

Modified general relativity

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Abstract

A modified Einstein equation of general relativity is obtained by using the principle of least action, a decomposition of symmetric tensors on a time oriented Lorentzian manifold, and a fundamental postulate of general relativity. The decomposition introduces a new symmetric tensor $\Phi_{\alpha\beta}$ which describes the energy-momentum of the gravitational field itself. It completes Einstein's equation and addresses the energy localization problem. The positive part of Φ , the trace of the new tensor with respect to the metric, describes dark energy. The cosmological constant must vanish and is dynamically replaced by Φ . A cyclic universe which developed after the Big Bang is described. The dark energy density provides a natural explanation of why the vacuum energy density is so small, and why it dominates the present epoch of the universe. The negative part of Φ describes the attractive self-gravitating energy of the gravitational field. $\Phi_{\alpha\beta}$ introduces two additional terms into the Newtonian radial force equation: the force due to dark energy and the $\frac{1}{r}$ "dark matter" force. When the dark energy force balances the Newtonian force, the flat rotation curves and the baryonic Tully-Fisher relation are obtained. The Newtonian rotation curves for galaxies with no flat orbital curves, and those with rising rotation curves for large radii are described as examples of the flexibility of the orbital rotation curve equation.

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1. Introduction

It has been over a century since Einstein [1, 2, 3] formulated general relativity (GR) in 1915. He was aware that the gravitational field must interact with itself, but was unable to produce a symmetric tensor to properly describe the energy-momentum of the gravitational field. Instead, a non-covariant pseudo-tensor was introduced. However, the difficulties associated with this pseudo-tensor led to the problem of the localization of energy in GR. Over the decades, other pseudo-tensors were developed and different approaches to describe the energy-momentum of the gravitational field were investigated, [4, 5, 6](and references therein) but the energy localization problem still exists today. Despite this deficiency, general relativity is one of the two cornerstones of physics.

GR was developed by Einstein on a four-dimensional Riemannian manifold with a metric that represented the gravitational field and described the geometry of spacetime. Today, we more properly describe spacetime on a time oriented Lorentzian manifold with metric. The Lorentzian metric can be associated with a Riemannian metric by using the line element field, $(\mathbf{X}, -\mathbf{X})$, that is known to exist. A classical result in Riemannian geometry, namely the Berger-Ebin theorem [7], can then be adapted to spacetime. This results in the Orthogonal Decomposition Theorem (ODT): an arbitrary second rank symmetric tensor on a time oriented Lorentzian manifold with a torsionless and metric compatible connection can be orthogonally decomposed into a divergenceless tensor and a new geometrical tensor, $\Phi_{\alpha\beta}$. It is a symmetric tensor constructed from the Lie derivative of both the metric and the unit vectors collinear with one of the pair of regular vectors in the line element field.

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The right hand side of Einstein's equation $\frac{8\pi G}{c^4}T_{\alpha\beta} = G_{\alpha\beta} + \Lambda g_{\alpha\beta}$ involves symmetric divergenceless tensors. The left hand side is defined by the variation of the action functional for all matter fields with respect to the metric. This generates a divergenceless symmetric tensor that must contain all interactions of the gravitational field with the matter fields, and with itself; otherwise, it would not be locally conserved. However, there is nothing in this definition that deals explicitly with the self-interaction of the gravitational field. If we define $\tilde{T}_{\alpha\beta}$ as a symmetric energy-momentum tensor generated from the matter fields without the requirement that it completely describes the self-gravitating field as well, it cannot be locally conserved and would not be divergenceless. Consequently, this second rank symmetric tensor can be locally decomposed by the ODT into a linear sum of divergenceless tensors and $\Phi_{\alpha\beta}$. Lovelock's theorem [8] proves that in four dimensions, the divergenceless tensors composed from the metric and its first two derivatives can only consist of the metric and the tensor named after Einstein, $G_{\alpha\beta}$. Therefore, Einstein's equation in a four-dimensional Lorentzian spacetime should be expressed more generally by including the $\Phi_{\alpha\beta}$ term.

It will be proved that $\frac{8\pi G}{c^4}\tilde{T}_{\alpha\beta} = G_{\alpha\beta} + \Lambda g_{\alpha\beta} + \Phi_{\alpha\beta}$ and that the tensor $T_{\alpha\beta} = \tilde{T}_{\alpha\beta} - \frac{c^4}{8\pi G}\Phi_{\alpha\beta}$ is divergenceless which allows Einstein's equation to be recovered. In that sense, $\Phi_{\alpha\beta}$ is hidden in GR. Thus, general relativity is not complete; it is possible to construct a symmetric tensor from the metric and a regular vector field that is independent of the energy-momentum tensor of the matter fields and represents the energy-momentum of the gravitational field itself.

This differs with the presently and generally accepted belief that GR is complete. However, if that notion was true, GR should be able to describe particular features of dark matter. That unfortunately is not the case and is the reason why physicists invented the generally well accepted theory of Lambda cold dark matter (Λ CDM) to explain, in particular, the flat rotation curves of some galaxies, while leaving GR intact. Modified general relativity can describe those and other galactic rotation curves as discussed in section 7.

Since Lie derivatives have the same form when expressed with covariant or partial derivatives, $\Phi_{\alpha\beta}$ does not vanish when the connection coefficients vanish. The metric can be locally Minkowskian, as in free fall, without forcing $\Phi_{\alpha\beta}$ to vanish. This contrasts free fall in GR where the connection coefficients vanish locally and the gravitational field locally disappears; hence the well known conception that the energy-momentum of the gravitational field is not localizable [9]. $\Phi_{\alpha\beta}$ has the structure to describe local gravitational energy-momentum. Its trace with respect to the metric, Φ , can describe the self-energy of the gravitational field. In free fall, the effective force of gravity disappears locally but the self-energy of the gravitational field is intact accordingly.

In section 2, the Orthogonal Decomposition Theorem is proved. In section 3, a modified equation of GR is derived by using the principle of least action, the ODT and a fundamental postulate of GR. $\Phi_{\alpha\beta}$ appears naturally alongside the Einstein tensor and introduces X^μ from the line element field and its collinear vector u^μ as dynamical variables independent of the Riemannian metric. Variation of the action functional with respect to u^μ leads to an expression for its co-vector in terms of the magnitude f of X^α and Φ : $u_\mu = \frac{3}{8}\partial_\mu f$.

Section 4 discusses the conservation equation for the divergenceless energy-momentum tensor $T_{\alpha\beta} = \tilde{T}_{\alpha\beta} - \frac{c^4}{8\pi G}\Phi_{\alpha\beta}$ where $\tilde{T}_{\alpha\beta}$ is the total matter energy-momentum tensor describing all types of matter including baryonic and dark matter, massive neutrinos, massive gravitons and any other possible particle; if dark matter particles exist. Some additional properties of $\Phi_{\alpha\beta}$ and X_μ are presented.

In section 5, another interesting result is apparent from the calculation of the interaction of the gravitational field with its source, the energy-momentum tensor. Using the global constraint $\int \Phi \sqrt{-g} d^4x = 0$, it is shown that the cosmological constant Λ must vanish, and is dynamically replaced with Φ .

Section 6 is a discussion of the modified Einstein equation of GR in the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, and dark energy. A gravitationally repulsive condition is described by $\Phi > -2\Lambda_d$ where Λ_d is the dark energy density. $\Phi > 0$ defines dark energy. Dark energy describes the inflation of the universe immediately after the Big Bang when no matter of any type was present. The dark energy density then tends to the present value of the vacuum energy density. A cyclic universe is born with maximum and minimum values of the cosmological scale factor in the FLRW metric. Dark energy explains the small value of the vacuum energy density and why it now dominates the expansion and acceleration of the present universe.

Cyclic universes have been reported in the literature [10, 11, 12, 13, 14]. Dark energy has been described by various scalar theories such as the quintessential [13], k-essence [14] (and references therein) phantom or quintom theories [17] (and further references therein). Dark energy in this article not a scalar theory; it is the positive part of the energy of the gravitational field.

