Electroweak physics reconstrued using null cone integrals

Julian Brown

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Abstract

We introduce a zero-dimensional analogue of the Lagrangian density, formed from integrals of scalar product terms over null cones. We demonstrate that application of the variational principle to this zero-dimensional object yields the familiar equations of motion of fundamental fermions and bosons. The (covariant) derivatives of the conventional theory do not make an explicit appearance, being subsumed into Lorenz gauge transformations in which charge appears on the same footing as bosonic field intensity. Whereas the conventional lagrangian is largely agnostic as to the underlying group structure of the particle fields, our treatment finds its most natural expression in $SO(2N)$ GUTs, more especially because the null cone geometry upon which it is based is itself generated by the spinorial dimensions of $SO(4)$.

1 Space-time anti-derivatives

Consider a complex spinor $\Lambda = \begin{pmatrix}\lambda_1 + i\lambda_2, & \lambda_3 + i\lambda_4\end{pmatrix}$ with a $U(1)$ degenerate mapping onto the past null cone with vertex at the origin of x:

$$x_\mu = \Lambda^* \sigma_\mu \Lambda$$

$$d\Lambda = \prod_{i=1}^{4} d\lambda_i = 2\pi\delta(t^2 - r^2)d^4x = 2\pi\frac{d^3\mathbf{r}}{r}$$

It can be shown that, for all $k^2 \neq 0$

$$\int_{-\infty}^{0} e^{-ik_\nu x^\nu} d\Lambda = \frac{1}{k^2} \tag{1}$$

and

$$\int_{-\infty}^{0} x_\mu e^{-ik_\nu x^\nu} d\Lambda = \frac{2ik_\mu}{k^4} \tag{2}$$

The product of two Λ cones maps onto $(x_\mu x^\mu > 0)$ space-time with a $U(1)_L \times SU(2) \times U(1)_R$ degeneracy:

$$x_\mu = a_\mu + b_\mu = (\Lambda_a^*, \Lambda_b^*)[\gamma_\mu \otimes \mathbf{1}](\Lambda_a, \Lambda_b)$$

$$\iint e^{-ik_\nu x^\nu} d\Lambda_a d\Lambda_b = \frac{1}{k^4} \tag{3}$$

$$d\Lambda_a d\Lambda_b = 4\pi^2 d^4 x$$

So, for an arbitrary field f:

$$f = \partial_\nu \partial^\nu \int f d\Lambda = \frac{1}{2}\partial^\mu \partial_\nu \partial^\nu \int f x_\mu d\Lambda = (\partial_\nu \partial^\nu)^2 \iint f d\Lambda_a d\Lambda_b \tag{4}$$

2 Clifford algebra of particles

As pointed out by Wilczek and Zee[1], the fermions of one family can be assigned to a complex spinor representation of $SO(10)$ and the leptons of one family to an $SO(6)$ subgroup. $|\epsilon_1, \epsilon_2, \epsilon_3\rangle$, $\epsilon_i = \pm 1$. The spinor representation of $SO(6)$ contains 8 states: $|\epsilon_1, \epsilon_2, \epsilon_3\rangle$, $\epsilon_i = \pm 1$. By adding a further (4th) rank to the Clifford algebra (i.e. $SO(8)$), we can incorporate ($2\times$) spin as an additional quantum number (not acted upon by the $\dagger$ charge conjugation operator). Representing the $SO(8)$ vacuum by $|0\rangle$ ($\equiv |0,0,0,0\rangle$) with the orthonormality property:

$$\langle 0| \tau_a \times \tau_b \times \tau_c \times \sigma_d |0\rangle = \delta_{0a}\delta_{0b}\delta_{0c}\delta_{0d}$$

