}$$

$$= \overbrace{\sum_i m_i[\boldsymbol{a}_i + \omega \times (\omega \times \mathbf{r}_i) + \dot{\omega} \times \mathbf{r}_i + 2\omega \times \boldsymbol{v}_i]}^{\text{Active force (forces relative to } S\text{-frame (inertial frame))}}$$

$$+ \overbrace{\sum_i m_i[(-\boldsymbol{a}'_i) + (-\omega \times (\omega \times \mathbf{r}'_i)) + (-\dot{\omega} \times \mathbf{r}'_i) + (-2\omega \times \boldsymbol{v}'_i)]}^{\text{Inertial force (forces relative to } S'\text{-frame (active frame))}} \quad (27)$$

and more generally,

$$\mathbf{F}_{total} = \mathbf{F}_{active} + \mathbf{F}_{inertial} \quad (28)$$

We generally found that every single force of the above forces, (equations (23) to (26)), is a synthesis of an *active* force and an *inertial* force and similarly the total force (Equation (27)).

If the coordinate origin (point Q) of the S'-frame has been chosen to be a center of mass of a rigid body, say the rigid body "A", that is, $\mathbf{r}'_i = \boldsymbol{\rho}'_i$ where $\boldsymbol{\rho}'_i$ is the position vector of the point i relative to the center of mass, and with the help of Equation (37) and its first and second derivative with respect to time ($\sum_i m_i \dot{\boldsymbol{\rho}}'_i = \sum_i m_i \boldsymbol{v}'_i = \mathbf{0}$ and $\sum_i m_i \ddot{\boldsymbol{\rho}}'_i = \sum_i m_i \boldsymbol{a}'_i = \mathbf{0}$ since $\boldsymbol{\rho}'_i$ is constant). Then equations (23) to (27) will give

Azimuthal force:

$$m\,(\dot{\omega} \times \mathbf{r}_{CM}) = \sum_i m_i\,(\dot{\omega} \times \mathbf{r}_i) \;+\; \mathbf{0} \quad (29)$$

Coriolis force:

$$m\,(2\omega \times \boldsymbol{v}_{CM}) = \sum_i m_i\,(2\omega \times \boldsymbol{v}_i) \;+\; \mathbf{0} \quad (30)$$

Centripetal force:

$$m\,(\omega \times (\omega \times \mathbf{r}_{CM})) = \sum_i m_i\,(\omega \times (\omega \times \mathbf{r}_i)) \;+\; \mathbf{0} \tag{31}$$

Rectilinear force:

$$m\boldsymbol{a}_{CM} = \sum_i m_i \boldsymbol{a}_i \;+\; \mathbf{0} \tag{32}$$

The combination of these forces gives the total force:

$$\begin{aligned} & m[\boldsymbol{a}_{CM} + \boldsymbol{\omega} \times (\omega \times \mathbf{r}_{CM}) + \dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM} + 2\omega \times \boldsymbol{v}_{CM}] \\ &\qquad = \sum_i m_i[\boldsymbol{a}_i + \boldsymbol{\omega} \times (\omega \times \mathbf{r}_i) + \dot{\boldsymbol{\omega}} \times \mathbf{r}_i + 2\omega \times \boldsymbol{v}_i] \;+\; \mathbf{0} \end{aligned} \tag{33}$$

The inertial forces have been neutralized due to the balanced distribution of the mass elements relative to the center of mass of the rigid body, or in another word the net inertial forces is equal to zero, nevertheless the inertial accelerations have not vanished.

2.4.1 The contribution of Euler force to the spin torque

If the rigid body "*A*" is *rotationally* accelerates from its center of mass then by cross multiplying Equation (29) by $\mathbf{r}_{CM}$, we will obtain the torque due to the tangential force. Hence, one finds (see appendix C, X)

$$\boldsymbol{\tau} = \boldsymbol{\tau}_R \;+\; \boldsymbol{\tau}_{Euler} \tag{34}$$

This force causes spin of the mass element m_i relative to the center of mass of the rigid body (origin of S'-frame) in a direction counter to the direction of rotation of the rigid body relative to the axis of rotation O (origin of S-frame).

Equation (34) is identically Equation (12) which has been derived earlier in Subsection (2.3). This implies that $\boldsymbol{\tau}_{Euler} = \boldsymbol{\tau}_S$, that is, the counter rotation of the rigid body relative to its center of mass occurs due to Euler inertial force.

2.4.2 The contribution of Coriolis force to the spin torque

If the rigid body "*A*" is *radial* translating while uniformly *rotate* from its center of mass, then by cross multiplying Equation (30) by $\mathbf{r}_{CM}$, we will obtain the torque due to the Coriolis force. Therefore, Equation (34) will update to (see appendix C, XI):

$$\boldsymbol{\tau} = \boldsymbol{\tau}_R + \overbrace{(\boldsymbol{\tau}_{\text{Euler}} + \boldsymbol{\tau}_{\text{Coriolis}})}^{\boldsymbol{\tau}_S} \tag{35}$$

This force also causes spin of the mass element m_i relative to the center of mass of the rigid body in a direction counter to the direction of rotation of the rigid body relative to the axis of rotation O.

We also will find the centripetal and rectilinear forces do *not* contribute to the spin of the rigid body (see appendix C, XII and XIII). Therefore, we can write the equation of motion which describes all forces coupling spin to rotation:

$$\begin{aligned} &\overbrace{\mathbf{r}_{CM} \times m(\dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM}) + \mathbf{r}_{CM} \times m(2\boldsymbol{\omega} \times \boldsymbol{v}_{CM})}^{\boldsymbol{\tau}} \\ &= \overbrace{\sum_i \mathbf{r}_i \times m_i(\dot{\boldsymbol{\omega}} \times \mathbf{r}_i) + \sum_i \mathbf{r}_i \times m_i(2\boldsymbol{\omega} \times \boldsymbol{v}_i)}^{\boldsymbol{\tau}_R} \\ &\quad + \overbrace{\left[\sum_i \boldsymbol{\rho}'_i \times m_i(-\dot{\boldsymbol{\omega}} \times \mathbf{r}_i) + \sum_i \boldsymbol{\rho}'_i \times m_i(-2\boldsymbol{\omega} \times \boldsymbol{v}_i)\right]}^{\boldsymbol{\tau}_S}, \end{aligned} \tag{36}$$

that is.

3 Discussion

The results of the analysis suggest that distant stars and celestial bodies have no significant effect on local inertial frames — see Mach's hypothesis [5–8, Berkeley, Mach, Sciama, and others], whereas inertial forces have been completely derived from Galilean-Newtonian set of concepts [*cf.* 9, Principia] and have been shown that it have a kinematic rather than a dynamical origin [*cf.* 10, p. 1476].

If the rest frame determines the inertial frames, it follow that inertia is not an intrinsic property of matter, but arises as a result of the interaction of matter with the rest frame [*cf.* 8, p. 35] —inertial forces are exerted by absolute space, not by matter and the whole of the inertial field must not be due to sources [*cf.* 10, p. 1476]. Therefore, Newton's laws of motion can be accurate without need of reference to the physical properties of the universe, such as the amount of matter it contains, which does not imply that matter has inertia only in the presence of other matter [*cf.* 8, p. 35].