The negative values of Φ represent the attractive energy of the gravitational field interacting with itself. This brings into question the subject of dark matter. Since Einstein's equation is incomplete without the tensor $\Phi_{\alpha\beta}$ describing the self-interactions of the gravitational field, the plausibility of dark matter is questionable; its existence is based on the assumption that general relativity is a complete theory. Although the self-interactions in a weak gravitational field may be extremely small, in the gravitational field of a galaxy, they may be significant enough to explain dark matter.

In section 7, the modified equation of GR is calculated with a spheroidal metric in a region of spacetime outside of matter with the assumption that dark matter does not exist. Two additional terms appear in the modified Newtonian force equation that provides it the flexibility to describe various types of galaxies. By balancing the dark energy force with the Newtonian force, the Tully-Fisher relation is established and the acceleration parameter in MOND is expressed in terms of the dark energy radial force parameter.

2. Orthogonal Decomposition of Symmetric Tensors

Curved spacetime is described by the four-dimensional time oriented Lorentzian manifold with a +2 signature metric, $(M, g_{\alpha\beta})$. The connection on the manifold is torsionless and metric compatible. The Lorentzian manifold is assumed to be compact with vanishing Euler-Poincaré characteristic. It admits a smooth regular line element field $(X^\beta, -X^\beta)$ [18, 19, 23].

The orthogonal decomposition of symmetric tensors on Riemannian manifolds has been documented in the literature [7, 20, 21, 22]. However, a decomposition of symmetric tensors on a time oriented Lorentzian manifold is required.

Theorem 2.1. *Orthogonal Decomposition Theorem (ODT): An arbitrary (0,2) symmetric tensor $w_{\alpha\beta}$ in the symmetric cotangent bundle S^2T^*M on an n -dimensional time oriented Lorentzian manifold $(M, g_{\alpha\beta})$ with a torsionless and metric compatible connection can be orthogonally decomposed as*

$$w_{\alpha\beta} = v_{\alpha\beta} + \Phi_{\alpha\beta} \quad (1)$$

where $\nabla^\alpha v_{\alpha\beta} = 0$ and $\Phi_{\alpha\beta} = \frac{1}{2}\mathcal{L}_X g_{\alpha\beta} + \mathcal{L}_X u_\alpha u_\beta$ with \mathbf{X} a regular vector field on M and \mathbf{u} a timelike unit vector collinear with \mathbf{X} .

Proof. Let the Lorentzian manifold $(M, g_{\alpha\beta})$ be paracompact, or compact and orientable with vanishing Euler-Poincaré characteristic. A smooth regular line element field $(\mathbf{X}, -\mathbf{X})$ exists as does a unit vector \mathbf{u} collinear with \mathbf{X} , where \mathbf{X} is one of the pair $(\mathbf{X}, -\mathbf{X})$. Let M be endowed with a smooth Riemannian metric $g_{\alpha\beta}^+$. The smooth Lorentzian metric $g_{\alpha\beta}$ is constructed from $g_{\alpha\beta}^+$ and the unit vector \mathbf{u} by setting [19, 23]

$$g_{\alpha\beta} = g_{\alpha\beta}^+ - 2u_\alpha u_\beta. \quad (2)$$

Let $w_{\alpha\beta}$ and $v_{\alpha\beta}$ belong to S^2T^*M , the cotangent bundle of symmetric (0,2) tensors on M . In the Riemannian open subset of S^2T^*M which contains $g_{\alpha\beta}^+$, an arbitrary (0,2) symmetric tensor $w_{\alpha\beta}$ can be orthogonally decomposed by the Berger-Ebin theorem [7] according to

$$w_{\alpha\beta} = v_{\alpha\beta} + \frac{1}{2}\mathcal{L}_X g_{\alpha\beta}^+ \quad (3)$$

where $\nabla^{+\alpha} v_{\alpha\beta} = 0$. Given the unit vector field \mathbf{u} collinear with \mathbf{X} , the Riemannian connection ∇^+ on M is the same as the Lorentzian connection ∇ on M because both connections have the same geodesics with the same parameterization and both connections are torsionless [22]. Hence,

$$\begin{aligned} w_{\alpha\beta} &= v_{\alpha\beta} + \frac{1}{2}\mathcal{L}_X g_{\alpha\beta} + \mathcal{L}_X u_\alpha u_\beta \\ &= v_{\alpha\beta} + \Phi_{\alpha\beta} \end{aligned} \quad (4)$$

where $\nabla^\alpha v_{\alpha\beta} = 0$ and

$$\Phi_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha X_\beta + \nabla_\beta X_\alpha) + u^\lambda(u_\alpha \nabla_\beta X_\lambda + u_\beta \nabla_\alpha X_\lambda). \quad (5)$$

□

3. Derivation of the modified equation of general relativity

A modified equation of general relativity of the form $C_{\alpha\beta} = 0$ is sought which contains a linear combination of symmetric tensors that define the Einstein equation, and a new tensor which can describe the energy-momentum of the gravitational field itself. This can be achieved by using the principle of least action, the Orthogonal Decomposition Theorem (1), and a fundamental postulate of GR.

Firstly, the field equations contained in $C_{\alpha\beta}$ which are sought to describe general relativity and the energy-momentum of the gravitational field, must be derivable from the action functional

$$S = S^F + S^{EH} + S^G \quad (6)$$

where S^F and L^F refer to the action and Lagrangian, respectively, for all types of matter fields including those of dark matter if dark matter particles exist. S^{EH} is the Einstein-Hilbert action for general relativity and S^G is the action for the energy-momentum of the gravitational field with Lagrangian L^G . The variation of S^F with respect to $g^{\alpha\beta}$

$$\delta S^F = \int \left(\frac{\delta L^F}{\delta g^{\alpha\beta}} - \frac{1}{2} L^F g_{\alpha\beta} \right) \delta g^{\alpha\beta} \sqrt{-g} d^4x \quad (7)$$

generates the symmetric energy-momentum tensor $\tilde{T}_{\alpha\beta}$ which represents the interaction of all types of matter fields and associated radiation in a gravitational field, but does not specifically include the self-interaction of the gravitational field:

$$\tilde{T}_{\alpha\beta} := -2c \left(\frac{\delta L^F}{\delta g^{\alpha\beta}} - \frac{1}{2} L^F g_{\alpha\beta} \right). \quad (8)$$

$C_{\alpha\beta}$ must then be expressed as

$$C_{\alpha\beta} = \frac{a}{c} \tilde{T}_{\alpha\beta} + b w_{\alpha\beta} \quad (9)$$

where $w_{\alpha\beta}$ is an unknown symmetric tensor independent of $\tilde{T}_{\alpha\beta}$; and a and b are arbitrary constants.

Secondly, $w_{\alpha\beta}$ can be orthogonally decomposed by the ODT into

$$w_{\alpha\beta} = v_{\alpha\beta} + \Phi_{\alpha\beta} \quad (10)$$

where $\Phi_{\alpha\beta}$ is given by (5) and $\nabla^\alpha v_{\alpha\beta} = 0$.

Thirdly, Einstein concluded [1] that the metric should describe both the geometry of spacetime and the gravitational field. He postulated the totality of the matter energy-momentum tensor and the energy-momentum of the gravitational field, to be the source of the gravitational field. Adhering to this philosophy, the energy-momentum tensor $T_{\alpha\beta}$ describing the totality of all types of matter and the energy-momentum of the gravitational field, is postulated to be the source of the gravitational field.