..where we have used σ_j rather than $\frac{1}{2}\tau_j$ in the 4th rank so as make a clarifying distinction between charge and spin spaces.
We have fermions:

$$\begin{cases} \hat{\mathbf{e}}_{\mathbf{R}\uparrow} |0\rangle = |+1,+1,-1,+1\rangle \\ \hat{\nu}_{\mathbf{L}\uparrow} |0\rangle = |-1,+1,-1,+1\rangle \\ \hat{\mathbf{e}}_{\mathbf{L}\uparrow} |0\rangle = |+1,-1,-1,+1\rangle \\ \hat{\nu}_{\mathbf{R}\uparrow} |0\rangle = |-1,-1,-1,+1\rangle \end{cases} \tag{5}$$

anti-fermions:

$$\begin{cases} \hat{\mathbf{e}}^\dagger_{\mathbf{R}\uparrow} |0\rangle = |-1,-1,+1,+1\rangle \\ \hat{\nu}^\dagger_{\mathbf{L}\uparrow} |0\rangle = |+1,-1,+1,+1\rangle \\ \hat{\mathbf{e}}^\dagger_{\mathbf{L}\uparrow} |0\rangle = |-1,+1,+1,+1\rangle \\ \hat{\nu}^\dagger_{\mathbf{R}\uparrow} |0\rangle = |+1,+1,+1,+1\rangle \end{cases}$$

and bosons:

$$\begin{cases} \hat{\mathcal{W}}_j^\pm = \tau^\mp \times \tau^\pm \times 1 \times \sigma_j \\ \hat{\mathcal{W}}_j^3 = \hat{\mathcal{W}}_j^+ \hat{\mathcal{W}}_j^- - \hat{\mathcal{W}}_j^- \hat{\mathcal{W}}_j^+ = \frac{1}{2}(1 \times \tau^3 \times 1 \times \sigma_j) - \frac{1}{2}(\tau^3 \times 1 \times 1 \times \sigma_j) \\ \hat{\mathcal{B}}_j = (1 \times 1 \times \tau^3 \times \sigma_j) - \frac{1}{2}(\tau^3 \times 1 \times 1 \times \sigma_j) - \frac{1}{2}(1 \times \tau^3 \times 1 \times \sigma_j) \\ \hat{\Phi}^\pm = 1 \times \tau^\pm \times 1 \times 1 \end{cases} \tag{6}$$

..where $j = \{0,1,2,3\}$
These 18 boson operators transform the (anti-)leptons amongst themselves, for example:

$$\hat{\mathcal{W}}_3^- \hat{\nu}_{\mathbf{L}\downarrow} = -\hat{\mathbf{e}}_{\mathbf{L}\downarrow} \ , \qquad \hat{\Phi}\hat{\mathbf{e}}_{\mathbf{R}\uparrow} = \hat{\mathbf{e}}_{\mathbf{L}\uparrow}$$

.. and more generally create and destroy units of charge and z-spin, for example:

$$\hat{\mathcal{W}}_3^+ |0\rangle = |-2,+2,0,0\rangle \ , \quad \hat{\mathcal{W}}_-^3 |0\rangle = |0,0,0,-2\rangle \ , \quad \hat{\Phi} |0\rangle = |0,-2,0,0\rangle$$

The hypercharge, isospin and electric charge are:

$$\begin{cases} Y =< \hat{\mathcal{B}}_0 >=< 1\times 1\times \tau_3 \times 1 > -\frac{1}{2} < 1\times \tau_3 \times 1\times 1 > -\frac{1}{2} < \tau_3 \times 1\times 1\times 1 >= \epsilon_3 - \frac{1}{2}(\epsilon_1+\epsilon_2) \\ T_3 =< \hat{\mathcal{W}}_0^3 >= \frac{1}{4} < 1\times \tau_3 \times 1\times 1 > -\frac{1}{4} < \tau_3 \times 1\times 1\times 1 >= \frac{1}{4}(\epsilon_2-\epsilon_1) \\ Q \equiv \frac{1}{2}Y + T_3 = \frac{1}{2} < 1\times 1\times \tau_3 \times 1 > -\frac{1}{2} < \tau_3 \times 1\times 1\times 1 >= \frac{1}{2}(\epsilon_3-\epsilon_1) \end{cases}$$

