# Modified general relativity

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**Abstract**

A modified Einstein equation of general relativity is obtained by using the principle of least action, a decomposition of symmetric tensors on a time oriented Lorentzian manifold, and a fundamental postulate of general relativity. The decomposition introduces a new symmetric tensor $\Phi_{\alpha\beta}$ which describes the energy-momentum of the gravitational field itself. It completes Einstein's equation and addresses the energy localization problem. The positive part of $\Phi$, the trace of the new tensor with respect to the metric, describes dark energy. The cosmological constant must vanish and is dynamically replaced by $\Phi$. A cyclic universe which developed after the Big Bang is described. The dark energy density provides a natural explanation of why the vacuum energy density is so small, and why it dominates the present epoch of the universe. The negative part of $\Phi$ describes the attractive self-gravitating energy of the gravitational field. $\Phi_{\alpha\beta}$ introduces two additional terms into the Newtonian radial force equation: the force due to dark energy and the $\frac{1}{r}$ "dark matter" force. When the dark energy force balances the Newtonian force, the flat rotation curves and the baryonic Tully-Fisher relation are obtained. The Newtonian rotation curves for galaxies with no flat orbital curves, and those with rising rotation curves for large radii are described as examples of the flexibility of the orbital rotation curve equation.

## 1. Introduction

It has been over a century since Einstein [1] formulated general relativity (GR) in 1915. He was aware that the gravitational field must interact with itself, but was unable to produce a symmetric tensor to properly describe the energy-momentum of the gravitational field. Instead, a non-covariant pseudo-tensor was introduced. However, the difficulties associated with this pseudo-tensor led to the problem of the localization of energy in GR. Over the decades, other pseudo-tensors were developed and different approaches to describe the energy-momentum of the gravitational field were investigated, [2, 3, 4](and references therein) but the energy localization problem still exists today. Despite this deficiency, general relativity is one of the two cornerstones of physics.

GR was developed by Einstein on a four-dimensional Riemannian manifold with a metric that represented the gravitational field and described the geometry of spacetime. Today, we more properly describe spacetime on a time oriented Lorentzian manifold with metric. The Lorentzian metric can be associated with a Riemannian metric by using the line element field, $(\boldsymbol{X}, -\boldsymbol{X})$, that is known to exist mathematically but is rarely used in physics. A classical result in Riemannian geometry, namely the Berger-Ebin theorem [5], can then be be adapted to spacetime. This results in the Orthogonal Decomposition Theorem (ODT): an arbitrary second rank symmetric tensor on a time oriented Lorentzian manifold with a torsionless and metric compatible connection can be orthogonally decomposed into a divergenceless part and a new geometrical tensor, $\Phi_{\alpha\beta}$. It is a symmetric tensor constructed from the Lie derivative of both the metric and the unit vectors collinear with one of the pair of regular vectors in the line element field.

The right hand side of Einstein's famous equation $\frac{8\pi G}{c^4}T_{\alpha\beta} = G_{\alpha\beta} + \Lambda g_{\alpha\beta}$ involves symmetric divergenceless tensors. The left hand side is defined by the variation of the action functional for all matter fields with respect to the metric. This generates a symmetric tensor that is assumed to contain all interactions of the gravitational field with the matter fields, and with itself. However, there is nothing

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in this definition that deals explicitly with the self-interaction of the gravitational field. If we define $\tilde{T}_{\alpha\beta}$ as a symmetric energy-momentum tensor generated from the matter fields without the assumption that it completely describes the self-gravitating field as well, it cannot be locally conserved and would not be divergenceless. Consequently, this second rank symmetric tensor can be locally decomposed by the ODT into a collection of divergenceless tensors and $\Phi_{\alpha\beta}$. Lovelock's theorem [6] proves that in four dimensions, the divergenceless tensors can only consist of the metric and the tensor named after Einstein, $G_{\alpha\beta}$. Therefore, Einstein's equation in a four-dimensional Lorentzian spacetime should be expressed more generally by including the $\Phi_{\alpha\beta}$ term.

It will be proved that $\frac{8\pi G}{c^4}\tilde{T}_{\alpha\beta} = G_{\alpha\beta} + \Lambda g_{\alpha\beta} + \Phi_{\alpha\beta}$ and that the tensor $T_{\alpha\beta} = \tilde{T}_{\alpha\beta} - \frac{c^4}{8\pi G}\Phi_{\alpha\beta}$ is divergenceless which allows Einstein's equation to be recovered. Thus, general relativity is not complete; it is possible to construct a symmetric tensor from the metric and a regular vector field that is independent of the energy-momentum tensor of the matter fields and represents the energy-momentum of the gravitational field itself.

This differs with the presently and generally accepted belief that an energy-momentum tensor for the gravitational field is not necessary and that GR is complete. However, if that notion was true, GR should be able to describe particular features of dark matter. That unfortunately is not the case and is the reason why physicists invented the generally well accepted theory of Lambda cold dark matter ($\Lambda$CDM) to explain, in particular, the flat rotation curves of some galaxies, while leaving GR intact. Modified general relativity can describe those and other galactic rotation curves as discussed in section 7.

Since Lie derivatives have the same form when expressed with covariant or partial derivatives, $\Phi_{\alpha\beta}$ does not vanish when the connection coefficients vanish. The metric can be locally Minkowskian, as in free fall, without affecting $\Phi_{\alpha\beta}$. This contrasts free fall in GR where the connection coefficients vanish locally and the gravitational field is forced to locally disappear; hence the well known notion that the energy of the gravitational field is not localizable [7]. $\Phi_{\alpha\beta}$ has the structure to describe local gravitational energy-momentum. Its trace with respect to the metric, $\Phi$, can describe the self-energy of the gravitational field. In free fall, the effective force of gravity disappears locally but the energy of the gravitational field is intact.

In section 2, the Orthogonal Decomposition Theorem is proved. In section 3, a modified equation of GR is derived by using the principle of least action, the ODT and a fundamental postulate of GR. $\Phi_{\alpha\beta}$ appears naturally alongside the Einstein tensor.

Section 4 discusses the conservation equation for the divergenceless energy-momentum tensor $T_{\alpha\beta} = \tilde{T}_{\alpha\beta} - \frac{c^4}{8\pi G}\Phi_{\alpha\beta}$ where $\tilde{T}_{\alpha\beta}$ is the total matter energy-momentum tensor describing all types of matter including baryonic and dark matter, massive neutrinos, massive gravitons and any other possible particle; if dark matter particles exist. Some additional properties of $\Phi_{\alpha\beta}$ are presented.

In section 5, another interesting result is apparent from the calculation of the interaction of the gravitational field with its source, the energy-momentum tensor. Using the global constraint $\int \Phi\sqrt{-g}d^4x = 0$, it is shown that the cosmological constant $\Lambda$ must vanish, and is dynamically replaced with $\Phi$.