$\Phi_{\alpha\beta}$ is independent of $\tilde{T}_{\alpha\beta}$ and is not divergenceless. $\Phi_{\alpha\beta}$ is therefore the sole candidate to describe the energy-momentum of the gravitational field. Thus,

$$T_{\alpha\beta} = \tilde{T}_{\alpha\beta} + \frac{bc}{a} \Phi_{\alpha\beta} \quad (11)$$

and the interaction of the gravitational field with its energy-momentum tensor can be defined with the action

$$S^G := -b \int g_{\alpha\beta} \Phi^{\alpha\beta} \sqrt{-g} d^4x. \quad (12)$$

It was proved by Lovelock [8] that the only tensors in a four-dimensional spacetime which are symmetric, divergence free, and a concomitant of the metric tensor together with its first two derivatives are the metric and the Einstein tensor, $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$. $v_{\alpha\beta}$ must therefore contain the Lovelock tensors.

$C_{\alpha\beta}$ is then formally decomposed as

$$C_{\alpha\beta} = \frac{a}{c}T_{\alpha\beta} + bv_{\alpha\beta} \quad (13)$$

with $\nabla_\alpha v^{\alpha\beta} = 0$ and $v_{\alpha\beta} := G_{\alpha\beta} + \Lambda g_{\alpha\beta}$. Λ is an integration constant (in hindsight identified as the cosmological constant). With the collection of tensors $C_{\alpha\beta}$ defined to vanish, we obtain the modified Einstein equation of general relativity with cosmological constant Λ and the gravitational energy-momentum term $\Phi_{\alpha\beta}$

$$-\frac{8\pi G}{c^4}\tilde{T}_{\alpha\beta} + R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta} + \Phi_{\alpha\beta} = 0 \quad (14)$$

by setting $a = -\frac{1}{2}$ and $b = \frac{c^3}{16\pi G}$.

Ma and Wang [22] obtained a similar result to (14) with $\Lambda = 0$, but with an entirely different $\Phi_{\alpha\beta} = \nabla_\alpha \partial_\beta \phi$ for some scalar ϕ by using a decomposition of symmetric tensors on a Riemannian manifold.

Equation (14) must be derived from the action functional (6). With (12):

$$S = \int L^F(A^\beta, \nabla^\alpha A^\beta, \dots, g^{\alpha\beta})\sqrt{-g}d^4x + b \int (R - 2\Lambda)\sqrt{-g}d^4x - b \int \Phi_{\alpha\beta}g^{\alpha\beta}\sqrt{-g}d^4x. \quad (15)$$

To calculate the variation of S^G with respect to the inverse metric $g^{\alpha\beta}$, the following results are used: $g^{\alpha\beta} = g^{+\alpha\beta} - 2u^\alpha u^\beta$ is the inverse of $g_{\alpha\beta}$; $\nabla_\mu(u_\alpha u_\beta) = 0$; $g^{+\alpha\beta}\delta g_{\alpha\beta} = -g_{\alpha\beta}\delta g^{+\alpha\beta}$; $\delta g^{+\rho\beta} = -g^{+\alpha\beta}g^{\lambda\rho}\delta g_{\alpha\lambda}^+$; $g_{\alpha\beta}\delta(u^\alpha u^\beta) = u_\alpha u_\beta \delta g^{\alpha\beta}$; and $\delta(u^\alpha u^\beta) = u_\lambda u^\beta \delta g^{\lambda\alpha}$. The variation of S with respect to $g^{\alpha\beta}$ is then

$$\delta S = \int \left[-\frac{1}{2c}\tilde{T}_{\alpha\beta} + b(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R) + b\Lambda g_{\alpha\beta} + b(\nabla_\alpha X_\beta + 2u^\lambda u_\beta \nabla_\alpha X_\lambda + \nabla_\mu X_\nu (-u_\alpha u_\beta g^{\mu\nu} + u^\mu u^\nu g_{\alpha\beta})) \right] \delta g^{\alpha\beta} \sqrt{-g} d^4x \quad (16)$$

after calculating $\delta\Gamma_{\alpha\beta}^\lambda$ induced by the variations in the inverse metric, and integrating by parts several times. The last term vanishes which follows by writing the tensor in brackets, $-u_\alpha u_\beta g^{\mu\nu} + u^\mu u^\nu g_{\alpha\beta}$, as its equivalent, $\frac{1}{2}(g^{+\mu\nu}g_{\alpha\beta} - g_{\alpha\beta}^+g^{\mu\nu})$; and choosing an orthonormal basis (e_α) at a point $p \in M$ for g^+ with $e_0 = u$. Then, $u^0 u_0 = 1$, $u^i u_i = 0$, $g_{\alpha\beta}^+ = \delta_{\alpha\beta}$, $g^{00} = -g^{+00}$ and $g_{00} = -g_{00}^+$, with all other components of the metric g equal to those of the metric g^+ . Since $\delta g^{\alpha\beta}$ is symmetric, the second last term can be expressed as $b\Phi_{\alpha\beta}$. With $\delta S = 0$ and arbitrary variations $\delta g^{\alpha\beta}$, we have

$$-\frac{1}{2c}\tilde{T}_{\alpha\beta} + b(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R) + b\Lambda g_{\alpha\beta} + b\Phi_{\alpha\beta} = 0. \quad (17)$$

Setting $b = \frac{c^3}{16\pi G}$ yields the modified Einstein equation described in (14).

X^β from the line element field and its collinear vector u^β are dynamical variables independent of the Riemannian metric in (2). Varying (15) with respect to u^μ yields the equation

$$-3u^\alpha \nabla_\mu X_\alpha - \Phi u_\mu + u^\alpha \nabla_\alpha X_\mu + 4u^\lambda u_\mu u^\alpha \nabla_\alpha X_\lambda = 0 \quad (18)$$

using (14). With $X^\alpha = f u^\alpha$, where $f > 0$ is a scalar representing the length of the regular vector X^α , the third term of (18) is a geodesic of the form $X^\alpha \nabla_\alpha X_\mu = k X_\mu$ where k is an arbitrary function on the geodesic curve. However, it vanishes if the curve is affinely reparameterized; and both of the geodesic terms in (18) can then be set to zero. Since

$$X^\alpha X_\alpha = -f^2, \quad (19)$$

it follows from (18) that in an affine parameterization

$$u_\mu = \frac{3}{\Phi} \partial_\mu f. \quad (20)$$

There is a myriad of regular vectors from the line element field for each Riemannian metric, and the associated Lorentzian metric is not unique. However, the variation of (15) with respect to u^μ restricted the line element fields to those given by (20) which in turn restricts the Lorentzian metric. This removes the ambiguity of the line element field in modified general relativity. Other properties of X_μ are discussed in the next section.

4. The conserved energy-momentum tensor $T_{\alpha\beta}$ and additional properties of $\Phi_{\alpha\beta}$

The invariance of the action functional describing gravity, its self-energy-momentum and total matter fields under the symmetry of diffeomorphisms demands a symmetric divergenceless energy-momentum tensor

$$T^{\alpha\beta} = \tilde{T}^{\alpha\beta} - \frac{c^4}{8\pi G} \Phi^{\alpha\beta}. \quad (21)$$

This follows from an analysis of each term in the action functional S defined in (6). The action S^{EH} is independently invariant under a diffeomorphism. Variation of the action S^F with respect to the metric contains only $\tilde{T}^{\alpha\beta}$ because the variations of S^F with respect to each field and its derivatives vanish with the corresponding Euler-Lagrange equations. Variation of S^G with respect to the metric yields $\Phi^{\alpha\beta}$. Therefore, we can write

$$\int \left(-\frac{1}{2c} \tilde{T}^{\alpha\beta} + b \Phi^{\alpha\beta} \right) \delta g_{\alpha\beta} \sqrt{-g} d^4x = 0 \quad (22)$$

where $b = \frac{c^3}{16\pi G}$. Under a diffeomorphism, the Lie derivative of the metric along a regular vector Y^β generates the infinitesimal change in the metric $\delta g_{\alpha\beta} = \nabla_\alpha Y_\beta + \nabla_\beta Y_\alpha$. Integrating by parts then gives

$$\int \nabla_\alpha \left(-\frac{1}{2c} \tilde{T}^{\alpha\beta} + b \Phi^{\alpha\beta} \right) Y_\beta \sqrt{-g} d^4x = 0 \quad (23)$$

which requires

$$\nabla_\alpha T^{\alpha\beta} = 0 \quad (24)$$

for diffeomorphisms generated by Y^β .