In opposite to the rectilinear motion where inertia presents as resistance of the mass to motion, the objects when move curvilinearly, their inertias present as spin of their masses. The finding that inertial forces cause the masses of objects to oscillate when set into acceleration by external force (to spin or rotate —*the opposite phenomenon as the law of conservation of momentum implies*— when move curvilinearly or to vibrate when move rectilinearly) is the counterpart of the Mach effect [*cf.* 11] (also referred to as Woodward effect) which "predicted fluctuations in the masses of things that change their internal energies as they are accelerated by external forces"[*see* 12, p. 4], whereas in Galilean-Newtonian version —as the findings says— the fluctuations show themselves as oscillatory mechanical motions of masses (spin, rotation and vibration) and not as a change in the magnitude of the accelerated mass. Another thing is that the fluctuations that occur in Galilean-Newtonian frame are measurable at the macroscopic mechanical systems which in opposite to the Mach effect where measurable effect needs to be driven at a high frequency.

The findings point also toward absorption of internal energy by masses of things that are accelerated by external forces[*see* eq. 10]. This energy obtainable at any point of the space[*cf.* 13, p. 58].

One of the important insights that we have gained here is that when we utilize (differentiate, substitute or both) the *position* transformation equation which couples[1] the position vectors of a point relative to space-fixed and moving frames of

[1] The magnitude of the position transformation squared –equation (4) or (15)– shows coupling term, i.e., $\mathbf{r}_Q^2 = \mathbf{r}_i^2 + \mathbf{r}'^2_i + 2\,\mathbf{r}_i \cdot -\mathbf{r}'_i$. (Compare this equation with equations (6) and (13).) Similar equation can be obtain from velocities, accelerations and forces transformations –equations (16) to (21) and (23) to (26)– and that by

reference[*see* eqs. 4, 15], we obtain the *velocity* and *accelerations* transformation equations[*see* eqs. 16–21] which couple those latter quantities in space-fixed and moving frames, whereas from these latter transformations we obtain transformation equations of forces, torques, angular momentum and rotational kinetic energy[*see* eqs. 23–26, 34, 35, 5, 10]. Whereas, the parallel axis theorem provides the transformation equation of moment of inertia between the space-fixed and moving frames. These transformations are what we perceive as inertial forces, momentums and energies, that is, it explain occurrence of inertial forces and since the space of the fixed and moving frames may coupled[2] therefore any change in position vector relative to any one of these frames will be faced by opposite change in position vector in the other one (appears as resistance to motion).

These transformations could have been noticed earlier if Galilean transformation was parameterized with constant acceleration instead of constant velocity, that is, $\mathbf{r}' = \mathbf{r} - \frac{1}{2}\mathbf{a}t^2$, the instantaneous position vector, and its first differentiation $\dot{\mathbf{r}}' = \dot{\mathbf{r}} - \mathbf{a}t$ is the instantaneous inertial frame transformation equation, and its second differentiation is $\ddot{\mathbf{r}}' = \ddot{\mathbf{r}} - \mathbf{a}$ which can be rearrange to $\mathbf{a} = \ddot{\mathbf{r}} + (-\ddot{\mathbf{r}}')$, where $-\ddot{\mathbf{r}}'$ is an inertial acceleration. Since a transformation for almost every kinematic and dynamic quantity has been obtained, a suitable substitution of it in classical equations (e.g., Newton's equations of motion, etc.) will help understanding the role of the space and inertial forces in physics.

The analysis cover only in detail the orbital motion whereas the case of coinciding of center of mass and center of rotation (pure spin) is not covered in this analysis and it need a special mathematical treatment to derive its inertial forces without causing cancellation of active forces by inertial forces (both are equal and opposite).

Finally, a crude observation of the reported phenomenon can be obtain easily by rotating a metallic solid disk pivoted at its center or by rotating a vessel of water containing ice cubes and it can be exercise with hands.

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dotting each one with itself.

[2]Since two frames can couple to form a third frame, which in its turn can couple to a fourth one to form a fifth, etc., then, the logical consequence is the formation of a master container one-frame, i.e., the statement of infinitely many inertial frames and absolute space becomes equivalent in the presence of coupling.

4 Conclusion

A rigid body that angularly moves in a curvilinear path, will spin under the influence of inertial forces, exclusively, Euler and Coriolis forces.

The inertial force supplies a curvilinearly moving rigid body with an additional rotational kinetic energy and angular momentum which are independent from the rotational kinetic energy and angular momentum that have been supplied by the active force.

The rotational (orbital) angular momentum of a rigid body which undergoes rotational acceleration is mutually exchange with its spin angular momentum and that happens in order to conserve the total angular momentum of the rigid body.

The angular motion of the center of mass of a rigid body relative to a fixed point is equivalent to superposition of angular motions of its mass elements relative to the that fixed point and relative to the center of mass. This is the spin-rotation coupling theorem which has been summarized from the preceding analysis.

The parallel axis theorem coupling of rotational (orbital) dynamics of a circularly accelerated rigid body to its spin dynamics. It maps moment of inertia of a rigid body to a moment of inertia of point mass.

Any mechanical force (rectilinear, azimuthal, centripetal and Coriolis) when acts over a rigid body, it de-synthesis into active and inertial forces. □

Appendices

Appendix A

I. Derivation of the coupling formula of spin and rotational angular momentums

Taking the first term in the RHS of Equation (3) and substitute the position vector equation $\mathbf{r}_i = \mathbf{r}_{CM} + \boldsymbol{\rho'}_i$ (This substitution is the main device which brings us to another level of analysis of these formulae and the results follow directly from it), and since $m = \sum_i m_i$ then we have

$$
\begin{aligned}
\mathbf{r}_{CM} \times m(\boldsymbol{\omega} \times \mathbf{r}_{CM}) &= \sum_i (\mathbf{r}_i - \boldsymbol{\rho'}_i) \times m_i(\boldsymbol{\omega} \times (\mathbf{r}_i - \boldsymbol{\rho'}_i)) \ , \\
&= \sum_i \mathbf{r}_i \times m_i(\boldsymbol{\omega} \times \mathbf{r}_i) - \sum_i \boldsymbol{\rho'}_i \times m_i(\boldsymbol{\omega} \times \mathbf{r}_i) - \sum_i \mathbf{r}_i \times m_i(\boldsymbol{\omega} \times \boldsymbol{\rho'}_i) \\
&\quad + \sum_i \boldsymbol{\rho'}_i \times m_i(\boldsymbol{\omega} \times \boldsymbol{\rho'}_i) \ , \\
&= \sum_i \mathbf{r}_i \times m_i(\boldsymbol{\omega} \times \mathbf{r}_i) - \sum_i \boldsymbol{\rho'}_i \times m_i(\boldsymbol{\omega} \times (\mathbf{r}_{CM} + \boldsymbol{\rho'}_i)) \\
&\quad - \sum_i (\mathbf{r}_{CM} + \boldsymbol{\rho'}_i) \times m_i(\boldsymbol{\omega} \times \boldsymbol{\rho'}_i) + \sum_i \boldsymbol{\rho'}_i \times m_i(\boldsymbol{\omega} \times \boldsymbol{\rho'}_i) \ , \\
&= \sum_i \mathbf{r}_i \times m_i(\boldsymbol{\omega} \times \mathbf{r}_i) - \sum_i m_i \boldsymbol{\rho'}_i \times (\boldsymbol{\omega} \times \mathbf{r}_{CM}) \\
&\quad - \sum_i \boldsymbol{\rho'}_i \times m_i(\boldsymbol{\omega} \times \boldsymbol{\rho'}_i) - \mathbf{r}_{CM} \times \left(\boldsymbol{\omega} \times \sum_i m_i \boldsymbol{\rho'}_i \right) \\
&\quad - \sum_i \boldsymbol{\rho'}_i \times m_i(\boldsymbol{\omega} \times \boldsymbol{\rho'}_i) + \sum_i \boldsymbol{\rho'}_i \times m_i(\boldsymbol{\omega} \times \boldsymbol{\rho'}_i) \ ,
\end{aligned}
$$