Section 6 is a discussion of the modified Einstein equation of GR in the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, and dark energy. A gravitationally repulsive condition is described by $\Phi > -2\Lambda_d$ where $\Lambda_d$ is the dark energy density. $\Phi > 0$ defines dark energy. Dark energy describes the inflation of the universe immediately after the Big Bang when no matter of any type was present. The dark energy density then tends to the present value of the vacuum energy density. A cyclic universe is born with maximum and minimum values of the cosmological scale factor in the FLRW metric. Dark energy explains the small value of the vacuum energy density and why it now dominates the expansion and acceleration of the present universe.

Cyclic universes have been reported in the literature [8, 9, 10, 11, 12]. Dark energy has been described by various scalar theories such as the quintessential [13], k-essence [14] (and references therein) phantom or quintom theories [15] (and further references therein). Dark energy in this article not a scalar theory; it is the positive part of the energy of the gravitational field.

The negative values of $\Phi$ represent the attractive energy of the gravitational field interacting with itself. This brings into question the subject of dark matter. Since Einstein's equation is incomplete without the tensor $\Phi_{\alpha\beta}$ describing the self-interactions of the gravitational field, the plausibility of dark matter is questionable; its existence is based on the assumption that general relativity is a complete theory.

Although the self-interactions in a weak gravitational field may be extremely small, in the gravitational field of a galaxy, they may be significant enough to explain dark matter.

In section 7, the modified equation of GR is calculated with a spheroidal metric in a region of spacetime outside of matter with the assumption that dark matter does not exist. Two additional terms appear in the modified Newtonian force equation that provide it the flexibility to describe various types of galaxies. By balancing the dark energy force with the Newtonian force, the Tully-Fisher relation is established and the acceleration parameter in MOND is expressed in terms of the dark energy radial force parameter.

## 2. Orthogonal Decomposition of Symmetric Tensors

Curved spacetime is described by the four-dimensional time oriented Lorentzian manifold with a +2 signature metric, $(M, g_{\alpha\beta})$. The connection on the manifold is torsionless and metric compatible. The Lorentzian manifold is assumed to be compact with vanishing Euler-Poincaré characteristic. It admits a smooth regular line element field $(X^\beta, -X^\beta)$ [16, 17, 21].

The orthogonal decomposition of symmetric tensors on Riemannian manifolds has been documented in the literature [5, 18, 19, 20]. Ma and Wang [20] extended their results on a Riemannian manifold to the decomposition of an arbitrary symmetric tensor $w_{\alpha\beta}$ on a four dimensional Minkowskian manifold into

$$w_{\alpha\beta} = v_{\alpha\beta} + \nabla_\alpha \partial_\beta \phi \tag{1}$$

where $v_{\alpha\beta} = v_{\beta\alpha}$, $\nabla^\alpha v_{\alpha\beta} = 0$, $\phi$ is a scalar and the first Betti number $b_1(M) = 0$. However, a more general decomposition of symmetric tensors on a time oriented Lorentzian manifold is required.

**Theorem 2.1.** *Orthogonal Decomposition Theorem (ODT): An arbitrary (0,2) symmetric tensor $w_{\alpha\beta}$ in the symmetric cotangent bundle $S^2T^*M$ on an n-dimensional time oriented Lorentzian manifold $(M, g_{\alpha\beta})$ with a torsionless and metric compatible connection can be orthogonally decomposed as*

$$w_{\alpha\beta} = v_{\alpha\beta} + \Phi_{\alpha\beta} \tag{2}$$

*where $\nabla^\alpha v_{\alpha\beta} = 0$ and $\Phi_{\alpha\beta} = \frac{1}{2}\pounds_X g_{\alpha\beta} + \pounds_X u_\alpha u_\beta$ with $\boldsymbol{X}$ a regular vector field on $M$ and $\boldsymbol{u}$ a timelike unit vector collinear with $\boldsymbol{X}$.*

*Proof.* Let the Lorentzian manifold $(M, g_{\alpha\beta})$ be paracompact, or compact and orientable with vanishing Euler-Poincaré characteristic. A smooth regular line element field $(\boldsymbol{X}, -\boldsymbol{X})$ exists as does a timelike unit vector $\boldsymbol{u}$ collinear with $\boldsymbol{X}$, where $\boldsymbol{X}$ is one of the pair $(\boldsymbol{X}, -\boldsymbol{X})$. Let M be endowed with a Riemannian metric $g^+_{\alpha\beta}$. In accordance with [17, 21], the Lorentzian metric $g_{\alpha\beta}$ is associated with an adapted Riemannian metric $g^+_{\alpha\beta}$ and unit vector $\boldsymbol{u}$ by setting

$$g_{\alpha\beta} = g^+_{\alpha\beta} - 2u_\alpha u_\beta. \tag{3}$$

Let $w_{\alpha\beta}$ and $v_{\alpha\beta}$ belong to $S^2T^*M$, the cotangent bundle of symmetric $(0,2)$ tensors on M. In the Riemannian open subset of $S^2T^*M$ which contains $g^+_{\alpha\beta}$, an arbitrary $(0,2)$ symmetric tensor $w_{\alpha\beta}$ can be orthogonally decomposed by the Berger-Ebin theorem [5] according to

$$w_{\alpha\beta} = v_{\alpha\beta} + \frac{1}{2}\pounds_X g^+_{\alpha\beta} \tag{4}$$

where ${\nabla^+}^\alpha v_{\alpha\beta} = 0$. Given the unit vector field $\boldsymbol{u}$ collinear with $\boldsymbol{X}$, the Riemannian connection $\nabla^+$ on M is the same as the Lorentzian connection $\nabla$ on M because both connections have the same geodesics with the same parameterization and both connections are torsionless [22]. Hence,

$$\begin{aligned} w_{\alpha\beta} &= v_{\alpha\beta} + \frac{1}{2}\pounds_X g_{\alpha\beta} + \pounds_X u_\alpha u_\beta \\ &= v_{\alpha\beta} + \Phi_{\alpha\beta} \end{aligned} \tag{5}$$

where $\nabla^\alpha v_{\alpha\beta} = 0$ and

$$\Phi_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha X_\beta + \nabla_\beta X_\alpha) + u^\lambda(u_\alpha \nabla_\beta X_\lambda + u_\beta \nabla_\alpha X_\lambda). \tag{6}$$

■ ■

## 3. Derivation of the modified equation of general relativity

A modified equation of general relativity of the form $C_{\alpha\beta} = 0$ is sought which contains a linear combination of symmetric tensors that define the Einstein equation, and a new tensor which can describe the energy-momentum of the gravitational field itself. This can be achieved by using the principle of least action, the Orthogonal Decomposition Theorem (2), and a fundamental postulate of GR.

