The Zeta Induction Theorem: The Simplest Equivalent to the Riemann Hypothesis?

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Abstract

This paper presents an uncommon variation of proof by induction. We call it deferred induction by recursion. To set up our proof, we state (but do not prove) the Zeta Induction Theorem. We then assume that theorem is true and provide an elementary proof of the Riemann Hypothesis (showing their equivalence).

1 Introduction

We define (but do not prove) the Zeta Induction Theorem below. Using that theorem, we provide a simple and (hopefully) interesting proof of the Riemann Hypothesis. Our proof uses “deferred induction by recursion”. To be clear, we make no claim as to the usefulness of the Zeta Induction Theorem to the theory of the Riemann Zeta Function.

2 Definitions

In all that follows, the definitions below are assumed:

Definition. For \( m,n \in \mathbb{N} \), define

\[
A_m = \sum_{j=1}^{m} \left( \frac{1}{2} \right)^j \quad \text{and} \quad B_{m,n} = \left( \frac{1}{2} \right)^{m+n}
\]

Definition. For \( t \in \mathbb{R}^>0 \), define \( \epsilon(t) = \frac{1}{8.463 \cdot \log(|t| + 2)} \).

Definition. Fix \( t \in \mathbb{R}^>0 \). For \( s \in \mathbb{C} \), define the following open rectangles:

\[
R_R(t) = \frac{1}{2} < \text{Re}(s) < 1; \quad |\text{Im}(s)| < t
\]

\[
R_L(t) = 0 < \text{Re}(s) < \frac{1}{2}; \quad |\text{Im}(s)| < t
\]

\[
R_e(t) = 1 - \epsilon(t) < \text{Re}(s) < 1; \quad |\text{Im}(s)| < t
\]

\[
R_{m,n}(t) = (A_m + B_{m,n}) < \text{Re}(s) < 1; \quad |\text{Im}(s)| < t
\]

Definition. \( \zeta(s) \) is as defined in Riemann[1].

Definition. We define \( \text{ChooseIndex}(S_k, \uparrow \text{ or } \downarrow, \rightarrow \text{ limit}, k, K, \delta) \) as follows. \( S_k \) is a real-valued sequence that is either monotone increasing (\( \uparrow \)) or monotone decreasing (\( \downarrow \)), with \( \lim_{k \to \infty} S_k = \text{limit} \). For the given \( S_k \), these facts are clear by inspection and are not separately proved. Therefore, for the given \( \delta > 0 \), there is a \( K \in \mathbb{N} \) such that for all \( k > K \) we have: (1) if monotone decreasing, then \( 0 \leq (S_k - \text{limit}) < \delta \), and (2) if monotone increasing, then \( (\text{limit} - \delta) < S_k \). We assume that the given \( K \) is the \( K \) needed for the given \( \delta \), and that the given \( k > K \).
3 The Zeta Induction Theorem

Theorem 1 (Zeta Induction Theorem). Let \( s \in \mathbb{C}; t \in \mathbb{R}_{>0} \). If we assume \( \zeta(s) \neq 0 \) when \( s \in R_{m,n}(t) \), then we have \( \zeta(s) \neq 0 \) when \( s \in R_{m,n+1}(t) \).

Proof. In this paper we assume (but do not prove) this theorem.

4 Lemma

This lemma pulls together various statements, with proofs that are either well-known or straightforward. That way, those statements can be used subsequently without detracting from the flow of the discussion.

Lemma 1. Let \( s \in \mathbb{C}\setminus\{1\} \); fix \( t \in \mathbb{R}_{>0} \). We have the following:

i If \( \zeta(s) \neq 0 \) for all \( s \in R_R(t) \), then \( \zeta(s) \neq 0 \) for all \( s \in R_L(t) \).

ii If \( \zeta(s) \neq 0 \) for \( s \in R_e(t) \).

iii \( (A_m + B_{m,1}) = A_{m+1} \).

iv There exists an \( M \in \mathbb{N} \) such that \( m > M \Rightarrow R_{m,1}(t) \subset R_e(t) \).

v There exists an \( N \in \mathbb{N} \) such that \( n > N \) and \( s \in R_{m,1}(t) \Rightarrow s \in R_{m+1,n}(t) \).