since $\boldsymbol{\rho'}_i$ is the vector position of mass element m_i relative to the center of mass therefore from the definition of the center of mass, we have[*see* 4, p. 98]

$$
\sum_i m_i \boldsymbol{\rho'}_i = \mathbf{0} \ , \tag{37}
$$

which implies that

$$\mathbf{r}_{CM} \times m(\boldsymbol{\omega} \times \mathbf{r}_{CM}) = \sum_i \mathbf{r}_i \times m_i(\boldsymbol{\omega} \times \mathbf{r}_i) - \sum_i \boldsymbol{\rho}'_i \times m_i(\boldsymbol{\omega} \times \boldsymbol{\rho}'_i) , \tag{38}$$

using the identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\,\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\,\mathbf{C} , \tag{39}$$

and using the facts that $\boldsymbol{\rho}'_i$ and $\boldsymbol{\omega}$ are mutually orthogonal and so are $\mathbf{r}_i$ and $\boldsymbol{\omega}$. Therefore, one finds

$$\mathbf{r}_{CM} \times m(\boldsymbol{\omega} \times \mathbf{r}_{CM}) = \sum_i m_i r_i^2 \boldsymbol{\omega} - \sum_i m_i \rho_i'^2 \boldsymbol{\omega} = I_o \boldsymbol{\omega} + I_{CM}(-\boldsymbol{\omega}) , \tag{40}$$

where

$$I_o = \sum_i m_i r_i^2 , \tag{41}$$

is the moment of inertia of the rigid body relative to the axis of rotation O, which is a perpendicular distance $\mathbf{r}_{CM}$ from the center of mass, and

$$I_{CM} = \sum_i m_i \rho_i'^2 , \tag{42}$$

is the moment of inertia of the rigid body relative to its center of mass[*see* 14, p. 246].

Therefore, Equation (3), the total angular momentum becomes

$$\mathbf{L} = I_o \boldsymbol{\omega} + I_{CM}(-\boldsymbol{\omega}) + I_{CM}\boldsymbol{\Omega} , \tag{43}$$

The term $I_{CM}(-\boldsymbol{\omega})$ is an additional angular momentum term relative to the center of mass of the rigid body (spin angular momentum) and occurs due to the rigid body rotational motion relative to the axis of rotation O. The term $I_{CM}\boldsymbol{\Omega}$ can be consider as the *initial* spin angular momentum that the rigid body acquired before it start its rotational motion and since $\boldsymbol{\Omega}$ is arbitrary, so that it can be zero and have not to be a mandatory term of Equation (43). Thus, one can writes $\mathbf{L} = I_o \boldsymbol{\omega} + I_{CM}(-\boldsymbol{\omega})$, and by writing $I_o \boldsymbol{\omega} = \mathbf{L}_R$ and $I_{CM}(-\boldsymbol{\omega}) = \mathbf{L}_S$, we obtain Equation (5). ■

II. Derivation of the parallel axis theorem from the coupling formula of spin and rotational angular momentums

Taking the dot product of Equation (5) with itself, we get

$$\mathbf{L}^2 = \mathbf{L}_R^2 + \mathbf{L}_S^2 + 2\,\mathbf{L}_R \cdot\ \mathbf{L}_S\ ,$$

using Identity (39) to simplify the term $\mathbf{L} = \mathbf{r}_{CM} \times (m\boldsymbol{\omega} \times \mathbf{r}_{CM})$, and since the motion is planar then $\mathbf{r}_{CM}$ and $\boldsymbol{\omega}$ are mutually orthogonal, so that we have

$$\begin{aligned}((\mathbf{r}_{CM} \cdot \mathbf{r}_{CM})\, m\boldsymbol{\omega})^2 &= (I_o\boldsymbol{\omega})^2 + (I_{CM}\,(-\boldsymbol{\omega}))^2 + 2\,(I_o\boldsymbol{\omega}) \cdot (I_{CM}\,(-\boldsymbol{\omega}))\ ,\\ m^2 r_{CM}^4\, \omega^2 &= I_o^2\omega^2 + I_{CM}^2\omega^2 - 2I_o I_{CM}\omega^2\ ,\\ \left(m r_{CM}^2\right)^2 \omega^2 &= (I_o - I_{CM})^2\, \omega^2\ ,\end{aligned} \tag{44}$$

dividing into ω^2 and then taking the square root and rearrange, we get $I_o = m r_{CM}^2 + I_{CM}$, which is the parallel axis theorem[*see* 14, p. 249]. ■

III. Derivation of the coupling formula of spin and rotational kinetic energies

Taking the first term in the RHS of Equation (9) and using the identity

$$(\mathbf{A} \times \mathbf{B} \cdot \mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})\,(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})\,(\mathbf{B} \cdot \mathbf{C})\ , \tag{45}$$

and the fact that $\mathbf{r}_{CM}$ and $\boldsymbol{\omega}$ are mutually orthogonal and accompany with suitable substitutions of Equation (4), one obtains

$$
\begin{aligned}
\tfrac{1}{2}m\,(\omega\times\mathbf{r}_{CM}\cdot\omega\times\mathbf{r}_{CM}) &= \tfrac{1}{2}m\,(\omega\cdot\omega)\,(\mathbf{r}_{CM}\cdot\mathbf{r}_{CM}) \;=\; \tfrac{1}{2}m\,(\omega\cdot\omega)\,(\mathbf{r}_i-\boldsymbol{\rho}'_i\cdot\mathbf{r}_i-\boldsymbol{\rho}'_i)\ ,\\
&= \tfrac{1}{2}m\,(\omega\cdot\omega)\,((\mathbf{r}_i\cdot\mathbf{r}_i)-2\,(\mathbf{r}_i\cdot\boldsymbol{\rho}'_i)+(\boldsymbol{\rho}'_i\cdot\boldsymbol{\rho}'_i))\ ,\\
&= \tfrac{1}{2}\sum_i m_i\,(\mathbf{r}_i\cdot\mathbf{r}_i)\,(\omega\cdot\omega)-\sum_i m_i\,(\mathbf{r}_i\cdot\boldsymbol{\rho}'_i)\,(\omega\cdot\omega)\\
&\quad+\tfrac{1}{2}\sum_i m_i\,(\boldsymbol{\rho}'_i\cdot\boldsymbol{\rho}'_i)\,(\omega\cdot\omega)\ ,\\
&= \tfrac{1}{2}\sum_i m_i\,(\mathbf{r}_i\cdot\mathbf{r}_i)\,(\omega\cdot\omega)-\sum_i m_i\,(\mathbf{r}_{CM}+\boldsymbol{\rho}'_i\cdot\boldsymbol{\rho}'_i)\,(\omega\cdot\omega)\\
&\quad+\tfrac{1}{2}\sum_i m_i\,(\boldsymbol{\rho}'_i\cdot\boldsymbol{\rho}'_i)\,(\omega\cdot\omega)\ ,\\
&= \tfrac{1}{2}\sum_i m_i\,(\mathbf{r}_i\cdot\mathbf{r}_i)\,(\omega\cdot\omega)-\left(\mathbf{r}_{CM}\cdot\sum_i m_i\boldsymbol{\rho}'_i\right)(\omega\cdot\omega)\\
&\quad-\sum_i m_i\,(\boldsymbol{\rho}'_i\cdot\boldsymbol{\rho}'_i)\,(\omega\cdot\omega)+\tfrac{1}{2}\sum_i m_i\,(\boldsymbol{\rho}'_i\cdot\boldsymbol{\rho}'_i)\,(\omega\cdot\omega)\ ,
\end{aligned}
\tag{46}
$$

