Refutation of the modal logic GL$_2$

Abstract: We evaluate the modal logic GL$_2$ in two axioms and for satisfiability. One of the two axioms is *not* tautologous, and the five formulas for PSpace complete satisfiability are *not* tautologous. Hence GL$_2$ is refuted.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated proof value, F as contradiction, N as truthity (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal. The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬;  + Or, ∨;  - Not Or;  & And, ∧;  \ Not And;
> Imply, greater than, →;  < Not Imply, less than, ∈
= Equivalent, ≡;  @ Not Equivalent, ≠;
% possibility, for one or some, ∃◊;  # necessity, for every or all, ∀□;
~( y < x) ( x ≤ y),  ( x ≤ y).


LET  p, q, r, s:  p, q, r, γ.

We also present a slightly more intuitive and concise axiomatization of PL$_2$ by the following two formulas:

(I)  p → □[¬p → □(p → □p)]  
\[p > #(¬p > #(p > #p))\]  \(\text{TNTN TNTN TNTN TNTN}\)  \(4.2.1\)

(II) □[(r ∧ q) → γ] → [(r ∧ q) → ◊(¬(r ∧ q) ∧ ◊□p ∧ ◊□¬p)]
\[\#((\text{r&q}>s)>((\text{r&q})>\%((\text{r&q})&%p&%¬p)))\]  \(\text{TTTT TTTT TTTT TTCC}\)  \(4.3.1\)

Where γ is the formula ◊□(p ∧ q) ∧ ◊□(¬p ∧ q) ∧ ◊□(p ∧ ¬q).
\(4.4.1\)

s=(%#(p&q)&(%#(~p&q)&%#(p&~q)))
\(\text{TTTT TTTT FFFF FFFF}\)  \(4.4.2\)

Substituting Eq. 4.4.1 into 4.3.1:
\[\#((\text{r&q})>(%\#(p&q)\&%\#(p&~q)))>\]
\[((\text{r&q})>\%((\text{r&q})&\%p&\%¬p))\]  \(\text{TTTT TTTT TTTT TTTT}\)  \(4.5.1\)

While Eq. 4.5.2 is tautologous as an axiom, Eq. 4.2.2 is not tautologous as an axiom. This means the
slightly more intuitive and concise axiomatization of PL\textsubscript{2} is refuted.

\begin{align*}
\text{LET} & \quad p, q, r, s, t: \varphi, \psi, r, m, e. \\
\text{Theorem 5.1.} & \quad \text{The satisfiability problem of our logic is PSpace complete.} \\
\text{Let } C & \text{ be the conjunction of these formulas:} \\
\text{r, m, e are disjoint and one of them holds at each world.} & \quad (5.0.1.1) \\
(r+s)+t; & \quad (5.0.1.2) \\
r \rightarrow \Diamond m & \quad (5.0.2.1) \\
r > % t ; & \quad (5.0.2.2) \\
m \rightarrow \Diamond e. & \quad (5.0.3.1) \\
s > % t ; & \quad (5.0.3.2) \\
C=(r \rightarrow \Diamond m) \& (m \rightarrow \Diamond e). & \quad (5.0.4.1) \\
((r+s)+t) \& ((r > % s) \& (s > % t)) ; & \quad (5.0.4.2) \\
r \land \square C \land \Diamond \varphi \land \square (m \lor e) \land \Diamond (m \lor e) \rightarrow \psi) & \text{is satisfiable in our logic.} & \quad (5.1.1.1) \\
((r & ((r+s)+t) \& ((r > % s) \& (s > % t))) \& (% (s > t) \& p)) \& ((s > t) > q) ; & \quad (5.1.1.2) \\
\text{Eq. 5.1.1.2 as rendered is not tautologous and not contradictory, differing from the state of contradiction by one value } N. \text{ This means PL}_2 \text{ is not satisfiable as PSpace complete.}
\end{align*}