Proof.

i From Riemann[1]: For \( 0 \leq Re(s) \leq 1 \), if \( \zeta(s) = 0 \), then \( \zeta(1-s) = 0 \) (we call them twin zeros). Now assume \( \zeta(s) = 0 \) for some \( s \in R_L(t) \cup R_R(t) \). We consider separately the real and imaginary parts of our twin zeros. Real Parts: \( Re(s) + Re(1-s) = 1 \).

Set \( \delta = \frac{1}{2} - Re(s) \), Then, \( Re(s) = (\frac{1}{2} - \delta) \) and \( Re(1-s) = (\frac{1}{2} + \delta) \). Imaginary Parts: \( |Im(s)| = |Im(1-s)| \). In all cases, we have one of the twin zeros in \( R_L(t) \) and the other in \( R_R(t) \). Thus, with no zeros in \( R_R(t) \) there can be no zeros in \( R_L(t) \).

ii From Ford[2]: \( \zeta(\beta + it) \neq 0 \) for \( |t| \geq 3 \) and \( 1 - \beta \leq \frac{1}{8.463 \cdot log(|t|)} \).

Ford’s statement still holds if we increase the size of the denominator, so \( \epsilon(t) \) was defined by replacing \( log(|t|) \) with \( log(|t|) + 2 \). For all increasing \( |t| \geq 0 \), it is easily verified that \( \epsilon(t) < 0.2 \) and monotone decreasing. As revised by \( \epsilon(t) \), Ford’s statement extends to all \( |t| \geq 0 \) because, from Brent[3], there are no zeros in the \( R_R(3) \) region. If \( s \in R_e(t) \), we have \( \epsilon(t) < \epsilon(Im(s)) \), and therefore \( \zeta(s) \neq 0 \).

iii As defined: \( (A_m + B_{m,1}) = \sum_{j=1}^{m+1} \left( \frac{1}{2} \right)^j + \left( \frac{1}{2} \right) \sum_{j=1}^{m+1} \left( \frac{1}{2} \right)^j = A_{m+1} \).

iv We ChooseIndex\( (A_{m+1}, \uparrow, 1, m + 1, M, \delta = \epsilon(t)) \). Thus \( 1 - \epsilon(t) < A_{m+1} \). Using (iii), we have \( 1 - \epsilon(t) < (A_m + B_{m,1}) \). Hence, \( R_{m,1}(t) \subset R_e(t) \).

v Fix \( s \in R_{m,1}(t) \) and fix \( \epsilon = Re(s) - (A_m + B_{m,1}) \). To set \( B_{m+1,n} < \epsilon \), we now ChooseIndex\( (B_{m+1,n}, \downarrow, 0, n, N, \delta = \epsilon) \). Using (iii), we have: \( (A_{m+1} + B_{m+1,n}) < (A_{m+1} + \epsilon) = ((A_m + B_{m,1}) + \epsilon) = Re(s) \). But \( (A_{m+1} + B_{m+1,n}) < Re(s) \) means \( s \in R_{m+1,n}(t) \).
5 The Riemann Hypothesis

Theorem 2 (Riemann Hypothesis). Let \( s \in \mathbb{C} \setminus \{1\} \), with \( \text{Re}(s) \in [0,1) \setminus \{\frac{1}{2}\} \). Then, \( \zeta(s) \neq 0 \).

Proof. From Hadamard\([4]\): \( \zeta(s) \neq 0 \) for \( \text{Re}(s) \in \{0,1\} \). So, we limit our proof to \( \text{Re}(s) \in (0,1) \setminus \{\frac{1}{2}\} \). Fix \( t \in \mathbb{R}_{>0} \). We first show \( \zeta(s) \neq 0 \) for \( s \in R_R(t) \cup R_L(t) \).

Step 1-A (The First Interval). We start by assuming that \( \zeta(s) \neq 0 \) when \( s \in R_{1,1}(t) \).