using equations (37), (41) and (42), we get

$$
\tfrac{1}{2}m\,(\omega\times\mathbf{r}_{CM}\cdot\omega\times\mathbf{r}_{CM}) = \tfrac{1}{2}I_o\,(\omega\cdot\omega)+\tfrac{1}{2}I_{CM}\,(-\omega\cdot\omega)\ , \tag{47}
$$

substituting Equation (47) back into Equation (9), the kinetic energy formula becomes

$$
T = \tfrac{1}{2}I_o\,(\omega\cdot\omega)+\tfrac{1}{2}I_{CM}\,(-\omega\cdot\omega)+\tfrac{1}{2}I_{CM}\,(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega})\ , \tag{48}
$$

The term $\tfrac{1}{2}I_{CM}\,(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega})$ can be consider as the *initial* spin kinetic energy which is the rigid body have before it start its rotational motion and since $\boldsymbol{\Omega}$ is arbitrary, so that it can be zero and have not to be a mandatory term of Equation (48). Therefore, we can write $T = \tfrac{1}{2}I_o\,(\omega\cdot\omega)+\tfrac{1}{2}I_{CM}\,(-\omega\cdot\omega)$, and by writing $\tfrac{1}{2}I_o\,(\omega\cdot\omega) = T_R$ and $\tfrac{1}{2}I_{CM}\,(-\omega\cdot\omega) = T_S$, we obtain Equation (10). ■

IV. Derivation of the parallel axis theorem from the coupling formula of spin and rotational kinetic energies

Taking Equation (10) and write it explicitly using Identity (45) to simplify the term T (see Equation (46)), so we have

$$\begin{aligned} T &= T_R + T_S \ , \\ \tfrac{1}{2}m\,(\boldsymbol{\omega}\cdot\boldsymbol{\omega})\,(\mathbf{r}_{CM}\cdot\mathbf{r}_{CM}) &= \tfrac{1}{2}I_o\,(\boldsymbol{\omega}\cdot\boldsymbol{\omega}) + \tfrac{1}{2}I_{CM}\,(-\boldsymbol{\omega}\cdot\boldsymbol{\omega}) \ , \\ mr_{CM}^2\left(\tfrac{1}{2}\omega^2\right) &= I_o\left(\tfrac{1}{2}\omega^2\right) - I_{CM}\left(\tfrac{1}{2}\omega^2\right) \ , \end{aligned} \tag{49}$$

dividing into $\tfrac{1}{2}\omega^2$ and rearrange, we obtain $I_o = mr_{CM}^2 + I_{CM}$, which is the parallel axis theorem. ■

V. Derivation of the coupling formula of spin and rotational acting forces

Substituting Equation (4) into (11), we have

$$\begin{aligned} \mathbf{r}_{CM}\times m(\dot{\boldsymbol{\omega}}\times\mathbf{r}_{CM}) &= (\mathbf{r}_i - \boldsymbol{\rho}'_i)\times m(\dot{\boldsymbol{\omega}}\times(\mathbf{r}_i - \boldsymbol{\rho}'_i)) \ , \\ &= \sum_i \mathbf{r}_i\times m_i(\dot{\boldsymbol{\omega}}\times\mathbf{r}_i) - \sum_i \boldsymbol{\rho}'_i\times m_i(\dot{\boldsymbol{\omega}}\times\mathbf{r}_i) \\ &\quad - \sum_i \mathbf{r}_i\times m_i(\dot{\boldsymbol{\omega}}\times\boldsymbol{\rho}'_i) + \sum_i \boldsymbol{\rho}'_i\times m_i(\dot{\boldsymbol{\omega}}\times\boldsymbol{\rho}'_i) \ , \\ &= \sum_i \mathbf{r}_i\times m_i(\dot{\boldsymbol{\omega}}\times\mathbf{r}_i) - \sum_i \boldsymbol{\rho}'_i\times m_i(\dot{\boldsymbol{\omega}}\times(\mathbf{r}_{CM} + \boldsymbol{\rho}'_i)) \\ &\quad - \sum_i (\mathbf{r}_{CM} + \boldsymbol{\rho}'_i)\times m_i(\dot{\boldsymbol{\omega}}\times\boldsymbol{\rho}'_i) + \sum_i \boldsymbol{\rho}'_i\times m_i(\dot{\boldsymbol{\omega}}\times\boldsymbol{\rho}'_i) \ , \\ &= \sum_i \mathbf{r}_i\times m_i(\dot{\boldsymbol{\omega}}\times\mathbf{r}_i) - \sum_i m_i\boldsymbol{\rho}'_i\times(\dot{\boldsymbol{\omega}}\times\mathbf{r}_{CM}) \\ &\quad - \sum_i \boldsymbol{\rho}'_i\times m_i(\dot{\boldsymbol{\omega}}\times\boldsymbol{\rho}'_i) - \mathbf{r}_{CM}\times\left(\dot{\boldsymbol{\omega}}\times\sum_i m_i\boldsymbol{\rho}'_i\right) \\ &\quad - \sum_i \boldsymbol{\rho}'_i\times m_i(\dot{\boldsymbol{\omega}}\times\boldsymbol{\rho}'_i) + \sum_i \boldsymbol{\rho}'_i\times m_i(\dot{\boldsymbol{\omega}}\times\boldsymbol{\rho}'_i) \ , \end{aligned}$$

using Equation (37), we obtain

$$\mathbf{r}_{CM}\times m(\dot{\boldsymbol{\omega}}\times\mathbf{r}_{CM}) = \sum_i \mathbf{r}_i\times m_i(\dot{\boldsymbol{\omega}}\times\mathbf{r}_i) + \sum_i \boldsymbol{\rho}'_i\times m_i(-\dot{\boldsymbol{\omega}}\times\boldsymbol{\rho}'_i) \ , \tag{50}$$

using Identity (39) accompany with the facts that $\boldsymbol{\rho}'_i$ and $\dot{\boldsymbol{\omega}}$ are mutually orthogonal and so are $\mathbf{r}_i$ and $\dot{\boldsymbol{\omega}}$ and then simplify using equations (41) and (42). Therefore we can write $\mathbf{r}_{CM} \times m(\dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM}) = I_O\dot{\boldsymbol{\omega}} + I_{CM}(-\dot{\boldsymbol{\omega}})$. Putting $I_O\dot{\boldsymbol{\omega}} = \boldsymbol{\tau}_R$ and $I_{CM}(-\dot{\boldsymbol{\omega}}) = \boldsymbol{\tau}_S$, we obtain Equation (12). ■