Firstly, the field equations contained in $C_{\alpha\beta}$ which are sought to describe general relativity, dark matter, dark energy and the energy-momentum of the gravitational field, must be derivable from the action functional

$$S = S^F + S^{EH} + S^G \tag{7}$$

where $S^F$ and $L^F$ refer to the action and Lagrangian, respectively, for all types of matter fields including those of dark matter if dark matter particles exist. $S^{EH}$ is the Einstein-Hilbert action for general relativity and $S^G$ is the action for the energy-momentum of the gravitational field with Lagrangian $L^G$. The variation of $S^F$ with respect to $g^{\alpha\beta}$

$$\delta S^F = \int (\frac{\delta L^F}{\delta g^{\alpha\beta}} - \frac{1}{2} L^F g_{\alpha\beta}) \delta g^{\alpha\beta} \sqrt{-g} d^4x \tag{8}$$

generates the symmetric energy-momentum tensor $\tilde{T}_{\alpha\beta}$ which represents the interaction of all types of matter fields and associated radiation in a gravitational field but does not specifically include the self-interaction of the gravitational field:

$$\tilde{T}_{\alpha\beta} := -2c(\frac{\delta L^F}{\delta g^{\alpha\beta}} - \frac{1}{2} L^F g_{\alpha\beta}). \tag{9}$$

$C_{\alpha\beta}$ must then be expressed as

$$C_{\alpha\beta} = \frac{a}{c} \tilde{T}_{\alpha\beta} + b w_{\alpha\beta} \tag{10}$$

where $w_{\alpha\beta}$ is an unknown symmetric tensor independent of $\tilde{T}_{\alpha\beta}$; and a and b are arbitrary constants.

Secondly, $w_{\alpha\beta}$ can be orthogonally decomposed by the ODT into

$$w_{\alpha\beta} = v_{\alpha\beta} + \Phi_{\alpha\beta} \tag{11}$$

where $\Phi_{\alpha\beta}$ is given by (6) and $\nabla^\alpha v_{\alpha\beta} = 0$.

Thirdly, Einstein concluded [1] that the metric should describe both the geometry of spacetime and the gravitational field. He postulated the totality of the matter energy-momentum tensor and the energy-momentum of the gravitational field, to be the source of the gravitational field. Adhering to this philosophy, the energy-momentum tensor $T_{\alpha\beta}$ describing the totality of all types of matter and the energy-momentum of the gravitational field, including dark energy, is postulated to be the source of the gravitational field.

$\Phi_{\alpha\beta}$ is independent of $\tilde{T}_{\alpha\beta}$ and is not divergenceless. $\Phi_{\alpha\beta}$ is therefore the sole candidate to describe the energy-momentum of the gravitational field. Thus,

$$T_{\alpha\beta} = \tilde{T}_{\alpha\beta} + \frac{bc}{a} \Phi_{\alpha\beta} \tag{12}$$

and the interaction of the gravitational field with its energy-momentum tensor can be defined with the action

$$S^G := -b \int g_{\alpha\beta} \Phi^{\alpha\beta} \sqrt{-g} d^4x. \tag{13}$$

It was proved by Lovelock [6] that the only tensors in a four-dimensional spacetime which are symmetric, divergence free, and a concomitant of the metric tensor together with its first two derivatives are the metric and the Einstein tensor, $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$. $v_{\alpha\beta}$ must therefore contain the Lovelock tensors.

$C_{\alpha\beta}$ is then formally decomposed as

$$C_{\alpha\beta} = \frac{a}{c} T_{\alpha\beta} + b v_{\alpha\beta} \tag{14}$$

with $\nabla_\alpha v^{\alpha\beta} = 0$ and $v_{\alpha\beta} := G_{\alpha\beta} + \Lambda g_{\alpha\beta}$. $\Lambda$ *is an integration constant* (in hindsight identified as the cosmological constant). With the collection of tensors $C_{\alpha\beta}$ defined to vanish, we obtain the modified Einstein equation of general relativity with cosmological constant $\Lambda$ and the gravitational energy-momentum term $\Phi_{\alpha\beta}$

$$-\frac{8\pi G}{c^4}\tilde{T}_{\alpha\beta} + R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta} + \Phi_{\alpha\beta} = 0 \tag{15}$$

by setting $a = -\frac{1}{2}$ and $b = \frac{c^3}{16\pi G}$.

Ma and Wang [23] obtained a similar result to (15) but with an entirely different $\Phi_{\alpha\beta}$. They postulated $\nabla_\alpha(\Phi^{\alpha\beta} + \frac{8\pi G}{c^4}T^{\alpha\beta}_{matter}) = 0$ with $\Phi_{\alpha\beta} = \nabla_\alpha\partial_\beta\phi$ for some scalar $\phi$ by using the decomposition (1).

Equation (15) must be derived from the action functional (7). With (13):

$$\begin{aligned} S &= S^F + S^{EH} + S^G \\ &= \int L^F(A^\beta, \nabla^\alpha A^\beta, ..., g^{\alpha\beta})\sqrt{-g}d^4x \quad + b\int (R - 2\Lambda)\sqrt{-g}d^4x - b\int \Phi_{\alpha\beta}g^{\alpha\beta}\sqrt{-g}d^4x. \end{aligned} \tag{16}$$

To calculate the variation of $S^G$ with respect to the inverse metric $g^{\alpha\beta}$, the following results are used: $g^{\alpha\beta} = g^{+\alpha\beta} - 2u^\alpha u^\beta$ is the inverse of $g_{\alpha\beta}$; $\nabla_\mu(u_\alpha u_\beta) = 0$; $g^{+\alpha\beta}\delta g_{\alpha\beta} = -g_{\alpha\beta}\delta g^{+\alpha\beta}$; $\delta g^{+\rho\beta} = -g^{+\alpha\beta}g^{\lambda\rho}\delta g^{+}_{\alpha\lambda}$; $g_{\alpha\beta}\delta(u^\alpha u^\beta) = u_\alpha u_\beta \delta g^{\alpha\beta}$; and $\delta(u^\alpha u^\beta) = u_\lambda u^\beta \delta g^{\lambda\alpha}$. The variation of S with respect to $g^{\alpha\beta}$ is then

$$\begin{aligned} \delta S = \int [-\frac{1}{2c}\tilde{T}_{\alpha\beta} + b(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R) + b\Lambda g_{\alpha\beta} + b(\nabla_\alpha X_\beta + 2u^\lambda u_\beta \nabla_\alpha X_\lambda \\ + \nabla_\mu X_\nu(-u_\alpha u_\beta g^{\mu\nu} + u^\mu u^\nu g_{\alpha\beta}))]\delta g^{\alpha\beta}\sqrt{-g}\ d^4x \end{aligned} \tag{17}$$

after calculating $\delta\Gamma^\lambda_{\alpha\beta}$ induced by the variations in the inverse metric, and integrating by parts several times. The last term vanishes which follows by writing the tensor in brackets, $-u_\alpha u_\beta g^{\mu\nu} + u^\mu u^\nu g_{\alpha\beta}$, as its equivalent, $\frac{1}{2}(g^{+\mu\nu}g_{\alpha\beta} - g^{+}_{\alpha\beta}g^{\mu\nu})$; and choosing an orthonormal basis $(e_\alpha)$ at a point $p \in M$ for $g^+$ with $e_0 = u$. Then, $u^0u_0 = 1$, $u^iu_i = 0$, $g^+_{\alpha\beta} = \delta_{\alpha\beta}$, $g^{00} = -g^{+00}$ and $g_{00} = -g^+_{00}$, with all other components of the metric g equal to those of the metric $g^+$. Since $\delta g^{\alpha\beta}$ is symmetric, the second last term can be expressed as $b\Phi_{\alpha\beta}$. With $\delta S = 0$ and arbitrary variations $\delta g^{\alpha\beta}$, we have

$$-\frac{1}{2c}\tilde{T}_{\alpha\beta} + b(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R) + b\Lambda g_{\alpha\beta} + b\Phi_{\alpha\beta} = 0. \tag{18}$$

Setting $b = \frac{c^3}{16\pi G}$ yields the modified Einstein equation described in (15).