Equation (24) is the local description of the conservation of energy and momentum in a modified theory of GR described by (14). The gravitational field has an intrinsic energy-momentum which is attributed to $\Phi_{\alpha\beta}$. Being independent of $\tilde{T}_{\alpha\beta}$, $\frac{c^4}{8\pi G} \Phi_{\alpha\beta}$ provides the additional energy and momentum from the self-interactions of the gravitational field necessary to complete the source $T_{\alpha\beta}$ of the geometry of spacetime. $\Phi_{\alpha\beta}$ completes the Einstein equation and leaves it intact in form:

$$\frac{8\pi G}{c^4} T_{\alpha\beta} = G_{\alpha\beta} + \Lambda g_{\alpha\beta}. \quad (25)$$

$\Phi_{\alpha\beta}$ is expressed in terms of the Lie derivatives by (4). Since Lie derivatives have the same form when expressed with covariant or partial derivatives, $\Phi_{\alpha\beta}$ does not vanish when the connection coefficients vanish. The metric can be locally Minkowskian, as in free fall, without affecting $\Phi_{\alpha\beta}$. It has the structure to describe local gravitational energy-momentum. $\Phi_{\alpha\beta}$ vanishes if the gravitational field is constant along X^β . In general this is not the case because the gravitational field interacts with itself; even weak gravitational fields self-gravitate.

It is straightforward to calculate the coupling of the gravitational field with its energy-momentum tensor:

$$\int g_{\alpha\beta} \Phi^{\alpha\beta} \sqrt{-g} d^4x = \int \Phi \sqrt{-g} d^4x = 0 \quad (26)$$

where $\Phi = \nabla_\alpha X_\beta (g^{\alpha\beta} + 2u^\alpha u^\beta)$. Equation (26) means the scalar Φ has local positive and negative values, all of which add to zero when integrated over the entire spacetime. Φ is globally conserved. Section 6 demonstrates that the positive values of Φ are attributed to the gravitationally repulsive properties of dark energy with the cosmological constant set to zero. The negative values represent the attractive energy of

the gravitational field interacting with itself. Φ is measurable; it can be expressed in terms of the density and pressure of total matter and the vacuum energy density as shown in section 6.

From (20), it follows that

$$\square f = -\frac{1}{9}\Phi^2 \quad (27)$$

which is the wave equation for the magnitude of the regular vector X_μ driven by the square of the self-energy of the gravitational field. The energy-momentum of the gravitational field is localizable and measurable.

5. Cosmological Constant

The metric cannot appear as a field variable alongside the cosmological constant. Λ is an integration constant and must vanish.

Theorem 5.1. *The cosmological constant Λ must vanish and is dynamically replaced by the trace of $\Phi_{\alpha\beta}$.*

Proof. The action S_g^{int} for the coupling of the energy-momentum tensor with the metric is:

$$\begin{aligned} S_g^{int} &= -\frac{1}{2c} \int T^{\alpha\beta} g_{\alpha\beta} \sqrt{-g} d^4x \\ &= \frac{c^3}{16\pi G} \int (R - 4\Lambda) \sqrt{-g} d^4x. \end{aligned} \quad (28)$$

Since the energy-momentum tensor for all types of matter and the energy-momentum of the gravitational field, is the source of the gravitational field and the geometry of spacetime, $S_g^{int} \equiv S^{EH}$. Comparing (28) to (15) requires the integration constant Λ to vanish.

Using (26), S^{EH} with $\Lambda = 0$ can then be written as

$$S^{EHG} = \frac{c^3}{16\pi G} \int (R - \Phi) \sqrt{-g} d^4x \quad (29)$$

which generates the modified Einstein equation with no cosmological constant from (15). If $\Phi = 2\Lambda$ *locally*, the Einstein equation with the cosmological constant is obtained accordingly. The trace of the tensor describing the energy-momentum of the gravitational field, dynamically replaces the cosmological constant but must obey the global equation (26). □

6. Energy-momentum of the gravitational field in the FLRW metric: Dark energy

Some properties of $\Phi_{\alpha\beta}$ in the Friedmann-Lemaître-Robertson-Walker metric are now investigated. The FLRW metric is typically used to describe a spatially maximal symmetric universe according to the cosmological principle [25] whereby the universe is homogeneous and isotropic when measured on a large scale. This metric is given by

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[\frac{1}{1 - \kappa r^2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right] \quad (30)$$

where $a(t)$ is the cosmological scale factor which satisfies $a > 0$ after the Big Bang at $t = 0$. κ is a constant used to describe a particular spatial geometry. The connection components of the FLRW metric are

$$\Gamma_{j0}^i = \frac{\dot{a}}{ca} \delta_j^i, \quad \Gamma_{ij}^0 = \frac{\dot{a}}{ca} g_{ij}, \quad \Gamma_{00}^\mu = 0 \quad (31)$$

where $j = 1, 2, 3$. The Ricci tensor components are

$$R_{00} = -3 \frac{\ddot{a}}{ac^2}, \quad R_{ij} = \left(\frac{\ddot{a}}{ac^2} + 2 \frac{\dot{a}^2}{a^2 c^2} + 2 \frac{\kappa}{a^2} \right) g_{ij} \quad (32)$$

and the Ricci scalar is

$$R = \frac{6}{a^2 c^2} (a\ddot{a} + \dot{a}^2 + \kappa c^2). \quad (33)$$

It was proved in [25] that a maximally spatial form invariant symmetric second rank tensor $B_{\alpha\beta}$ has components in the form

$$B_{00} = \varrho(t), \quad B_{0j} = 0, \quad B_{ij} = p(t)g_{ij} \quad (34)$$

where $\varrho(t)$ and $p(t)$ are arbitrary functions of time. We therefore set,

$$\tilde{T}_{00} = c^2 \varrho, \quad \tilde{T}_{ij} = pg_{ij}, \quad \tilde{T}_{\mu}^{\mu} = -c^2 \varrho + 3p \quad (35)$$

where $\varrho(t)$ and $p(t)$ are designated as the mass density and pressure functions, respectively, of total matter including dark matter; if dark matter particles exist. Similarly,

$$\Phi_{00} = \Lambda_d, \quad \Phi_{ij} = \frac{P_d}{c^2} g_{ij}, \quad \Phi_{\mu}^{\mu} = -\Lambda_d + 3\frac{P_d}{c^2} \quad (36)$$

where $\Lambda_d(t)$ and $P_d(t)$ refer to the energy density and pressure, respectively, of the tensor describing the energy-momentum of the gravitational field.

To obtain the Friedmann equations, we use the trace of the modified Einstein equation

$$-\frac{8\pi G}{c^4} \tilde{T} - R + \Phi = 0 \quad (37)$$

to rewrite the modified Einstein equation as

$$R_{\alpha\beta} = \frac{8\pi G}{c^4} (\tilde{T}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \tilde{T}) + \frac{1}{2} g_{\alpha\beta} \Phi - \Phi_{\alpha\beta} \quad (38)$$

from which we obtain

$$3\frac{\ddot{a}}{a} = -4\pi G(\varrho + \frac{3p}{c^2}) + \frac{1}{2}c^2\Lambda_d + \frac{3}{2}P_d \quad (39)$$

from the R_{00} component. The R_{11} component gives

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2\kappa c^2}{a^2} = \frac{c^2}{2}(-\Lambda_d + \frac{P_d}{c^2}) + 4\pi G(\varrho - \frac{p}{c^2}) \quad (40)$$

and the conservation law for $T^{\alpha\beta}$ yields

$$\dot{\varrho} - \frac{c^2}{8\pi G}\dot{\Lambda}_d = -3\frac{\dot{a}}{a}(\varrho + \frac{p}{c^2} - \frac{c^2}{8\pi G}(\Lambda_d + \frac{P_d}{c^2})). \quad (41)$$

Inserting (39) into (40) produces a simpler equation

$$\frac{\dot{a}^2}{a^2} + \frac{\kappa c^2}{a^2} = \frac{8\pi G}{3}\varrho - \frac{1}{3}c^2\Lambda_d. \quad (42)$$

Equations (39) and (42) are the Friedmann equations modified with $\Phi_{\alpha\beta}$.