In the following, the fermion spin index ↑,↓ is implicit and we will distinguish the space-time fields of the bosons from their quanta, summed over j, thus:

$$\begin{cases} \mathcal{W}^{\pm}(x) = w_j^{\pm}(x)\cdot \hat{\mathcal{W}}_j^{\pm} \\ \mathcal{W}^{3}(x) = w_j^{3}(x)\cdot \hat{\mathcal{W}}_j^{3} \\ \mathcal{B}(x) = b_j(x)\cdot \hat{\mathcal{B}}_j \\ \Phi^{\pm}(x) = \phi^{\pm}(x)\cdot \hat{\Phi}^{\pm} \end{cases} \tag{7}$$

3 Propagators and interactions

We will now demonstrate explicitly how all empirically verified propagators and interactions of the Standard Model can be derived from a certain zero-dimensional scalar functional ($\mathcal{Q}$) without recourse to any of the derivative operators appearing in the conventional Lagrangian density $\mathcal{L}$. As one would expect, the field amplitudes of the leptons (e,ν) and Higgs field have dimension $-\frac{3}{2}$ and -1 respectively. The boson field amplitudes $\mathcal{W},\mathcal{B}$ however have dimension 0 rather than the -1 of the conventional vector gauge fields (to which they are related in a manner to be shown). Another difference from the Lagrangian approach is that most of the various scalar products are integrated over one or two null cones - so as to form zero dimensional Lorentz scalars - *before* invocation of the principle that $\mathcal{Q}$ be invariant w.r.t. variations in each of its constituent fields.

For reasons of brevity and clarity, this treatment is restricted to one family of leptons, but the extension to $SU(3)_C$, quarks and family mixing is fairly trivial and obvious.

3.1 Leptons

If neutrino mass is neglected[1] there are just 5 possible zero-dimensional Lorentz scalar products involving lepton operators of one family, no gauge bosons and no more than two cone integrations *viz*:

$$\mathcal{Q}_f \equiv \int x^{\alpha}\big[\nu_L^{\dagger}\sigma_{\alpha}\nu_L + e_L^{\dagger}\sigma_{\alpha}e_L + e_R^{\dagger}\tilde{\sigma}_{\alpha}e_R\big]d\Lambda \\ + y_e \iint \big[e_L^{\dagger}\Phi^{+}e_R + e_R^{\dagger}\Phi^{-}e_L\big]d\Lambda_a d\Lambda_b \tag{8}$$

$\mathcal{Q}_f$ is clearly isomorphic to

$$\mathcal{L}_f = \nu_L^{\dagger}\sigma_{\alpha}\partial_{\alpha}\nu_L + e_L^{\dagger}\sigma_{\alpha}\partial_{\alpha}e_L + e_R^{\dagger}\tilde{\sigma}_{\alpha}\partial_{\alpha}e_R + y_e[e_L^{\dagger}\phi e_R + e_R^{\dagger}\phi e_L] \tag{9}$$

[1] Addition of $\mathbf{M_{GUT}}\nu_{\mathbf{R}}^{\dagger}\nu_{\mathbf{R}}$ and $y_{\nu}[\nu_{\mathbf{L}}^{\dagger}\mathbf{\Phi}^{+}\nu_{\mathbf{R}} + \nu_{\mathbf{R}}^{\dagger}\mathbf{\Phi}^{-}\nu_{\mathbf{L}}]$ terms to (8), leads to $m_{\nu_L} \ll 1eV$ via the type-I see-saw mechanism