$\Phi_{\alpha\beta}$ adds $X^\beta$ from the line element field as a dynamical variable independent of the metric. The action $S^G$ from (13) is then

$$S^G = \int [\nabla_\alpha X^\mu(g^{\alpha\beta} + 2u^\alpha u^\beta)g_{\beta\mu} - \lambda(u^\alpha u^\mu g_{\mu\alpha} + 1)]\sqrt{-g}d^4x \tag{19}$$

where $u^\beta$ is collinear with $X^\beta$ and $\lambda > 0$ is a Lagrange multiplier. Varying with respect to $X^\mu$ yields the equation

$$X^\beta\nabla_\mu X_\beta + X^\alpha\nabla_\alpha X_\mu - \lambda X_\mu = 0. \tag{20}$$

The second term of (20) is a geodesic of the form $X^\alpha\nabla_\alpha X_\mu = \kappa X_\mu$ where $\kappa$ is an arbitrary function on the geodesic curve. However, it vanishes if the curve is affinely reparameterized and the geodesic term can be set to zero. By setting $X^\beta = fu^\beta$, where $f > 1$ is a scalar representing the length of the regular vector $X^\beta$, the constraint

$$X^\beta X_\beta = -f^2 \tag{21}$$

must be satisfied because $u^\beta u_\beta = -1$. Using (21), it follows that in an affine parameterization

$$X_\mu = -\frac{1}{\lambda}f\partial_\mu f = -\partial_\mu F \tag{22}$$

where

$$F := \frac{1}{2\lambda}f^2. \tag{23}$$

Some properties of $F$ are discussed in the next section.

## 4. The conserved energy-momentum tensor

The invariance of the action functional describing gravity, dark energy and total matter fields under the symmetry of diffeomorphisms demands a symmetric divergenceless energy-momentum tensor

$$T^{\alpha\beta} = \tilde{T}^{\alpha\beta} - \frac{c^4}{8\pi G}\varPhi^{\alpha\beta}. \tag{24}$$

This follows from an analysis of each term in the action functional $S$ defined in (7). The action $S^{EH}$ is independently invariant under a diffeomorphism. Variation of the action $S^F$ with respect to the metric contains only $\tilde{T}^{\alpha\beta}$ because the variations of $S^F$ with respect to each field and its derivatives vanish with the corresponding Euler-Lagrange equations. Variation of $S^G$ with respect to the metric yields $\varPhi^{\alpha\beta}$. Therefore, we can write

$$\int (-\frac{1}{2c}\tilde{T}^{\alpha\beta} + b\varPhi^{\alpha\beta})\delta g_{\alpha\beta}\sqrt{-g}d^4x = 0 \tag{25}$$

where $b = \frac{c^3}{16\pi G}$. As the Lie derivative along a regular vector $Y^\beta$ generates the infinitesimal change in a tensor under a diffeomorphism, $\delta g_{\alpha\beta} = \pounds_Y g_{\alpha\beta} = \nabla_\alpha Y_\beta + \nabla_\beta Y_\alpha$. Integrating by parts then gives

$$\int \nabla_\alpha(-\frac{1}{2c}\tilde{T}^{\alpha\beta} + b\varPhi^{\alpha\beta})Y_\beta\sqrt{-g}d^4x = 0 \tag{26}$$

which requires

$$\nabla_\alpha T^{\alpha\beta} = 0 \tag{27}$$

for diffeomorphisms generated by $Y^\beta$.

Equation (27) is the local description of the conservation of energy and momentum in a modified theory of GR described by (15). The gravitational field has an intrinsic energy-momentum which is attributed to $\varPhi_{\alpha\beta}$. Being independent of $\tilde{T}_{\alpha\beta}$, $\frac{c^4}{8\pi G}\varPhi_{\alpha\beta}$ provides the additional energy and momentum from the gravitational field necessary to complete the source $T_{\alpha\beta}$ of the geometry of spacetime. $\varPhi_{\alpha\beta}$ completes the Einstein equation and leaves it intact in form:

$$\frac{8\pi G}{c^4}T_{\alpha\beta} = G_{\alpha\beta} + \Lambda g_{\alpha\beta}. \tag{28}$$

$\varPhi_{\alpha\beta}$ is expressed in terms of the Lie derivatives by (5). Since Lie derivatives have the same form when expressed with covariant or partial derivatives, $\varPhi_{\alpha\beta}$ does not vanish when the connection coefficients vanish. The metric can be locally Minkowskian, as in free fall, without affecting $\varPhi_{\alpha\beta}$. It has the structure to describe local gravitational energy-momentum. $\varPhi_{\alpha\beta}$ vanishes if the gravitational field is constant along an integral curve in spacetime. In general this is not the case because the gravitational field can interact with itself; even weak gravitational fields can self-gravitate.

It is straightforward to calculate the coupling of the gravitational field with its energy-momentum tensor:

$$\int g_{\alpha\beta}\varPhi^{\alpha\beta}\sqrt{-g}d^4x = \int \Phi\sqrt{-g}d^4x = 0 \tag{29}$$

where $\Phi = \nabla_\alpha X_\beta(g^{\alpha\beta} + 2u^\alpha u^\beta)$. Equation (29) means the scalar $\Phi$ has local positive and negative values, all of which add to zero when integrated over the entire spacetime. $\Phi$ is globally conserved. Section 6 demonstrates that the positive values of $\Phi$ are attributed to the gravitationally repulsive properties of dark energy with the cosmological constant set to zero. The negative values represent the attractive energy of the gravitational field interacting with itself. $\Phi$ is measurable; it can be expressed in terms of the density and pressure of total matter and the vacuum energy density as shown in section 6. The energy of the gravitational field is therefore localizable.