From (39), we immediately see that $\Phi + 2\Lambda_d = \Lambda_d + \frac{3}{c^2}P_d > 0$ tends to accelerate the universe; while all types of matter combined, with a positive mass density and pressure, tend to decelerate the universe. $\Phi > -2\Lambda_d$ is a gravitationally repulsive condition which relates dark energy to Λ_d . Hence, Λ_d is called the dark energy density and P_d the dark energy pressure. Φ tends to accelerate or decelerate the universe but has a net zero effect on it. $\Phi_{\alpha\beta}$ and therefore Φ , provide the flexibility to describe various eras in the evolution of the universe. The cosmological constant Λ , on the other hand, can be expressed as a fixed negative energy density which would have tended to accelerate the universe during *all* epochs.

One of the recent challenges in cosmology has been to find a natural mechanism that describes a small but positive vacuum energy density to explain the observed acceleration of the present universe. Dark energy provides a natural explanation of this challenge without the need of a cosmological constant.

After the discovery in 1929 by Hubble [26] that the universe was expanding, Λ was not required to obtain a static solution to the Einstein equations with a positive mass density. Since the cosmological

constant was vastly smaller than any value predicted by particle theory, most particle theorists simply assumed, that for some unknown reason, this quantity was zero [27]. This was widely believed to be true until the discovery of the presently accelerating universe in 1998-99 [28, 29]. Λ was then considered to be associated with the dark energy conundrum. However, it is just an integration constant in the modified Einstein equation. Λ *must* vanish and is replaced by Φ as proved in theorem 5.1. This is readily verified by restricting the dark energy variables to the constant values $\Lambda_d = -\Lambda$ and $P_d = c^2\Lambda$ in (39) and (42). The Friedmann equations with the cosmological constant Λ are then recovered in accordance with theorem 5.1.

The Friedmann equations are now considered with $\kappa = 1$ describing a closed universe:

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - \frac{c^2}{3}\Lambda_d a^2 - c^2 \quad (43)$$

and

$$\ddot{a} = -\frac{4\pi G}{3}a\left(\rho + \frac{3p}{c^2}\right) + \frac{ac^2}{6}\left(\Lambda_d + \frac{3}{c^2}P_d\right). \quad (44)$$

To avoid confusion with Λ , we will denote the constant vacuum energy density as Λ_v with the property $\Lambda_v > 0$. In the present epoch, Λ_v is measured to be $\approx 1.1 \times 10^{-52} m^{-2}$. By defining

$$\tilde{\rho} = 8\pi G\rho - c^2\Lambda_d \quad (45)$$

and

$$\tilde{p} = -\frac{4\pi G}{c^2}p + \frac{1}{2}P_d, \quad (46)$$

these equations can be simplified to

$$\dot{a}^2 = \frac{\tilde{\rho}a^2}{3} - c^2, \quad (47)$$

and

$$\ddot{a} = a\left(-\frac{\tilde{\rho}}{6} + \tilde{p}\right) \quad (48)$$

with the conservation equation

$$\dot{\tilde{\rho}} = -3\frac{\dot{a}}{a}(\tilde{\rho} - 2\tilde{p}). \quad (49)$$

Unless otherwise stated, $\rho > 0$ and $p > 0$. Equation (47) requires $\tilde{\rho} > 0$.

It is interesting to explore how the energy-momentum of the gravitational field can describe critical features of a Big Bang universe. Immediately after the event of the Big Bang, the universe violently accelerates and $\dot{a} > 0$. For a very short time, there is no matter; $\rho = 0$ and $p = 0$. In this very early stage of the evolution of the universe, it is possible that the constant vacuum energy density developed. If we set $\rho = 0$ in (43), the inequality

$$\Lambda_d < -\frac{3}{a^2} \quad (50)$$

must hold. From (41) and (44) with $\dot{a} \neq 0$,

$$\frac{d}{da}\Lambda_d = -\frac{2\Lambda_d}{a} - \frac{6\ddot{a}}{a^2c^2}. \quad (51)$$

If $\Lambda_d \rightarrow -\Lambda_v$ and $\frac{P_d}{c^2} \rightarrow \Lambda_v$ just after the Big Bang, (44) requires $\frac{\dot{a}}{a}$ to be constant. With those assumptions, equation (51) has the solution

$$\Lambda_d = \frac{c_1}{a^2} - \frac{3\ddot{a}}{ac^2} \quad (52)$$

where c_1 is an arbitrary constant. Setting $c_1 = -3$ and $\Lambda_v = \frac{3\ddot{a}}{ac^2}$,

$$\Lambda_d = -\frac{3}{a^2} - \Lambda_v \quad (53)$$

which satisfies (50) and tends to $-\Lambda_v$ as the universe expands. Dark energy can generate Λ_v during this epoch of the universe. The expansion of the universe is then described by

$$\dot{a}^2 = \frac{1}{3}a^2c^2\Lambda_v. \quad (54)$$

The pressure density of dark energy is $P_d = \frac{c^2}{a^2} + \Lambda_v c^2$ and the acceleration of the universe is

$$\ddot{a} = \frac{ac^2\Lambda_v}{3}. \quad (55)$$

The scalar $\Phi = \frac{6}{a^2} + 4\Lambda_v$ is positive. $\Phi > 0$ is the condition to be satisfied for an expanding and accelerating universe when no matter is present. Because this result depends entirely on dark energy, $\Phi > 0$ defines dark energy.

With all types of matter appearing after the initial inflation, Λ_d must obey the constraint

$$\Lambda_d < -\frac{3}{a^2} + \frac{8\pi G\rho}{c^2}. \quad (56)$$

With constant total matter, the equation

$$\frac{d}{da}\Lambda_d = -\frac{2\Lambda_d}{a} + \frac{16\pi G\rho}{ac^2} - \frac{6\ddot{a}}{a^2c^2} \quad (57)$$

is obtained from (41) and (44) with $\dot{a} \neq 0$. Since $\frac{\ddot{a}}{a} = \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) + \left(\frac{\dot{a}}{a}\right)^2$, a slowly varying non-zero Hubble parameter $\frac{\dot{a}}{a}$ requires $\frac{\ddot{a}}{a}$ to be approximately constant. With that assumption, equation (57) has the solution

$$\Lambda_d = -\frac{3}{a^2} + \frac{8\pi G\rho}{c^2} - \Lambda_v \quad (58)$$

with

$$\Lambda_v = \frac{3\ddot{a}}{ac^2}. \quad (59)$$

The dark energy pressure is

$$P_d = \frac{c^2}{a^2} + \frac{8\pi Gp}{c^2} + c^2\Lambda_v. \quad (60)$$

A pure dark energy effect returns (54) and (55) as the expansion and acceleration, respectively. In a universe with essentially constant matter, which is assumed to be the case of the present era, this demonstrates why Λ_v is important. As expected, $\Phi = \frac{6}{a^2} + 4\Lambda_v + \frac{8\pi G}{c^2}(-\rho + \frac{3p}{c^2})$ is positive or negative.