VI. Derivation of the parallel axis theorem from coupling formula of spin and rotational acting forces

Taking the dot product of Equation (12) with itself, we get

$$\boldsymbol{\tau}^2 = \boldsymbol{\tau}_R^2 + \boldsymbol{\tau}_S^2 + 2\,\boldsymbol{\tau}_R \cdot \boldsymbol{\tau}_S \ ,$$

using Identity (39) to simplify the term $\boldsymbol{\tau} = \mathbf{r}_{CM} \times m(\dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM})$, and since the motion is planar therefore $\mathbf{r}_{CM}$ and $\dot{\boldsymbol{\omega}}$ are mutually orthogonal, so that we have

$$\begin{aligned}((\mathbf{r}_{CM} \cdot \mathbf{r}_{CM})\, m\dot{\boldsymbol{\omega}})^2 &= (I_O\dot{\boldsymbol{\omega}})^2 + (I_{CM}\,(-\dot{\boldsymbol{\omega}}))^2 + 2\,(I_O\dot{\boldsymbol{\omega}}) \cdot (I_{CM}\,(-\dot{\boldsymbol{\omega}})) \ , \\ m^2 r_{CM}^4\, \dot{\omega}^2 &= I_O^2\dot{\omega}^2 + I_{CM}^2\dot{\omega}^2 - 2I_O I_{CM}\dot{\omega}^2 \ , \\ \left(m r_{CM}^2\right)^2 \dot{\omega}^2 &= (I_o - I_{CM})^2\, \dot{\omega}^2 \ , \end{aligned} \tag{51}$$

dividing into $\dot{\omega}^2$ and then taking the square root and rearrange, we get $I_o = m r_{CM}^2 + I_{CM}$, which is the parallel axis theorem. ■

Appendix B

VII. Calculation of the first change in position vector $\mathbf{r}_i = \mathbf{r}_Q + \mathbf{r}'_i$

Taking the first change in position vector (Equation (15)) with respect to time, yields[*see* 3, p. 97]

$$\begin{aligned}\dot{\mathbf{r}}_i &= \dot{\mathbf{r}}_Q + \dot{\mathbf{r}}'_i \ , \\ \boldsymbol{v}_i + \boldsymbol{\omega}_i \times \mathbf{r}_i &= (\boldsymbol{v}_Q + \boldsymbol{\omega} \times \mathbf{r}_Q) + (\boldsymbol{v}'_i + \boldsymbol{\omega}'_i \times \mathbf{r}'_i) \ ,\end{aligned}$$

The angular velocities $\boldsymbol{\omega}, \boldsymbol{\omega}_i$ and $\boldsymbol{\omega}'_i$ are due to the rotation of the vectors position $\mathbf{r}_Q, \mathbf{r}_i$ and $\mathbf{r}'_i$ respectively, and since the motion is happening to a rigid body therefore we have $\omega_i = \omega'_i = \omega$. Hence, we can write

$$\boldsymbol{v}_i + \boldsymbol{\omega} \times \mathbf{r}_i = (\boldsymbol{v}_Q + \boldsymbol{\omega} \times \mathbf{r}_Q) + (\boldsymbol{v}'_i + \boldsymbol{\omega} \times \mathbf{r}'_i) \,, \tag{52}$$

and with the help of Equation (15), Equation (52) gives

$$\boldsymbol{v}_Q - \boldsymbol{v}_i + \boldsymbol{v}'_i = -\boldsymbol{\omega} \times (\mathbf{r}_Q - \mathbf{r}_i + \mathbf{r}'_i) = \mathbf{0} \,, \tag{53}$$

therefore we have $\boldsymbol{v}_Q = \boldsymbol{v}_i - \boldsymbol{v}'_i$, which is Equation (16) and $\boldsymbol{\omega} \times \mathbf{r}_Q = \boldsymbol{\omega} \times \mathbf{r}_i - \boldsymbol{\omega} \times \mathbf{r}'_i$, which is Equation (17). ■

VIII. Retrieving of the coupling formula of spin and rotational angular momentums

In Equation (17), let $\mathbf{r}_Q = \mathbf{r}_{CM}$ and $\mathbf{r}'_i = \boldsymbol{\rho}'_i$, then by cross multiplying by $\mathbf{r}_{CM}$ and multiply by m using the fact that $m = \sum_i m_i$, one obtains

$$\mathbf{r}_{CM} \times m(\boldsymbol{\omega} \times \mathbf{r}_{CM}) = \mathbf{r}_{CM} \times \left(\boldsymbol{\omega} \times \sum_i m_i \mathbf{r}_i\right) - \mathbf{r}_{CM} \times \left(\boldsymbol{\omega} \times \sum_i m_i \boldsymbol{\rho}'_i\right) \,,$$

using equations (37) and (15), gives

$$\begin{aligned}
\mathbf{r}_{CM} \times m(\boldsymbol{\omega} \times \mathbf{r}_{CM}) &= \sum_i (\mathbf{r}_i - \boldsymbol{\rho}'_i) \times m_i(\boldsymbol{\omega} \times \mathbf{r}_i) - \mathbf{r}_{CM} \times (\mathbf{0}) \,, \\
&= \sum_i \mathbf{r}_i \times m_i(\boldsymbol{\omega} \times \mathbf{r}_i) - \sum_i \boldsymbol{\rho}'_i \times m_i(\boldsymbol{\omega} \times (\mathbf{r}_{CM} + \boldsymbol{\rho}'_i)) \,, \\
&= \sum_i \mathbf{r}_i \times m_i(\boldsymbol{\omega} \times \mathbf{r}_i) - \sum_i m_i \boldsymbol{\rho}'_i \times (\boldsymbol{\omega} \times \mathbf{r}_{CM}) \\
&\quad - \sum_i \boldsymbol{\rho}'_i \times m_i(\boldsymbol{\omega} \times \boldsymbol{\rho}'_i) \,,
\end{aligned}$$

which reduces by Equation (37) to Equation (38) and then reduces by Identity (39) to $\mathbf{r}_{CM} \times m(\boldsymbol{\omega} \times \mathbf{r}_{CM}) = I_O\boldsymbol{\omega} + I_{CM}(-\boldsymbol{\omega})$, which is Equation (40) or (5). ■