$X_\mu$ in (22) is a regular covector normal to the hypersurface $F = \frac{1}{2\lambda}f^2$ which can be expressed in terms of the self-energy of the gravitational field $\Phi$, as follows. From (22),

$$\nabla_\alpha X_\beta = \nabla_\beta X_\alpha \tag{30}$$

and $\Phi_{\alpha\beta} = \nabla_\alpha X_\beta + \frac{1}{f} u_\alpha X^\lambda \nabla_\lambda X_\beta + \frac{1}{f} u_\beta X^\lambda \nabla_\lambda X_\alpha$, which simplifies to

$$\Phi_{\alpha\beta} = \nabla_\alpha X_\beta \tag{31}$$

in the affine parameterization used in (20). The condition for $X_\beta$ to be hypersurface orthogonal is $X_{[\alpha;\beta} X_{\lambda]} = 0$ which is satisfied by $\nabla_\alpha X_\beta = \frac{1}{2}(X_\alpha u_\beta + X_\beta u_\alpha)$. Then

$$\Phi_{\alpha\beta} \Phi^{\alpha\beta} = f^2. \tag{32}$$

The tensor $\Phi_{\alpha\beta}\Phi_{\mu\nu} g^{\mu\alpha} g^{\nu\beta} - \Phi_{\alpha\beta}\Phi_{\mu\nu} g^{\alpha\beta} g^{\mu\nu}$ vanishes in Minkowski spacetime, so in general

$$F = \frac{1}{2\lambda}\Phi^2, \ \mid \Phi \mid > 1. \tag{33}$$

The entropy of a continuous probability density function $p(f)$ of a random variable $f$ on an interval $I$ is defined by

$$h(p) = -\int_I p ln p df. \tag{34}$$

With a probablility distribution function defined by the truncated Gaussian

$$p(f) = \begin{cases} e^{-\frac{1}{2}f^2}, \text{if } f \text{ is in I} \\ 0, \text{otherwise} \end{cases} \tag{35}$$

and $I = [f^\star - \Delta, f^\star + \Delta]$ being the closed interval around $f^\star$, the mean value theorem demands

$$h(p) = \Delta e^{-\frac{1}{2}{f^\star}^2} {f^\star}^2. \tag{36}$$

Choosing $\Delta = \frac{1}{2} e^{\frac{1}{2}{f^\star}^2}$ gives

$$h(p) = -\ln p^\star \tag{37}$$

where $p^\star := p(f^\star)$. Thus, $\lambda F$ is identical to (37) if $f^\star$ is the magnitude of the vector that the metric traverses in the interval $I$, which is always possible given the myriad of regular vectors in $I$. The Lie derivative with respect to $X^\beta$ is then the flow of the metric along $X^\beta$ with a magnitude of $f = f^\star$:

$$F = -\frac{1}{\lambda}\ln p(f). \tag{38}$$

$F$ has a similar probability structure to that of a microcanonical ensemble but has a nonlinear probability density function which generates the nonlinear term in the self-energy of the gravitational field as compared to the linear energy term in the microcanonical ensemble. The nonlinear characteristic of $\Phi$ in $F$ is to be expected because the gravitational field gravitates. It follows from properties (22) and (38) that $F$ is an extensive variable that depends on the self-energy of the gravitational field; and $dF > 0$. Thus, $F$ may be interpreted as the entropy of the gravitational field.

## 5. Cosmological Constant

The metric cannot appear as a field variable alongside the cosmological constant. $\Lambda$ is an integration constant and must vanish.

**Theorem 5.1.** *The cosmological constant $\Lambda$ must vanish and is dynamically replaced by the trace of $\Phi_{\alpha\beta}$.*

*Proof.* The action $S_g^{int}$ for the coupling of the energy-momentum tensor with the metric is:

$$\begin{aligned} S_g^{int} &= -\frac{1}{2c}\int T^{\alpha\beta} g_{\alpha\beta}\sqrt{-g}d^4x \\ &= \frac{c^3}{16\pi G}\int (R - 4\Lambda)\sqrt{-g}d^4x. \end{aligned} \tag{39}$$

Since the energy-momentum tensor for all types of matter and the energy-momentum of the gravitational field, is the source of the gravitational field and the geometry of spacetime, $S_g^{int} \equiv S^{EH}$. Comparing (39) to (16) requires the integration constant $\Lambda$ to vanish.

Using (29), $S^{EH}$ with $\Lambda = 0$ can then be written as

$$S^{EHG} = \frac{c^3}{16\pi G}\int (R - \Phi)\sqrt{-g}d^4x \tag{40}$$

which generates the modified Einstein equation with no cosmological constant from (16). If $\Phi = 2\Lambda$ *locally*, the Einstein equation with the cosmological constant is obtained accordingly. The trace of the tensor describing the energy-momentum of the gravitational field, dynamically replaces the cosmological constant but must obey the global equation (29). ■ ■

## 6. Energy-momentum of the gravitational field in the FLRW metric: Dark energy

Some properties of $\Phi_{\alpha\beta}$ in the Friedmann-Lemaître-Robertson-Walker metric are now investigated. The FLRW metric is typically used to describe a spatially maximal symmetric universe according to the cosmological principle [24] whereby the universe is homogeneous and isotropic when measured on a large scale. This metric is given by

$$ds^2 = -c^2dt^2 + a(t)^2[\frac{1}{1-kr^2}dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2)] \tag{41}$$

where $a(t)$ is the cosmological scale factor which satisfies $a > 0$ after the Big Bang at $t = 0$. $k$ is a constant used to describe a particular spatial geometry. The connection components of the FLRW metric are

$$\Gamma^i_{j0} = \frac{\dot{a}}{ca}\delta^i_j,\ \ \Gamma^0_{ij} = \frac{\dot{a}}{ca}g_{ij},\ \ \Gamma^\mu_{00} = 0 \tag{42}$$

where $j = 1, 2, 3$. The Ricci tensor components are

$$R_{00} = -3\frac{\ddot{a}}{ac^2},\ \ R_{ij} = (\frac{\ddot{a}}{ac^2} + 2\frac{\dot{a}^2}{a^2c^2} + 2\frac{k}{a^2})g_{ij} \tag{43}$$

and the Ricci scalar is

$$R = \frac{6}{a^2c^2}(a\ddot{a} + \dot{a}^2 + kc^2). \tag{44}$$

It was proved in [24] that a maximally spatial form invariant symmetric second rank tensor $B_{\alpha\beta}$ has components in the form

$$B_{00} = \varrho(t),\ \ B_{0j} = 0,\ \ B_{ij} = p(t)g_{ij} \tag{45}$$

where $\varrho(t)$ and $p(t)$ are arbitrary functions of time. We therefore set,

$$\tilde{T}_{00} = c^2\varrho,\ \ \tilde{T}_{ij} = pg_{ij},\ \ \tilde{T}^\mu_\mu = -c^2\varrho + 3p \tag{46}$$

where $\varrho(t)$ and $p(t)$ are designated as the mass density and pressure functions, respectively, of total matter including dark matter; if dark matter particles exist. Similarly,

$$\Phi_{00} = \Lambda_d,\ \ \Phi_{ij} = \frac{P_d}{c^2}g_{ij},\ \ \Phi^\mu_\mu = -\Lambda_d + 3\frac{P_d}{c^2} \tag{47}$$

where $\Lambda_d(t)$ and $P_d(t)$ refer to the energy density and pressure, respectively, of the tensor describing the energy-momentum of the gravitational field.