Riess et al. [30] used the Hubble telescope "to provide the first conclusive evidence for cosmic deceleration that preceded the current epoch of cosmic acceleration". Given the violent acceleration after the Big Bang, this observation evidences the cyclic nature of the universe to this point in time. The cosmological scale factor must have had maximum and minimum values in the past because of the observed changes in sign of its second derivative; there were extremums at $\dot{a} = 0$. In general, this requires $\Lambda_d = -\frac{3}{a^2} + \frac{8\pi G\rho}{c^2}$ from equation (43). The Hubble parameter vanishes and (58) must change because (59) is not constant at the extremum. Dark energy in the amount of Λ_v must be transferred to Λ_d from the dark energy pressure; $\frac{P_d}{c^2}$ in (60) decreases by $-\Lambda_v$ with an offsetting change by that amount to Λ_d in (58). This allows an extremum to occur while keeping Φ unchanged. Then, cosmic acceleration can change to a decelerating epoch, and conversely with the opposite exchange of dark energy.

The maxima or minima of the cosmological scale factor follows directly from equations (47) and (48). The second derivative of a must satisfy

$$\ddot{a} = a\left(-\frac{c^2}{2a^2} + \tilde{p}\right) \quad (61)$$

when $\dot{a} = 0$. The value of \tilde{p} in equation (61) governs the condition for a maximum or minimum of a . With $-\Lambda_d$ having a small fixed value of Λ_v determined early in the evolution of the universe, the variation in Φ is determined by P_d . The constraint (26) on Φ can force P_d to change, which can change the sign of \tilde{p} . Near the end of an acceleration phase, if the dark energy pressure decreases so that $P_d \leq \frac{8\pi Gp}{c^2}$, \tilde{p} changes from

positive to zero or negative, and the scale factor has a maximum value at a_{max} ; $\tilde{p} \leq 0$ is satisfied in (61). The acceleration phase ends and the universe undergoes a deceleration. The scale factor then decreases toward a minimum value a_{min} at which the dark energy pressure increases enough to satisfy $\tilde{p} > \frac{c^2}{2a^2}$. The deceleration phase changes to that of an acceleration and the cyclic process continues indefinitely. Φ , governed by (26), smoothly controls the maximum and minimum values that the cosmological scale factor can have. The global constraint on Φ keeps the universe gravitationally in balance. This model of the universe starts with the Big Bang and then cycles to eternity. It does not suffer the catastrophes of the Big Crunch or the Big Rip.

Although recent data and analysis [31] suggests the observable universe is flat, the data likely represents a small fraction of the presently unknown *entire* universe. If the entire universe has a positive curvature, a measurement of it will appear to be nearly flat if data from large enough distances is not available. Therefore, at this time, the conjecture of a flat universe which expands forever based on observational evidence is less likely than the cyclic universe described and observed after the Big Bang and into this epoch.

Dark energy thus provides a natural explanation of why the vacuum energy density is so small, and why it dominates the present epoch of the universe.

7. Energy-momentum of the gravitational field: Dark matter

The Λ CDM model describes the formation of galaxies after the Big Bang from cooled baryonic matter gravitationally attracted into a dark matter skeleton. Dark matter in the Λ CDM model also provides the additional mass required to describe the flat rotation curves observed in many galaxies. However, no dark matter particles have been detected and there have been several attempts to explain the flat rotational curves without dark matter.

The leading candidate is a *phenomenological* model of Modified Newtonian dynamics (MOND) introduced by Milgrom [32]. The Newtonian force F is modified according to

$$F = m\mu\left(\frac{a}{a_0}\right)a \quad (62)$$

where a_0 is a fundamental acceleration $\approx 1.2 \times 10^{-10} m/s^2$. μ is a function of the ratio of the acceleration relative to a_0 which tends to one for $a \gg a_0$ and tends to $\frac{a}{a_0}$ for $a \ll a_0$. MOND successfully explains many, but not all, mass discrepancies observed in galactic data. However, it has no covariant roots in Einstein's equation or cosmological theory. MOND and Λ CDM were thoroughly discussed by McGaugh in [33].

Other alternatives to dark matter were reviewed by Mannheim in [34] with references therein. In particular, Moffat [35] used a nonsymmetric gravitational theory without dark matter to obtain the flat rotation curves of some galaxies. The bimetric theory of Milgrom [36] involved two metrics as independent degrees of freedom to obtain a relativistic formulation of MOND.

Different approaches to the missing matter problem include dipolar dark matter, which was introduced by Bernard, Blanchet and Heisenberg in [37] to solve the problems of cold dark matter at galactic scales and reproduce the phenomenology of MOND. The theory involves two different species of dark matter particles which are separately coupled to the two metrics of bigravity and are linked together by an internal vector field. In [38], a theory of emergent gravity (EG) which claims a possible breakdown in general relativity, was introduced by Verlinde that provided an explanation for Milgrom's phenomenological fitting formula in reproducing the flattening of rotation curves. Campigotto, Diaferio and Fatibene [39] showed conformal gravity cannot describe galactic rotation curves without the aid of dark matter. On the other hand, a logical analysis based on observational data was presented by Kroupa in [38] to support the conjecture that dark matter does not exist.

The existence of dark matter is based on the assumption that general relativity is correct. However, Einstein's equation is incomplete without the tensor $\tilde{\Phi}_{\alpha\beta}$ describing the self-interactions of the gravitational field. The validity of modified general relativity is now tested with the attempt to describe the additional gravitational attraction in various galaxies without dark matter.

7.1. Modified GR in a spheroidal spacetime

It is assumed dark matter does not exist and that baryonic matter and other possible sources of matter such as neutrinos, produce the gravitational field. In a region of spacetime where there is no matter, $\tilde{T}_{\alpha\beta} = 0$ and the field equations must satisfy

$$G_{\alpha\beta} + \Phi_{\alpha\beta} = 0. \quad (63)$$

Spheroidal solutions to these nonlinear equations are now investigated. The spheroidal behaviour of the metric is to be determined from a particular solution to (63) in a spacetime described by a metric of the form

$$ds^2 = -e^\nu c^2 dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (64)$$

where ν and λ are functions of t , r and θ . The non-zero connection coefficients (Christoffel symbols) are:

$$\begin{aligned} \Gamma_{00}^0 &= \frac{1}{2}\partial_0\nu, & \Gamma_{01}^0 &= \frac{1}{2}\partial_1\nu, & \Gamma_{02}^0 &= \frac{1}{2}\partial_2\nu, & \Gamma_{11}^0 &= \frac{1}{2}\partial_0\lambda e^{\lambda-\nu}, & \Gamma_{00}^1 &= \frac{1}{2}\partial_1\nu e^{\nu-\lambda}, & \Gamma_{01}^1 &= \frac{1}{2}\partial_0\lambda, \\ \Gamma_{11}^1 &= \frac{1}{2}\partial_1\lambda, & \Gamma_{12}^1 &= \frac{1}{2}\partial_2\lambda, & \Gamma_{22}^1 &= -r e^{-\lambda}, & \Gamma_{33}^1 &= -r \sin^2\theta e^{-\lambda}, & \Gamma_{00}^2 &= \frac{1}{2r^2}\partial_2\nu e^\nu, \\ \Gamma_{11}^2 &= \frac{1}{2r^2}\partial_2\lambda e^\lambda, & \Gamma_{12}^2 &= \frac{1}{r}, & \Gamma_{33}^2 &= -\sin\theta \cos\theta, & \Gamma_{13}^3 &= \frac{1}{r}, & \Gamma_{23}^3 &= \cot\theta. \end{aligned}$$

The unit vectors w^β satisfy

$$u^\alpha u_\alpha = -1. \quad (65)$$

As a first step to understand this highly nonlinear set of equations given by (63) with the property (65) in this metric, u_3 is chosen to vanish. This requires

$$X_3 = 0 \quad (66)$$

because u_α is collinear with X_α . All other components of X_α are non-zero.