Now set \( m = 1 \) and apply the Zeta Induction Theorem. It follows by induction that, for all \( n \in \mathbb{N}, \zeta(s) \neq 0 \) when \( s \in R_{1,n}(t) \).

Step 1-B (The Right Strip). Fix \( s \in R_R(t) \) and fix \( \varepsilon = ( \text{Re}(s) - A_1) > 0 \). Now \( \text{ChooseIndex}(B_{1,n}, \varepsilon \rightarrow 0, n, N, \delta = \varepsilon) \). Then \( A_1 = \frac{1}{2} < (A_1 + B_{1,n}) < (A_1 + \varepsilon) = \text{Re}(s) \).

But \( (A_1 + B_{1,n}) < \text{Re}(s) \) implies \( s \in R_{1,n}(t) \), so by Step 1-A we have \( \zeta(s) \neq 0 \). Hence, \( \zeta(s) \neq 0 \) for all \( s \in R_R(t) \).

Step 1-C (The Left Strip). From Lemma 1(i): \( \zeta(s) \neq 0 \) for \( s \in R_L(t) \).

Step 2 (The Second Interval). One problem remains. We assumed that \( \zeta(s) \neq 0 \) for \( s \in R_{1,1}(t) \). To prove that, we will now assume that \( \zeta(s) \neq 0 \) for \( s \in R_{2,1}(t) \). Now set \( m = 2 \) and apply the Zeta Induction Theorem. It follows by induction that, for all \( n \in \mathbb{N}, \zeta(s) \neq 0 \) when \( s \in R_{2,n}(t) \). We have therefore shown that \( \zeta(s) \neq 0 \) when \( s \in R_{1,1}(t) \) because by Lemma 1(v) there is an \( n \) such that \( s \in R_{1,1}(t) \) implies \( s \in R_{2,n}(t) \).

Step 3 (Recursion). We can continue our recursive augment as many times as we like. To prove that \( \zeta(s) \neq 0 \) when \( s \in R_{m,1}(t) \) we need only assume \( \zeta(s) \neq 0 \) when \( s \in R_{m+1,1}(t) \) and then apply the Zeta Induction Theorem and Lemma 1(v). But our desired result is eventually established by Lemma 1(ii) and (iv), with no further need for recursion, because there exists an \( M \) such that for \( m > M, R_{m,1}(t) \subset R_c(t) \), and we have \( \zeta(s) \neq 0 \) for \( s \in R_c(t) \).

Step 4 (Wrapping Up). We have established the theorem for \( s \in R_R(t) \cup R_L(t) \). But \( t \) was arbitrarily chosen, so the result holds for all \( t \in \mathbb{R}_{>0} \).

6 Discussion

Our “proof” of the Riemann Hypothesis (RH) uses deferred induction by recursion, with each inductive step depending, recursively, on a subsequent inductive step. An alternate (but less interesting) approach is also possible. We can recurse in the opposite direction (without deferred induction). We select \( m \) using Lemma 1(iv) and have \( \zeta(s) \neq 0 \) for \( s \in R_{m,1}(t) \subset R_c(t) \). By the Zeta Induction Theorem (ZI), \( \zeta(s) \neq 0 \) for all \( R_{m,n}(t) \). Then, using Lemma 1(v) we have \( \zeta(s) \neq 0 \) for \( s \in R_{m-1,1}(t) \). Again applying ZI, we recurse until we reach \( R_{1,1}(t) \) and \( R_{1,n}(t) \), thereby covering all of \( R_R(t) \).

ZI is just one short step away from simply assuming RH. So it should come as no surprise that ZI and RH are equivalent. Proof/disproof of one proves/disproves the other. We showed ZI implies RH. Clearly, RH implies ZI because \( \zeta(s) \neq 0 \) for all \( R_{m,n+1}(t) \). Both are disproved only if \( \zeta(s) = 0 \) for some \( \text{Re}(s) \in (0,1) \setminus \{\frac{1}{2}\} \). That said, proof of ZI almost certainly requires direct proof of RH.

References


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