To obtain the Friedmann equations, we use the trace of the modified Einstein equation

$$-\frac{8\pi G}{c^4}\tilde{T} - R + \Phi = 0 \tag{48}$$

to rewrite the modified Einstein equation as

$$R_{\alpha\beta} = \frac{8\pi G}{c^4}(\tilde{T}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\tilde{T}) + \frac{1}{2}g_{\alpha\beta}\Phi - \varPhi_{\alpha\beta} \tag{49}$$

from which we obtain

$$3\frac{\ddot{a}}{a} = -4\pi G(\varrho + \frac{3p}{c^2}) + \frac{1}{2}c^2\Lambda_d + \frac{3}{2}P_d \tag{50}$$

from the $R_{00}$ component. The $R_{11}$ component gives

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2kc^2}{a^2} = \frac{c^2}{2}(-\Lambda_d + \frac{P_d}{c^2}) + 4\pi G(\varrho - \frac{p}{c^2}) \tag{51}$$

and the conservation law for $T^{\alpha\beta}$ yields

$$\dot{\varrho} - \frac{c^2}{8\pi G}\dot{\Lambda}_d = -3\frac{\dot{a}}{a}(\varrho + \frac{p}{c^2} - \frac{c^2}{8\pi G}(\Lambda_d + \frac{P_d}{c^2})). \tag{52}$$

Inserting (50) into (51) produces a simpler equation

$$\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\varrho - \frac{1}{3}c^2\Lambda_d. \tag{53}$$

Equations (50) and (53) are the Friedmann equations modified with $\varPhi_{\alpha\beta}$.

From (50), we immediately see that $\Phi + 2\Lambda_d = \Lambda_d + \frac{3}{c^2}P_d > 0$ tends to accelerate the universe; while all types of matter combined, with a positive mass density and pressure, tend to decelerate the universe. $\Phi > -2\Lambda_d$ is a gravitationally repulsive condition which relates dark energy to $\Lambda_d$. Hence, $\Lambda_d$ is called the dark energy density and $P_d$ the dark energy pressure. $\Phi$ tends to accelerate or decelerate the universe but has a net zero effect on it. $\varPhi_{\alpha\beta}$ and therefore $\Phi$, provide the flexibility to describe various eras in the evolution of the universe. The cosmological constant $\Lambda$, on the other hand, can be expressed as a fixed negative energy density which would have tended to accelerate the universe during *all* epochs.

One of the recent challenges in cosmology has been to find a natural mechanism that describes a small but positive vacuum energy density to explain the observed acceleration of the present universe. Dark energy provides a natural explanation of this challenge without the need of a cosmological constant.

After the discovery in 1929 by Hubble [25] that the universe was expanding, $\Lambda$ was not required to obtain a static solution to the Einstein equations with a positive mass density. Since the cosmological constant was vastly smaller than any value predicted by particle theory, most particle theorists simply assumed, that for some unknown reason, this quantity was zero [26]. This was widely believed to be true until the discovery of the presently accelerating universe in 1998-99 [27, 28]. $\Lambda$ was then considered to be associated with the dark energy conundrum. However, it is just an integration constant in the modified Einstein equation. $\Lambda$ *must* vanish and is replaced by $\Phi$ as proved in theorem 5.1. This is readily verified by restricting the dark energy variables to the constant values $\Lambda_d = -\Lambda$ and $P_d = c^2\Lambda$ in (50) and (53). The Friedmann equations with the cosmological constant $\Lambda$ are then recovered in accordance with theorem 5.1.

The Friedmann equations are now considered with $k = 1$ describing a closed universe:

$$\dot{a}^2 = \frac{8\pi G}{3}\varrho a^2 - \frac{c^2}{3}\Lambda_d a^2 - c^2 \tag{54}$$

and

$$\ddot{a} = -\frac{4\pi G}{3}a(\varrho + \frac{3p}{c^2}) + \frac{ac^2}{6}(\Lambda_d + \frac{3}{c^2}P_d). \tag{55}$$

To avoid confusion with $\Lambda$, we will denote the constant vacuum energy density as $\Lambda_v$ with the property $\Lambda_v > 0$. In the present epoch, $\Lambda_v$ is measured to be $\approx 1.3 \times 10^{-52}m^{-2}$. By defining

$$\tilde{\varrho} = 8\pi G\varrho - c^2\Lambda_d \tag{56}$$

and

$$\tilde{p} = -\frac{4\pi G}{c^2}p + \frac{1}{2}P_d, \tag{57}$$

these equations can be simplified to

$$\dot{a}^2 = \frac{\tilde{\varrho} a^2}{3} - c^2, \tag{58}$$

and

$$\ddot{a} = a(-\frac{\tilde{\varrho}}{6} + \tilde{p}) \tag{59}$$

with the conservation equation

$$\dot{\tilde{\varrho}} = -3\frac{\dot{a}}{a}(\tilde{\varrho} - 2\tilde{p}). \tag{60}$$

Unless otherwise stated, $\varrho > 0$ and $p > 0$. Equation (58) requires $\tilde{\varrho} > 0$.

It is interesting to explore how the energy-momentum of the gravitational field can describe critical features of a Big Bang universe. Immediately after the event of the Big Bang, the universe violently accelerates and $\dot{a} > 0$. For a very short time, there is no matter; $\varrho = 0$ and $p = 0$. In this very early stage of the evolution of the universe, it is possible that the constant vacuum energy density developed. If we set $\varrho = 0$ in (54), the inequality

$$\Lambda_d < -\frac{3}{a^2} \tag{61}$$

must hold. From (52) and (55) with $\dot{a} \neq 0$,

$$\frac{d}{da}\Lambda_d = -\frac{2\Lambda_d}{a} - \frac{6\ddot{a}}{a^2c^2}. \tag{62}$$

If $\Lambda_d \longrightarrow -\Lambda_v$ and $\frac{P_d}{c^2} \longrightarrow \Lambda_v$ just after the Big Bang, (55) requires $\frac{\ddot{a}}{a}$ to be constant. With those assumptions, equation (62) has the solution

$$\Lambda_d = \frac{c_1}{a^2} - \frac{3\ddot{a}}{ac^2} \tag{63}$$

where $c_1$ is an arbitrary constant. Setting $c_1 = -3$ and $\Lambda_v = \frac{3\ddot{a}}{ac^2}$,

$$\Lambda_d = -\frac{3}{a^2} - \Lambda_v \tag{64}$$

which satisfies (61) and tends to $-\Lambda_v$ as the universe expands. Dark energy can generate $\Lambda_v$ during this epoch of the universe. The expansion of the universe is then described by

$$\dot{a}^2 = \frac{1}{3}a^2c^2\Lambda_v. \tag{65}$$

The pressure density of dark energy is $P_d = \frac{c^2}{a^2} + \Lambda_v c^2$ and the acceleration of the universe is

$$\ddot{a} = \frac{ac^2\Lambda_v}{3}. \tag{66}$$

The scalar $\Phi = \frac{6}{a^2} + 4\Lambda_v$ is positive. $\Phi > 0$ is the condition to be satisfied for an expanding and accelerating universe when no matter is present. Because this result depends entirely on dark energy, $\Phi > 0$ defines dark energy.