IX. Calculation of the second change in position vector $\mathbf{r}_i = \mathbf{r}_Q + \mathbf{r}'_i$

Taking the second change in position vector (Equation (15)) with respect to time, yields

$$\begin{aligned}\ddot{\mathbf{r}}_i &= \ddot{\mathbf{r}}_Q + \ddot{\mathbf{r}}'_i \,,\\ \dot{\boldsymbol{v}}_i + \dot{\omega}\times\mathbf{r}_i + \omega\times\dot{\mathbf{r}}_i &= (\dot{\boldsymbol{v}}_Q + \dot{\omega}\times\mathbf{r}_Q + \omega\times\dot{\mathbf{r}}_Q)\\ &\quad + (\dot{\boldsymbol{v}}'_i + \dot{\omega}\times\mathbf{r}'_i + \omega\times\dot{\mathbf{r}}'_i)\,,\end{aligned}$$

substituting $\dot{\mathbf{r}}_i$, $\dot{\mathbf{r}}_Q$ and $\dot{\mathbf{r}}'_i$ from Equation (52), gives[*see* 3; 15, p. 97 ;p. 250]

$$\begin{aligned}&\dot{\boldsymbol{v}}_i + \dot{\omega}\times\mathbf{r}_i + \omega\times\boldsymbol{v}_i + \omega\times(\omega\times\mathbf{r}_i)\\ &\qquad = (\dot{\boldsymbol{v}}_Q + \dot{\omega}\times\mathbf{r}_Q + \omega\times\boldsymbol{v}_Q + \omega\times(\omega\times\mathbf{r}_Q))\\ &\qquad\quad + (\dot{\boldsymbol{v}}'_i + \dot{\omega}\times\mathbf{r}'_i + \omega\times\boldsymbol{v}'_i + \omega\times(\omega\times\mathbf{r}'_i))\,,\end{aligned}$$

substituting the values of $\dot{\boldsymbol{v}}_i$, $\dot{\boldsymbol{v}}_Q$, $\dot{\boldsymbol{v}}'_i$, gives

$$\begin{aligned}&\boldsymbol{a}_i + \dot{\omega}\times\mathbf{r}_i + 2\omega\times\boldsymbol{v}_i + \omega\times(\omega\times\mathbf{r}_i)\\ &\qquad = (\boldsymbol{a}_Q + \dot{\omega}\times\mathbf{r}_Q + 2\omega\times\boldsymbol{v}_Q + \omega\times(\omega\times\mathbf{r}_Q))\\ &\qquad\quad + (\boldsymbol{a}'_i + \dot{\omega}\times\mathbf{r}'_i + 2\omega\times\boldsymbol{v}'_i + \omega\times(\omega\times\mathbf{r}'_i))\,,\end{aligned} \tag{54}$$

Regrouping by the types of accelerations, we obtain

$$\begin{aligned}\mathbf{0} = {}&[\boldsymbol{a}_Q - (\boldsymbol{a}_i - \boldsymbol{a}'_i)] + [\dot{\omega}\times\mathbf{r}_Q - (\dot{\omega}\times\mathbf{r}_i - \dot{\omega}\times\mathbf{r}'_i)]\\ &+ [2\omega\times\boldsymbol{v}_Q - (2\omega\times\boldsymbol{v}_i - 2\omega\times\boldsymbol{v}'_i)]\\ &+ [\omega\times(\omega\times\mathbf{r}_Q) - \{\omega\times(\omega\times\mathbf{r}_i) - \omega\times(\omega\times\mathbf{r}'_i)\}]\,,\end{aligned} \tag{55}$$

with the aid of Equation (15) we find that the grouped terms in the second and fourth square brackets in Equation (55) are equal to zeros, that is

$$\dot{\omega}\times(\mathbf{r}_Q - \mathbf{r}_i + \mathbf{r}'_i) = \mathbf{0}\,, \text{and} \tag{56}$$

$$\omega\times(\omega\times(\mathbf{r}_Q - \mathbf{r}_i + \mathbf{r}'_i)) = \mathbf{0}\,, \tag{57}$$

and Equation (16) implies that the grouped terms in the third square brackets in Equation (55) are equal to zero, that is

$$2\omega\times(\boldsymbol{v}_Q - \boldsymbol{v}_i + \boldsymbol{v}'_i) = \mathbf{0}\,, \tag{58}$$

and thus equations (56), (57) and (58) imply that the grouped terms in the first square brackets in Equation (55) are also equal to zero, that is

$$\boldsymbol{a}_Q - \boldsymbol{a}_i + \boldsymbol{a}'_i = \mathbf{0}\ . \tag{59}$$

Therefore, equations (56) to (59) can be rewritten as following:

$$\dot{\omega} \times \mathbf{r}_Q = \dot{\omega} \times \mathbf{r}_i + (-\dot{\omega} \times \mathbf{r}'_i) \tag{60}$$

$$2\omega \times \boldsymbol{v}_Q = 2\omega \times \boldsymbol{v}_i + (-2\omega \times \boldsymbol{v}'_i) \tag{61}$$

$$\omega \times (\omega \times \mathbf{r}_Q) = \omega \times (\omega \times \mathbf{r}_i) + (-\omega \times (\omega \times \mathbf{r}'_i)) \tag{62}$$

$$\boldsymbol{a}_Q = \boldsymbol{a}_i + (-\boldsymbol{a}'_i) \tag{63}$$

The sum of equations (60) to (63) is equals to Equation (54):

$$\begin{aligned}
&\boldsymbol{a}_Q + \omega \times (\omega \times \mathbf{r}_Q) + \dot{\omega} \times \mathbf{r}_Q + 2\omega \times \boldsymbol{v}_Q \\
&\quad = \boldsymbol{a}_i + \omega \times (\omega \times \mathbf{r}_i) + \dot{\omega} \times \mathbf{r}_i + 2\omega \times \boldsymbol{v}_i \\
&\quad + \\
&\quad (-\boldsymbol{a}'_i) + (-\omega \times (\omega \times \mathbf{r}'_i)) + (-\dot{\omega} \times \mathbf{r}'_i) + (-2\omega \times \boldsymbol{v}'_i)\ ,
\end{aligned}$$

which is Equation (22). ■

Appendix C

X. Calculation of the contribution of Euler force to the spin torque

Cross multiply Equation (29) by $\mathbf{r}_{CM}$, and with the aid of Equation (15) after letting $\mathbf{r}_Q = \mathbf{r}_{CM}$ and $\mathbf{r}'_i = \boldsymbol{\rho}'_i$ then one finds

$$\begin{aligned}
\mathbf{r}_{CM} \times m(\dot{\omega} \times \mathbf{r}_{CM}) &= \sum_i (\mathbf{r}_i - \boldsymbol{\rho}'_i) \times m_i(\dot{\omega} \times \mathbf{r}_i) + \mathbf{r}_{CM} \times (\mathbf{0})\ , \\
&= \sum_i \mathbf{r}_i \times m_i(\dot{\omega} \times \mathbf{r}_i) + \sum_i \boldsymbol{\rho}'_i \times m_i(-\dot{\omega} \times \mathbf{r}_i)\ , \\
&= \sum_i \mathbf{r}_i \times m_i(\dot{\omega} \times \mathbf{r}_i) + \sum_i \boldsymbol{\rho}'_i \times \mathbf{F}_{\text{Euler},i}\ , \\
&= \sum_i \mathbf{r}_i \times m_i(\dot{\omega} \times \mathbf{r}_i) + \boldsymbol{\tau}_{\text{Euler}}\ .
\end{aligned} \tag{64}$$