.. of the conventional formulation. Using (2):

$$\frac{\partial \mathcal{Q}_f}{\partial \nu_L^\dagger} = 0 \quad \Longrightarrow \int x^\alpha \sigma_\alpha \nu_L d\Lambda = 0 \Longrightarrow i\sigma_\alpha \partial^\alpha \nu_L = 0$$

.. which describes a freely propagating massless neutrino. Similarly,

$$\frac{\partial \mathcal{Q}_f}{\partial e_L^\dagger} = 0 \quad \Longrightarrow \int x^\alpha \sigma_\alpha e_L d\Lambda - y_e \iint \Phi_0^+ e_R d\Lambda_a d\Lambda_b$$

$$\Longrightarrow i\sigma_\alpha \partial^\alpha e_L - y_e \Phi_0^+ e_R = 0$$

Similarly,

$$\frac{\partial \mathcal{Q}_f}{\partial e_R^\dagger} = 0 \Longrightarrow i\tilde{\sigma}_\alpha \partial^\alpha e_R - y_e \Phi_0^- e_L = 0$$

.. from which we conclude that (8) describes an electron propagating with mass $m_e = y_e|\Phi_0|$

3.1.1 Coupling of bosons to fermions

We posit that the spin-1 boson fields $(\mathcal{B}, \mathcal{W})$ can be represented collectively by a single local transformation of the $SO(8)$ charge space, e^Θ, that does not change $\mathcal{Q}$. We further posit that the $SU(2)$ and $U(1)$ subgroups have differing low-energy effective coupling constants g and g' respectively.

In the subspace spanned by $\begin{pmatrix} \nu_L \\ e_L \\ \nu_R \\ e_R \end{pmatrix}$, according to (5), (6) and (7) we have

$$\Theta \equiv \begin{pmatrix} \Theta_L & 0 \\ 0 & \Theta_R \end{pmatrix} \equiv \frac{1}{2} \begin{pmatrix} g\mathcal{W}^3 - g'\mathcal{B} & g\mathcal{W}^+ & 0 & 0 \\ g\mathcal{W}^- & -g\mathcal{W}^3 - g'\mathcal{B} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2g'\mathcal{B} \end{pmatrix}$$

and, anticipating a non-zero v.e.v. v for the Higgs field:

$$\Phi_0^+ = \frac{1}{\sqrt{2}}[v + h(x)] \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \qquad \Phi_0 = \frac{1}{\sqrt{2}}[v + h(x)] \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

..where the Yukawa coupling constants (y_f) are particular to each species of fermion and make their appearance in a extra factor like:

$$y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & y_e & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & y_e \end{pmatrix}$$

We define transformed fermion fields:

$$\psi_L \equiv e^{\Theta_L} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} , \qquad \psi_L^\dagger \equiv \begin{pmatrix} \nu_L^\dagger & e_L^\dagger \end{pmatrix} e^{-\Theta_L} \quad \psi_R \equiv e^{\Theta_R} \begin{pmatrix} \nu_R \\ e_R \end{pmatrix} , \qquad \psi_R^\dagger \equiv \begin{pmatrix} \nu_R^\dagger & e_R^\dagger \end{pmatrix} e^{-\Theta_R}$$

and rewrite (8) in terms of these transformed fields (ψ_L, ψ_R), which, together with Θ and Φ_0, are taken as the independent variables for the purposes of the variational principle[2]:

$$\mathcal{Q}'_f \equiv \int x^\alpha \big[\psi_L^\dagger e^{\Theta_L}\sigma_\alpha e^{-\Theta_L}\psi_L + \psi_R^\dagger e^{\Theta_R}\tilde{\sigma}_\alpha e^{-\Theta_R}\psi_R\big] d\Lambda \\ + y_e \iint [\psi_L^\dagger e^{\Theta_L}\Phi^+ e^{-\Theta_R}\psi_R + \psi_R^\dagger e^{\Theta_R}\Phi_0^- e^{-\Theta_L}\psi_L] d\Lambda_a d\Lambda_b \quad (10)$$