With all types of matter appearing after the initial inflation, $\Lambda_d$ must obey the constraint

$$\Lambda_d < -\frac{3}{a^2} + \frac{8\pi G\varrho}{c^2}. \tag{67}$$

With constant total matter, the equation

$$\frac{d}{da}\Lambda_d = -\frac{2\Lambda_d}{a} + \frac{16\pi G\varrho}{ac^2} - \frac{6\ddot{a}}{a^2c^2} \tag{68}$$

is obtained from (52) and (55) with $\dot{a} \neq 0$. Since $\frac{\ddot{a}}{a} = \frac{d}{dt}(\frac{\dot{a}}{a}) + (\frac{\dot{a}}{a})^2$, a slowly varying non-zero Hubble parameter $\frac{\dot{a}}{a}$ requires $\frac{\ddot{a}}{a}$ to be approximately constant. With that assumption, equation (68) has the solution

$$\Lambda_d = -\frac{3}{a^2} + \frac{8\pi G\varrho}{c^2} - \Lambda_v \tag{69}$$

with

$$\Lambda_v = \frac{3\ddot{a}}{ac^2}. \tag{70}$$

The dark energy pressure is

$$P_d = \frac{c^2}{a^2} + \frac{8\pi G p}{c^2} + c^2\Lambda_v. \tag{71}$$

A pure dark energy effect returns (65) and (66) as the expansion and acceleration, respectively. In a universe with essentially constant matter, which is assumed to be the case of the present era, this demonstrates why $\Lambda_v$ is important. As expected, $\Phi = \frac{6}{a^2} + 4\Lambda_v + \frac{8\pi G}{c^2}(-\varrho + \frac{3p}{c^2})$ is positive or negative; the attraction of the gravitational field itself is evident.

Riess et al. [29] used the Hubble telescope to provide the first conclusive evidence for cosmic deceleration that preceded the current epoch of cosmic acceleration. Given the violent acceleration after the Big Bang, this observation evidences the cyclic nature of the universe to this point in time. The cosmological scale factor must have had maximum and minimum values in the past because of the observed changes in sign of its second derivative; there were extremums at $\dot{a} = 0$. In general, this requires $\Lambda_d = -\frac{3}{a^2} + \frac{8\pi G\varrho}{c^2}$ from equation (54). The Hubble parameter vanishes and (69) must change because (70) is not constant at the extremum. Dark energy in the amount of $\Lambda_v$ must be transferred to $\Lambda_d$ from the dark energy pressure; $\frac{P_d}{c^2}$ in (71) decreases by $-\Lambda_v$ with an offsetting change by that amount to $\Lambda_d$ in (69). This allows an extremum to occur while keeping $\Phi$ unchanged. Then, cosmic acceleration can change to a decelerating epoch, and conversely with the opposite exchange of dark energy.

The maxima or minima of the cosmological scale factor follows directly from equations (58) and (59). The second derivative of $a$ must satisfy

$$\ddot{a} = a(-\frac{c^2}{2a^2} + \tilde{p}) \tag{72}$$

when $\dot{a} = 0$. The value of $\tilde{p}$ in equation (72) governs the condition for a maximum or minimum of $a$. With $-\Lambda_d$ having a small fixed value of $\Lambda_v$ determined early in the evolution of the universe, the variation in $\Phi$ is determined by $P_d$. The constraint (29) on $\Phi$ can force $P_d$ to change, which can change the sign of $\tilde{p}$. Near the end of an acceleration phase, if the dark energy pressure decreases so that $P_d \leq \frac{8\pi G p}{c^2}$, $\tilde{p}$ changes from positive to zero or negative, and the scale factor has a maximum value at $a_{max}$; $\tilde{p} \leq 0$ is satisfied in (72). The acceleration phase ends and the universe undergoes a deceleration. The scale factor then decreases toward a minimum value $a_{min}$ at which the dark energy pressure increases enough to satisfy $\tilde{p} > \frac{c^2}{2a^2}$. The deceleration phase changes to that of an acceleration and the cyclic process continues indefinitely. $\Phi$, governed by (29), smoothly controls the maximum and minimum values that the cosmological scale factor can have. The global constraint on $\Phi$ keeps the universe gravitationally in balance. This model of the universe starts with the Big Bang and then cycles to eternity. It does not suffer the catastrophes of the Big Crunch or the Big Rip.

Although recent data and analysis [30] suggests the observable universe is flat, the data likely represents a small fraction of the presently unknown *entire* universe. If the entire universe has a positive curvature, a measurement of it will appear to be nearly flat if data from large enough distances is not available. Therefore, at this time, the conjecture of a flat universe which expands forever based on observational evidence is less likely than the cyclic universe described and observed after the Big Bang and into this epoch.

Dark energy thus provides a natural explanation of why the vacuum energy density is so small, and why it dominates the present epoch of the universe.

## 7. Energy-momentum of the gravitational field: Dark matter

The ΛCDM model describes the formation of galaxies after the Big Bang from cooled baryonic matter gravitationally attracted into a dark matter skeleton. Dark matter in the ΛCDM model also provides the additional mass required to describe the flat rotation curves observed in many galaxies. However, no dark matter particles have been detected and there have been several attempts to explain the flat rotational curves without dark matter.

The leading candidate is a *phenomenological* model of Modified Newtonian dynamics (MOND) introduced by Milgrom [31]. The Newtonian force F is modified according to

$$F = m\mu(\frac{a}{a_0})a \tag{73}$$

where $a_0$ is a fundamental acceleration $\approx 1.2\times10^{-10}m/s^2$. $\mu$ is a function of the ratio of the acceleration relative to $a_0$ which tends to one for $a \gg a_0$ and tends to $\frac{a}{a_0}$ for $a \ll a_0$. MOND successfully explains many, but not all, mass discrepancies observed in galactic data. However, it has no covariant roots in Einstein's equation or cosmological theory. MOND and ΛCDM were thoroughly discussed by McGaugh in [32].

Other alternatives to dark matter were reviewed by Mannheim in [33] with references therein. In particular, Moffat [34] used a nonsymmetric gravitational theory without dark matter to obtain the flat rotation curves of some galaxies. The bimetric theory of Milgrom [35] involved two metrics as independent degrees of freedom to obtain a relativistic formulation of MOND.

Different approaches to the missing matter problem include dipolar dark matter, which was introduced by Bernard, Blanchet and Heisenberg in [36] to solve the problems of cold dark matter at galactic scales and reproduce the phenomenology of MOND. The theory involves two different species of dark matter particles which are separately coupled to the two metrics of bigravity and are linked together by an internal vector field. In [37], a theory of emergent gravity (EG) which claims a possible breakdown in general relativity, was introduced by Verlinde that provided an explanation for Milgrom's phenomenological fitting formula in reproducing the flattening of rotation curves. Campigotto, Diaferio and Fatibenec [38] showed conformal gravity cannot describe galactic rotation curves without the aid of dark matter. On the other hand, a logical analysis based on observational data was presented by Kroupa in [39] to support the conjecture that dark matter does not exist.