where $\boldsymbol{\tau}_{\text{Euler}}$ is an inertial torque occurs due to Euler force[*see* 15; 16, p. 251; p. 469] $\mathbf{F}_{\text{Euler},i} = m_i(-\dot{\boldsymbol{\omega}} \times \mathbf{r}_i)$ and which acts on the mass element m_i. Then, with the help of equations (15) and (37), we obtain (see steps to Equation (50))

$$\boldsymbol{\tau}_{\text{Euler}} = \sum_i \boldsymbol{\rho}'_i \times m_i(-\dot{\boldsymbol{\omega}} \times \mathbf{r}_i) = \overbrace{\sum_i \boldsymbol{\rho}'_i \times m_i(-\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}'_i)}^{\boldsymbol{\tau}_S} , \tag{65}$$

which leads to Equation (34). ■

XI. Calculation of the contribution of Coriolis force to the spin torque

Cross multiplying Equation (30) by $\mathbf{r}_{CM}$, and with the help of equations (15), (37) and (16) we will obtain the torque due to Coriolis force, that is

$$\begin{aligned}
\mathbf{r}_{CM} \times m(2\boldsymbol{\omega} \times \boldsymbol{v}_{CM}) &= \sum_i (\mathbf{r}_i - \boldsymbol{\rho}'_i) \times m_i(2\boldsymbol{\omega} \times \boldsymbol{v}_i) + \mathbf{r}_{CM} \times (\mathbf{0}) , \\
&= \sum_i \mathbf{r}_i \times m_i(2\boldsymbol{\omega} \times \boldsymbol{v}_i) + \sum_i \boldsymbol{\rho}'_i \times m_i(-2\boldsymbol{\omega} \times \boldsymbol{v}_i) , \\
&= \sum_i \mathbf{r}_i \times m_i(2\boldsymbol{\omega} \times \boldsymbol{v}_i) + \sum_i \boldsymbol{\rho}'_i \times \mathbf{F}_{\text{Coriolis},i} , \\
&= \sum_i \mathbf{r}_i \times m_i(2\boldsymbol{\omega} \times \boldsymbol{v}_i) + \boldsymbol{\tau}_{\text{Coriolis}} .
\end{aligned} \tag{66}$$

where $\boldsymbol{\tau}_{\text{Coriolis}}$ is an inertial torque occurs due to Coriolis force[*see* 15; 17, p. 251; p. 233] $\mathbf{F}_{\text{Coriolis},i} = m_i(-2\boldsymbol{\omega} \times \boldsymbol{v}_i)$ and which acts on the mass element m_i. Then, with the help of equations (16) and (37), one finds

$$\boldsymbol{\tau}_{\text{Coriolis}} = \sum_i \boldsymbol{\rho}'_i \times m_i(-2\boldsymbol{\omega} \times \boldsymbol{v}_i) = \sum_i \boldsymbol{\rho}'_i \times m_i(-2\boldsymbol{\omega} \times \boldsymbol{v}'_i) , \tag{67}$$

which leads to Equation (35). ■

XII. The null contribution of centripetal force to the spin torque

Cross multiplying Equation (31) by $\mathbf{r}_{CM}$ and accompany with the help of equations (15) and (37), we will obtain the torque due to centripetal force, that is

$$\begin{aligned}
\mathbf{r}_{CM} \times m(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{CM})) &= \sum_i (\mathbf{r}_i - \boldsymbol{\rho}'_i) \times m_i(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i)) + \mathbf{r}_{CM} \times (\mathbf{0}) \,, \\
&= \sum_i \mathbf{r}_i \times m_i(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i)) + \sum_i \boldsymbol{\rho}'_i \times m_i(-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i)) \,, \\
&= \sum_i \mathbf{r}_i \times m_i(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i)) + \sum_i \boldsymbol{\rho}'_i \times \mathbf{F}_{\text{Centrifugal},i} \,, \\
&= \sum_i \mathbf{r}_i \times m_i(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i)) + \boldsymbol{\tau}_{\text{Centrifugal}} \,. \qquad (68)
\end{aligned}$$

where $\boldsymbol{\tau}_{\text{Centrifugal}}$ is an inertial torque occurs due to centrifugal force[*see* 15, p. 251] $\mathbf{F}_{\text{Centrifugal},i} = m_i(-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i))$ and which acts on the mass element m_i. Then, with the help of equations (15) and (37), one obtains

$$\boldsymbol{\tau}_{\text{Centrifugal}} = \sum_i \boldsymbol{\rho}'_i \times m_i(-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i)) = \sum_i \boldsymbol{\rho}'_i \times m_i(-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}'_i)) \qquad (69)$$

Using Identity (39) and the fact that the motion is planar, Equation (68) gives

$$\mathbf{r}_{CM} \times m(-\omega^2 \mathbf{r}_{CM}) = \sum_i \mathbf{r}_i \times m_i(-\omega^2 \mathbf{r}_i) - \sum_i \boldsymbol{\rho}'_i \times m_i(-\omega^2 \boldsymbol{\rho}'_i) \,, \qquad (70)$$

since the cross product of a vector with itself is zero then all terms of Equation (70) are zeros, that is, the centripetal force does *not* contribute to the counter spin of the rigid body. ■

XIII. The null contribution of rectilinear force to the spin torque

Cross multiplying Equation (32) by $\mathbf{r}_{CM}$, we will obtain the torque due to the rectilinear force, that is

$$\begin{aligned}
\mathbf{r}_{CM} \times m\boldsymbol{a}_{CM} &= \sum_i (\mathbf{r}_i - \boldsymbol{\rho}'_i) \times m_i\boldsymbol{a}_i + \mathbf{r}_{CM} \times (\mathbf{0}) \,, \\
&= \sum_i \mathbf{r}_i \times m_i\boldsymbol{a}_i + \sum_i \boldsymbol{\rho}'_i \times m_i(-\boldsymbol{a}_i) \,, \\
&= \sum_i \mathbf{r}_i \times m_i\boldsymbol{a}_i + \sum_i \boldsymbol{\rho}'_i \times \mathbf{F}_{\text{Rectilinear},i} \,, \\
&= \sum_i \mathbf{r}_i \times m_i\boldsymbol{a}_i + \boldsymbol{\tau}_{\text{Rectilinear}} \,, \qquad (71)
\end{aligned}$$

where $\boldsymbol{\tau}_{\text{Rectilinear}}$ is an inertial torque occurs due to rectilinear force $\mathbf{F}_{\text{Rectilinear},i} = m_i(-\boldsymbol{a}_i)$ and which acts on the mass element m_i. Then, with the help of equations (59) and (37), one finds

$$\boldsymbol{\tau}_{\text{Rectilinear}} = \sum_i \boldsymbol{\rho}'_i \times m_i(-\boldsymbol{a}_i) = \sum_i \boldsymbol{\rho}'_i \times m_i(-\boldsymbol{a}'_i) \,, \qquad (72)$$

therefore Equation (71) becomes

$$\mathbf{r}_{CM} \times m\boldsymbol{a}_{CM} = \sum_i \mathbf{r}_i \times m_i\boldsymbol{a}_i + \sum_i \boldsymbol{\rho}'_i \times m_i(-\boldsymbol{a}'_i) \,, \qquad (73)$$

since the vectors $\mathbf{r}_{CM}$ and $\boldsymbol{a}_{CM}$ are parallel and so are $\mathbf{r}_i$, $\boldsymbol{a}_i$ and $\boldsymbol{\rho}'_i, \boldsymbol{a}'_i$ then all terms of Equation (73) are zeros due to the cross product of parallel vectors. Therefore, the rectilinear force also does *not* contribute to the counter spin of the rigid body.