Static solutions to (63) are sought which require the components of the line element field to satisfy

$$\partial_0 X_\alpha = 0, \quad (67)$$

and from the metric,

$$\partial_0\lambda = 0, \quad \partial_0\nu = 0. \quad (68)$$

The components of $\Phi_{\alpha\beta}$ to be considered are then:

$$\Phi_{00} = (1 + 2u_0 u^0) \left(-\frac{1}{2} e^{\nu-\lambda} \nu' X_1 - \frac{1}{2r^2} e^\nu \partial_2\nu X_2 \right), \quad (69)$$

$$\Phi_{11} = (1 + 2u_1 u^1) \left(X_1' - \frac{1}{2} \lambda' X_1 - \frac{1}{2r^2} e^\lambda \partial_2\lambda X_2 \right), \quad (70)$$

$$\Phi_{22} = (1 + 2u_2 u^2) (\partial_2 X_2 + r e^{-\lambda} X_1), \quad (71)$$

$$\Phi_{33} = r \sin^2\theta e^{-\lambda} X_1 + \sin\theta \cos\theta X_2, \quad (72)$$

the Ricci scalar, which from (63) equals Φ , is

$$R = e^{-\lambda} \left(-\nu'' - \frac{1}{2} \nu'^2 + \frac{1}{2} \lambda' \nu' - \frac{2}{r} \nu' + \frac{2}{r} \lambda' - \frac{2}{r^2} \right) + \frac{1}{r^2} \left(-\frac{1}{2} \partial_2\nu^2 - \partial_2\partial_2\nu - \partial_2\nu \cot\theta + 2 \right), \quad (73)$$

and the corresponding components of the Einstein tensor are:

$$G_{00} = \frac{1}{r^2} e^{\nu-\lambda} (r\lambda' - 1 + e^\lambda) + \frac{e^\nu}{4r^2} \partial_2\nu \partial_2\lambda, \quad (74)$$

$$G_{11} = \frac{1}{r^2} (1 + r\nu' - e^\lambda) + \frac{e^\lambda}{2r^2} [\partial_2\partial_2\lambda + \partial_2\partial_2\nu + \frac{1}{2}(\partial_2\lambda)^2 + \frac{1}{2}(\partial_2\nu)^2 + \frac{1}{2}\partial_2\lambda\partial_2\nu + \cot\theta(\partial_2\lambda + \partial_2\nu)], \quad (75)$$

$$G_{22} = \frac{r^2 e^{-\lambda}}{2} [\nu'' + (\frac{1}{2}\nu' + \frac{1}{r})(\nu' - \lambda')] - \frac{1}{2}\partial_2\partial_2\lambda - \frac{1}{4}(\partial_2\lambda)^2 + \frac{1}{2}\cot\theta\partial_2\nu, \quad (76)$$

$$G_{33} = \sin^2\theta \left[\frac{r^2 e^{-\lambda}}{2} \left(-\frac{\lambda'}{r} + \frac{\nu'}{r} + \nu'' + \frac{1}{2}\nu'^2 - \frac{1}{2}\lambda'\nu' \right) + \frac{1}{4}(\partial_2\nu)^2 + \frac{1}{2}\partial_2\partial_2\nu - \frac{1}{2}\cot\theta\partial_2\lambda \right] \quad (77)$$

where the prime denotes ∂_1 .

These equations are greatly simplified by setting

$$\nu = -\lambda. \quad (78)$$

Thus, a class of static spheroidal solutions to (63) are sought with the restrictions (66),(67),(68) and (78).

Since $e^{\lambda-\nu}(\Phi_{00} + G_{00}) + \Phi_{11} + G_{11} = 0$ from (63),

$$\left(\frac{\lambda'}{2}X_1 + \frac{1}{2r^2}e^\lambda\partial_2\lambda X_2 \right)(1 + 2u_0u^0) + \left(X_1' - \frac{\lambda'}{2}X_1 - \frac{1}{2r^2}e^\lambda\partial_2\lambda X_2 \right)(1 + 2u_1u^1) = 0. \quad (79)$$

If we choose $\frac{\lambda'}{2}X_1 = X_1' - \frac{\lambda'}{2}X_1$,

$$X_1 = a_1 e^\lambda \quad (80)$$

where a_1 is an arbitrary but non-zero constant with the dimensions of L^{-1} . Then (79) becomes

$$\left(\frac{X_1'}{2} + \frac{1}{2r^2}e^\lambda\partial_2\lambda X_2 \right)(1 + 2u_0u^0) + \left(\frac{X_1'}{2} - \frac{1}{2r^2}e^\lambda\partial_2\lambda X_2 \right)(1 + 2u_1u^1) = 0 \quad (81)$$

from which

$$\partial_2\lambda = a_1\lambda'r^2 \frac{u_2u^2}{(u_0u^0 - u_1u^1)X_2} \quad (82)$$

using $1 + u_0u^0 + u_1u^1 = -u_2u^2$.

From (63) and (80) in the interval $0 < \theta < \pi$, $G_{22} + \Phi_{22} = 0$ gives

$$(-\lambda'' + \lambda'^2 - \frac{2}{r}\lambda') + \frac{e^\lambda}{r^2} \left(-\frac{1}{2}(\partial_2\lambda)^2 - \partial_2\partial_2\lambda - \partial_2\lambda \cot\theta \right) + \frac{2e^\lambda}{r^2}(\partial_2X_2 + a_1r)(1 + 2u_2u^2) = 0 \quad (83)$$

and $G_{33} + \Phi_{33} = 0$ yields

$$-\lambda'' + \lambda'^2 - \frac{2}{r}\lambda' + \frac{e^\lambda}{r^2} \left(\frac{1}{2}(\partial_2\lambda)^2 - \partial_2\partial_2\lambda - \partial_2\lambda \cot\theta \right) + \frac{2e^\lambda}{r^2}(a_1r + X_2 \cot\theta) = 0. \quad (84)$$

Subtracting (83) from (84) requires

$$\cot\theta X_2 - \partial_2X_2 + \frac{1}{2}(\partial_2\lambda)^2 - 2u_2u^2(\partial_2X_2 + a_1r) = 0. \quad (85)$$

Choosing $\cot\theta X_2 = \partial_2X_2$ gives

$$X_2 = a_2 \sin\theta \quad (86)$$

where $a_2 \neq 0$ is an otherwise arbitrary dimensionless constant, and demands

$$(\partial_2\lambda)^2 = 4u_2u^2(a_2 \cos\theta + a_1r). \quad (87)$$

Equation (84) can now be expressed as

$$-\lambda'' + \lambda'^2 - \frac{2}{r}\lambda' + \frac{e^\lambda}{r^2} \left(\frac{1}{2}(\partial_2\lambda)^2 - \partial_2\partial_2\lambda - \partial_2\lambda \cot\theta \right) + \frac{2e^\lambda}{r^2}(a_1r + a_2 \cos\theta) = 0. \quad (88)$$

From (82) and (87), the derivative terms in $\partial_2\lambda$ can be neglected if u_2u^2 is restricted to be very small but non-zero. Assuming $\partial_2\partial_2\lambda$ can also be neglected, equation (88) is then approximated by

$$-\lambda'' + \lambda'^2 - \frac{2}{r}\lambda' + \frac{2e^\lambda}{r^2}(a_1r + a_2 \cos\theta) = 0, \quad 0 < \theta < \pi \quad (89)$$

which, for a fixed value of $\cos\theta$, has the exact solution

$$\lambda = -\ln\left(\frac{c_1}{r} + c_2 - a_1r - 2a_2 \cos\theta \ln r\right), \quad 0 < \theta < \pi, \quad 0 < r < \infty \quad (90)$$

where c_1 and c_2 are arbitrary constants.