Proceeding as before:

$$\frac{\partial \mathcal{Q}'_f}{\partial \psi_R^\dagger} = 0 \quad \Longrightarrow \int x^\alpha e^{\Theta_R}\sigma_\alpha e^{-\Theta_R}\psi_R d\Lambda + y_e \iint e^{\Theta_R}\Phi^- e^{-\Theta_L}\psi_L d\Lambda_a d\Lambda_b = 0$$

$$\Longrightarrow e^{\Theta_R}\big[i\tilde{\sigma}_\alpha \overleftrightarrow{\partial}_\alpha e_R + y_e \Phi_0^- e_L\big] = 0$$

$$\Longrightarrow \tilde{\sigma}_\alpha[i\partial_\alpha - g' B_\alpha] e_R + y_e \Phi_0^- e_L = 0$$

.. where the conventional gauge vector field components B_α are related to the dimensionless amplitudes b_j by:

$$B_0 = i\partial_j b_j \qquad B_j = i\epsilon_{jkl}\partial_k b_l - i\partial_0 b_j \quad (11)$$

$$\Longrightarrow \partial^\mu B_\mu = i\partial^0\partial_j b_j + i\epsilon_{jkl}\partial^j\partial_k b_l - i\partial_0\partial^j b_j \equiv 0 \quad (12)$$

Similarly,

$$\frac{\partial \mathcal{Q}'_f}{\partial \psi_L^\dagger} = 0 \quad \Longrightarrow \int x^\alpha e^{\Theta_L}\sigma_\alpha e^{-\Theta_L}\psi_L d\Lambda + y_e \iint e^{\Theta_L}\Phi_0^+ e^{-\Theta_R}\psi_R d\Lambda_a d\Lambda_b = 0$$

$$\Longrightarrow e^{\Theta_L}\left[\begin{pmatrix} i\sigma_\alpha \overleftrightarrow{\partial}_\alpha & 0 \\ 0 & i\sigma_\alpha \overleftrightarrow{\partial}_\alpha \end{pmatrix}\begin{pmatrix}\nu_L \\ e_L\end{pmatrix} + y_e \Phi_0^+ \begin{pmatrix} 0 \\ e_R \end{pmatrix}\right] = 0$$

$$\Longrightarrow \begin{pmatrix} \sigma_\alpha[i\partial_\alpha + \frac{g}{2}W_\alpha^3 - \frac{g'}{2}B_\alpha] & \frac{g}{2}W_\alpha^+ \\ \frac{g}{2}W_\alpha^- & \sigma_\alpha[i\partial_\alpha - \frac{g}{2}W_\alpha^3 - \frac{g'}{2}B_\alpha] \end{pmatrix}\begin{pmatrix}\nu_L \\ e_L\end{pmatrix} + y_e\Phi_0^+\begin{pmatrix} 0 \\ e_R\end{pmatrix} = 0$$

.. where, as in (11), the conventional gauge vector field components W_α are related to the dimensionless amplitudes w_j by:

$$W_0^{\{+,-,3\}} = i\partial_j w_j^{\{+,-,3\}} \qquad W_j^{\{+,-,3\}} = i\epsilon_{jkl}\partial_k w_l^{\{+,-,3\}} - i\partial_0 w_j^{\{+,-,3\}} \quad (13)$$

$$\Longrightarrow \partial^\mu W_\mu^{\{+,-,3\}} \equiv 0 \quad (14)$$

So the transformation of variables entailed in going from (8) to (10) yields the $SU(2)_L \times U(1)$ Dirac equation in the presence of vector fields B_μ, W_μ. This is clearly analogous to what happens when we impose invariance of the conventional leptonic Lagrangian (9) under local gauge transformations, with the important difference that with $\mathcal{Q}_f$, there is no choice of gauge possible in the resulting equations of motion: the derivative vector fields obey the Lorenz gauge condition (12,14) by construction.