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We now can summarize all *inertial* forces $\mathbf{F}_i$ which relative to the *space*-fixed coordinate system S and act over mass element m_i :

$$\mathbf{F}_i = m_i[(-\boldsymbol{a}_i) + (-\omega \times (\omega \times \mathbf{r}_i)) + (-\dot{\omega} \times \mathbf{r}_i) + (-2\omega \times \boldsymbol{v}_i)] \,, \qquad (74)$$

and all *inertial* forces $\mathbf{F}'_i$ which relative to the body-fixed coordinate system S' with origin at center of mass:

$$\begin{aligned}
\mathbf{F}'_i &= m_i[(-\boldsymbol{a}'_i) + (-\omega \times (\omega \times \boldsymbol{\rho}'_i)) + (-\dot{\omega} \times \boldsymbol{\rho}'_i) + (-2\omega \times \boldsymbol{v}'_i)] \,, \\
&= m_i[\ \mathbf{0}\ + (-\omega \times (\omega \times \boldsymbol{\rho}'_i)) + (-\dot{\omega} \times \boldsymbol{\rho}'_i) + \ \mathbf{0}\ ] \,, \qquad (75)
\end{aligned}$$

where $\boldsymbol{v}'_i = \dot{\boldsymbol{\rho}}'_i = \mathbf{0}$ and $\boldsymbol{a}'_i = \ddot{\boldsymbol{\rho}}'_i = \mathbf{0}$ since $\boldsymbol{\rho}'_i$ is constant. Hence and despite the fact that both rectilinear and Coriolis inertial forces which are relative to the body-fixed coordinate system S' are predicted by an observer on S'-frame to be equal to zero, this observer still observes action of similar forces over the rigid body on his or her frame, specially the Coriolis effect where the rigid body spins while the S'-frame transpose-rotates relative to an origin fixed in space (see appendix C, XI). Therefore, that gives a clue that inertial forces which are relative to the body-fixed coordinate system and that which are relative to the space-fixed coordinate system are not one in the same and in spite of equations (65),(67),(69) and (72) are hinted to the opposite of that meaning[3]. A similar pattern can be recognized in Equation (38) where the negative linear momentum in the second term of RHS can be attributed to motion that relative to the space-fixed coordinate system, that is, $\sum_i \boldsymbol{\rho}'_i \times m_i(-\boldsymbol{\omega} \times \boldsymbol{\rho}'_i) = \sum_i \boldsymbol{\rho}'_i \times m_i(-\boldsymbol{\omega} \times \mathbf{r}_i)$; the additional acquired momentum is due to the motion relative to the space.

[3]Here, the cause (*space*) of the effect (*inertial forces*) have been isolated. The effect occurs due to the interaction of the matter and space, or *superposition* of space-fixed and body-fixed frames of reference.

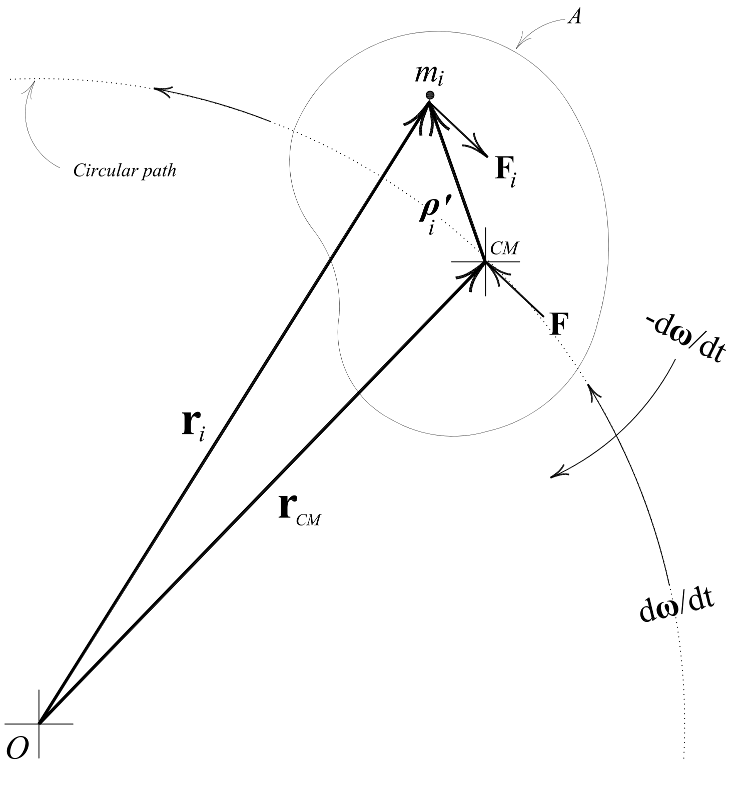

Figure 1: Rotation of the rigid body "*A*" relative to the fixed point *O*.

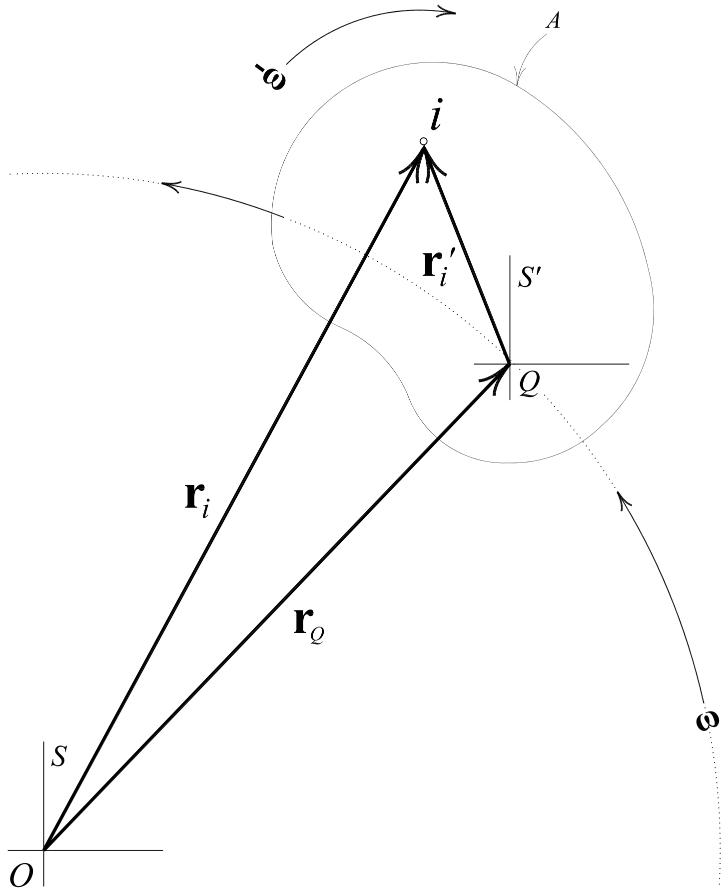

Figure 2: Body-fixed S' and space-fixed coordinate systems S.

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