7.2. The radial force and galactic rotation curves

The radial force on an object of mass m can now be calculated from (90). Using the conventional relationship of the Newtonian potential ϕ to g_{00} ,

$$\phi = \frac{c^2}{2}(e^\nu - 1), \quad (91)$$

the radial force F_r is

$$\begin{aligned} F_r &= -m\partial_1\phi \\ &= \frac{mc^2}{2}\left(\frac{c_1}{r^2} + \frac{2a_2 \cos\theta}{r} + a_1\right). \end{aligned} \quad (92)$$

Choosing

$$c_1 = -\frac{2GM}{c^2}, \quad (93)$$

where M represents the total mass of the galaxy composed of mainly baryonic matter and no dark matter, we arrive at the modified Newtonian force

$$F_r = -\frac{GMm}{r^2} + \frac{mc^2 a_2 \cos\theta}{r} + \frac{mc^2 a_1}{2}. \quad (94)$$

The correction terms to the Newtonian force come from the non-zero components of the line element field in the energy-momentum tensor $\Phi_{\alpha\beta}$. The components of the line element field can change their sign, which means a_j can change to $-a_j$ with $j=1,2$ in this restricted metric. Thus the middle term is gravitationally attractive and represents the "dark matter" correction if $a_2 \cos\theta < 0$ in the interval $0 < \theta < \pi$. It is the term that gives rise to the flat rotation curves. The third constant term is positive and repulsive if $a_1 > 0$. This describes the repulsive dark energy force in the present epoch. However, during a part of the previous decelerating epoch observed by Riess et al. [30], $a_1 < 0$. They used the Hubble telescope to provide the first conclusive evidence for cosmic deceleration that preceded the current epoch of cosmic acceleration.

Assuming a circular orbit about a point mass, it follows that the orbital velocity of a star rotating in the galaxy satisfies

$$v^2 = v_N^2 - a_2 c^2 \cos\theta - \frac{a_1 c^2}{2} r \quad (95)$$

where v_N^2 is the Newtonian term

$$v_N^2 = \frac{GM}{r}. \quad (96)$$

Equation (95) demands an upper limit to r describing a large but finite galaxy.

Because $a_1 \neq 0$, it is possible for the Newtonian force to balance the dark energy force, which requires

$$v_N^2 - \frac{a_1 c^2}{2} r = 0. \quad (97)$$

Then,

$$v^2 = -a_2 c^2 \cos\theta, \quad a_2 \cos\theta < 0 \quad (98)$$

describes a specific class of galaxies with a flat orbital rotation curve. From (96) and (97), we obtain the Tully-Fisher relation

$$v_N^4 = \frac{GMc^2 a_1}{2}, \quad a_1 > 0. \quad (99)$$

This result holds for any finite r in contrast to EG which holds only for large r as determined by Lelli, McGaugh and Schombert [41]. With $\frac{c^2 a_1}{2} := a_0$, the Tully-Fisher relation in MOND is evident.

The importance of the radial acceleration relative to the rotation curves of galaxies was discussed by Lelli, McGaugh, Schombert, and Pawlowski in [42] where it was determined that late time galaxies (spirals and irregulars), early time galaxies (ellipticals and lenticulars), and the most luminous dwarf spheroidals follow a tight radial acceleration relation which correlates well with that due to the distribution of baryons.

Equation (94), which does not include dark matter in this analysis, is general enough to describe the rotation curves of many types of galaxies. For example, galaxy NGC4261 has a relatively flat rotation curve but starts to rise at larger radii, reaching velocities of 700 km s^{-1} at 100 kpc [42]. That requires a_1 in (95) to be negative which was interpreted above. As another example, both $c^2 a_1 r$ and $c^2 a_2$ could be small enough relative to $\frac{GM}{r}$ so that the Newtonian term is dominant. Galaxies with no flat rotation curves have recently been observed by van Dokkum et.al [43]. It should also be remembered that equation (95) came from an approximation to equation (88) which could be used to model galaxies in greater detail. Furthermore, equation (88) is a restricted version of the general equation (63) which provides additional variables that may explain even more aspects of cosmology now attributed to dark matter.

However, it is still possible that dark matter particles may exist. As a part of (14) in the total matter energy-momentum tensor $\tilde{T}_{\alpha\beta}$, they would contribute to the gravitational field outside of its source along with baryonic matter in equation (63) and therefore in (94). But any dark matter contribution to the gravitational field would play a much lesser role because of the existence of $\Phi_{\alpha\beta}$.

8. Conclusion

The results in this article stem from the association of a Lorentzian metric with a Riemannian metric using the line element field $(X^\beta, -X^\beta)$. This allowed a classical result of Riemannian geometry to be adapted to the geometry of spacetime. An orthogonal decomposition of symmetric tensors on a time oriented Lorentzian manifold could then be developed. This introduced a new tensor, $\Phi_{\alpha\beta}$. Variation of the action functional for all matter fields with respect to the metric generated the symmetric energy-momentum tensor $\tilde{T}_{\alpha\beta}$ that contains all interactions of the gravitational field with the matter fields, but not with itself. $\Phi_{\alpha\beta}$ then represented the energy-momentum of the gravitational field. By requiring the field equations to be determined from an action functional, and adhering to Einstein's postulate requiring $T_{\alpha\beta}$, the totality of the matter energy-momentum and the energy-momentum of the gravitational field, to be the source of the gravitational field, a modified equation of general relativity was obtained. It was proved that $T_{\alpha\beta} = \tilde{T}_{\alpha\beta} - \frac{c^4}{8\pi G} \Phi_{\alpha\beta}$ is divergenceless and Einstein's equation is recovered in form. Thus, general relativity is complete with the addition of the geometrical tensor $\Phi_{\alpha\beta}$.

The line element field introduced X^μ and its collinear vector u^μ as dynamical variables independent of the Riemannian metric. Variation of the action functional with respect to u^μ generated the equation $u_\mu = \frac{3}{\Phi} \partial_\mu f$ from which the wave equation $\square f = -\frac{1}{9} \Phi^2$ was obtained. The energy-momentum of the gravitational field is localizable and measureable. Thus, $\Phi_{\alpha\beta}$, a symmetric tensor constructed from the Lie derivatives of both the metric and the unit vectors collinear with one of the pair of regular vectors in the line element field, provides a solution to the energy localization problem.

$\Phi > 0$ defines dark energy and $\Phi < 0$ the attractive energy of the gravitational field interacting with itself. Φ has the global property $\int g^{\alpha\beta} \Phi_{\alpha\beta} \sqrt{-g} d^4x = 0$. Φ tends to accelerate or decelerate the universe but has a net zero affect on it. The cosmological constant Λ is an integration constant and must vanish. It is dynamically replaced by Φ .

Important features attributed to dark energy resulted from the investigation of the modified Einstein equation in the FLRW metric. The dark energy pressure explained the observed cyclic nature of the universe after the Big Bang. The dark energy density explained the initial inflation of the universe and provided a natural explanation of why the vacuum energy density is so small and why it now dominates the expansion and acceleration of the present universe.

The self-interactions of the energy-momentum of the gravitational field are important in the description of dark matter. An exact static solution was obtained from the modified Einstein equation in a restricted spheroidal metric describing the gravitational field outside of its source, which does not contain dark matter. The modified Newtonian force contained two additional terms: one represented the constant dark energy force which depends on the parameter a_1 of the radial component X_1 of the line element field in $\Phi_{\alpha\beta}$ but not the radial variable; the other represented the $\frac{1}{r}$ "dark matter" force which depends on the parameter a_2 of X_2 in $\Phi_{\alpha\beta}$, and $\cos\theta$. The baryonic Tully-Fisher relation was obtained by balancing the dark energy force with the Newtonian force. This condition described the class of galaxies associated with MOND. The Newtonian rotation curves for galaxies with no flat orbital curves, and those with rising rotation curves for large radii were described as examples of the flexibility of the

orbital rotation curve equation. The results obtained from the *complete* Einstein equation thus far are able to substantially describe the missing mass problem attributed to dark matter. Further mathematical and detailed numerical analyses to explore the ability of the energy-momentum tensor of the gravitational field to replace dark matter in cosmology, are fully warranted. This rigorous analysis with comparison to astronomical data may still point to the existence of dark matter to some extent. But even if that is the case, the gravitational role of dark matter is substantially reduced by the impact of the energy-momentum tensor of the gravitational field.

Thus, $\Phi_{\alpha\beta}$ is the symmetric tensor that Einstein sought many years ago. It represents the energy-momentum of the gravitational field itself and explains particular features of dark energy and dark matter.

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