[2]The reader can readily verify that $\mathcal{Q}'_f = \mathcal{Q}_f$

3.2 Higgs

We now consider terms involving just the Higgs field:

$$\mathcal{Q}_H \equiv \int \langle 0| e^{-\Theta}\Phi_0^-\Phi_0^+ e^{\Theta} |0\rangle \, d\Lambda + \iint \langle 0| e^{-\Theta}\left[\mu^2\Phi_0^-\Phi_0^+ + \lambda(\Phi_0^-\Phi_0^+)^2\right] e^{\Theta} |0\rangle \, d\Lambda_a d\Lambda_b$$

$\mathcal{Q}'_f$ also depends upon Φ_0, so the relevant variational constraint is

$$\frac{\partial[\mathcal{Q}_H + \mathcal{Q}'_f]}{\partial \Phi_0^-} = 0 \quad \Longrightarrow \Phi_0^+ = -\int \left[\mu^2\Phi_0^+ + 2\lambda\Phi_0^+\Phi_0^-\Phi_0^+ + y_e e_R^\dagger e_L]\right] d\Lambda \quad (15)$$

The RHS is essentially infinite unless Φ_0^+ makes small excursions around a non-zero vacuum expectation value such that $\Phi_0^+ = \frac{1}{\sqrt{2}}(h(x)+v)\hat{\Phi}_0^+$, $v = \sqrt{-\mu^2/\lambda}$. We then have

$$\partial_\mu \partial^\mu h = 2\mu^2 h - \lambda(3vh^2 + h^3) - y_e e_R^\dagger e_L$$

..which describes a scalar particle with mass $M_H = \sqrt{2\lambda}v$ and couplings to the leptons and itself.

3.2.1 Boson mass eigenstates

The final contributions to $\mathcal{Q}$ that we will consider in this paper are pure gauge boson terms

$$\mathcal{Q}_{\mathcal{B}} = \langle 0| \mathcal{B}^2 + \mathcal{W}^-\mathcal{W}^+ + \mathcal{W}^3\mathcal{W}^3 |0\rangle = \sum_{j=0}^{3}[b_j b_j + w_j^+ w_j^- + w_j^3 w_j^3]$$

$$\frac{\partial[\mathcal{Q}_B + \mathcal{Q}_H]}{\partial w_j^-} = 0$$

$$\begin{aligned} \Longrightarrow w_j^+ &= g\int \langle 0| e^{-\Theta}\left\{\hat{\mathcal{W}}_j^+, \Phi_0^-\Phi_0^+\right\}_- e^{\Theta} |0\rangle \, d\Lambda \\ &- \iint \langle 0| e^{-\Theta}\left\{\hat{\mathcal{W}}_j^+, [\mu^2\Phi_0^-\Phi_0^+ + \lambda(\Phi_0^-\Phi_0^+)^2]\right\}_- e^{\Theta} |0\rangle \, d\Lambda_a d\Lambda_b \\ &= g^2\int w_j^+ v^2 d\Lambda + ... \quad (16) \end{aligned}$$

$$\Longrightarrow M_W = gv$$

The mass eigenstates

$$\mathcal{Z} \equiv \frac{1}{\sqrt{g^2+g'^2}}(g\mathcal{W}^3 - g'\mathcal{B}) \qquad \mathcal{A} \equiv \frac{1}{\sqrt{g^2+g'^2}}(g'\mathcal{W}^3 + g\mathcal{B}) \quad (17)$$

have

$$M_Z = v\sqrt{g^2+g'^2} \qquad M_A = 0$$

The complete eletroweak leptonic scalar is

$$\mathcal{Q} = \mathcal{Q}'_f + \mathcal{Q}_H + \mathcal{Q}_B$$

References

[1] F.Wilczek, A.Zee, "Families from Spinors", *Phys. Rev.* D**25**, 553 (1982)

[2] cds.cern.ch\record\2681012