The existence of dark matter is based on the assumption that general relativity is correct. However, Einstein's equation is incomplete without the tensor $\Phi_{\alpha\beta}$ describing the self-interactions of the gravitational field. The validity of modified general relativity is now tested with the attempt to describe the additional gravitational attraction in various galaxies without dark matter.

## 7.1. Modified GR in a spheroidal spacetime

It is assumed dark matter does not exist and that baryonic matter and other possible sources of matter such as neutrinos, produce the gravitational field. In a region of spacetime where there is no matter, $\tilde{T}_{\alpha\beta} = 0$ and the field equations must satisfy

$$G_{\alpha\beta} + \Phi_{\alpha\beta} = 0. \tag{74}$$

Spheroidal solutions to these nonlinear equations are now investigated. The spheroidal behaviour of the metric is to be determined from a particular solution to (74) in a spacetime described by a metric of the form

$$ds^2 = -e^\nu c^2dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2) \tag{75}$$

where $\nu$ and $\lambda$ are functions of t, r and $\theta$. The non-zero connection coefficients (Christoffel symbols) are:

$$\Gamma^0_{00} = \frac{1}{2}\partial_0\nu,\ \Gamma^0_{01} = \frac{1}{2}\partial_1\nu,\ \Gamma^0_{02} = \frac{1}{2}\partial_2\nu,\ \Gamma^0_{11} = \frac{1}{2}\partial_0\lambda e^{\lambda-\nu},\ \Gamma^1_{00} = \frac{1}{2}\partial_1\nu e^{\nu-\lambda},\ \Gamma^1_{01} = \frac{1}{2}\partial_0\lambda,$$
$$\Gamma^1_{11} = \frac{1}{2}\partial_1\lambda,\ \Gamma^1_{12} = \frac{1}{2}\partial_2\lambda,\ \Gamma^1_{22} = -re^{-\lambda},\ \Gamma^1_{33} = -r\sin^2\theta e^{-\lambda},\ \Gamma^2_{00} = \frac{1}{2r^2}\partial_2\nu e^\nu,$$
$$\Gamma^2_{11} = \frac{1}{2r^2}\partial_2\lambda e^\lambda,\ \Gamma^2_{12} = \frac{1}{r},\ \Gamma^2_{33} = -\sin\theta\cos\theta,\ \Gamma^3_{13} = \frac{1}{r},\ \Gamma^3_{23} = \cot\theta.$$

The unit vectors $u^\beta$ satisfy

$$u^\alpha u_\alpha = -1. \tag{76}$$

As a first step to understand this highly nonlinear set of equations given by (74) with the constraint (76) in this metric, $u_3$ is chosen to vanish. This requires

$$X_3 = 0 \tag{77}$$

because $u_\alpha$ is collinear with $X_\alpha$. All other components of $X_\alpha$ are non-zero.

Static solutions to (23) are sought which require the components of the line element field to satisfy

$$\partial_0 X_\alpha = 0, \tag{78}$$

and from the metric,

$$\partial_0 \lambda = 0, \ \ \partial_0 \nu = 0. \tag{79}$$

The components of $\Phi_{\alpha\beta}$ to be considered are then:

$$\Phi_{00} = (1 + 2u_0 u^0)(-\frac{1}{2}e^{\nu-\lambda}\nu' X_1 - \frac{1}{2r^2}e^{\nu}\partial_2\nu X_2), \tag{80}$$

$$\Phi_{11} = (1 + 2u_1 u^1)(X_1{}' - \frac{1}{2}\lambda' X_1 - \frac{1}{2r^2}e^{\nu}\partial_2\lambda X_2), \tag{81}$$

$$\Phi_{22} = (1 + 2u_2 u^2)(\partial_2 X_2 + re^{-\lambda}X_1), \tag{82}$$

$$\Phi_{33} = r\sin^2\theta e^{-\lambda}X_1 + \sin\theta\cos\theta X_2, \tag{83}$$

the Ricci scalar, which from (74) equals $\Phi$, is

$$R = e^{-\lambda}(-\nu'' - \frac{1}{2}\nu'^2 + \frac{1}{2}\lambda'\nu' - \frac{2}{r}\nu' + \frac{2}{r}\lambda' - \frac{2}{r^2}) + \frac{1}{r^2}(-\frac{1}{2}\partial_2\nu^2 - \partial_2\partial_2\nu - \partial_2\nu\cot\theta + 2), \tag{84}$$

and the corresponding components of the Einstein tensor are:

$$G_{00} = \frac{1}{r^2}e^{\nu-\lambda}(r\lambda' - 1 + e^{\lambda}) + \frac{e^{\nu}}{4r^2}\partial_2\nu\partial_2\lambda, \tag{85}$$

$$G_{11} = \frac{1}{r^2}(1 + r\nu' - e^{\lambda}) + \frac{e^{\lambda}}{2r^2}[\partial_2\partial_2\lambda + \partial_2\partial_2\nu + \frac{1}{2}(\partial_2\lambda)^2 + \frac{1}{2}(\partial_2\nu)^2 + \frac{1}{2}\partial_2\lambda\partial_2\nu + \cot\theta(\partial_2\lambda) + \partial_2\nu], \tag{86}$$

$$G_{22} = \frac{r^2e^{-\lambda}}{2}[\nu'' + (\frac{1}{2}\nu' + \frac{1}{r})(\nu' - \lambda')] - \frac{1}{2}\partial_2\partial_2\lambda - \frac{1}{2}(\partial_2\lambda)^2 + \frac{1}{2}\cot\theta\partial_2\nu, \tag{87}$$

$$G_{33} = \sin\theta^2[\frac{r^2e^{-\lambda}}{2}(-\frac{\lambda'}{r} + \frac{\nu'}{r} + \nu'' + \frac{1}{2}\nu'^2 - \frac{1}{2}\lambda'\nu') + \frac{1}{4}(\partial_2\nu)^2 + \frac{1}{2}\partial_2\partial_2\nu - \frac{1}{2}\cot\theta\partial_2\lambda] \tag{88}$$

where the prime denotes $\partial_1$.

These equations are greatly simplified by setting

$$\nu = -\lambda. \tag{89}$$

Thus, a class of static spheroidal solutions to (74) are sought with the restrictions (77),(78),(79) and (89).

Since $e^{\lambda-\nu}(\Phi_{00} + G_{00}) + \Phi_{11} + G_{11} = 0$ from (23),

$$(\frac{\lambda'}{2}X_1 + \frac{1}{2r^2}e^{\lambda}\partial_2\lambda X_2)(1 + 2u_0u^0) + (X_1' - \frac{\lambda'}{2}X_1 - \frac{1}{2r^2}e^{\lambda}\partial_2\lambda X_2)(1 + 2u_1u^1) = 0. \tag{90}$$

If we choose $\frac{\lambda'}{2}X_1 = X_1' - \frac{\lambda'}{2}X_1$,

$$X_1 = a_1 e^{\lambda} \tag{91}$$

where $a_1$ is an arbitrary but non-zero constant with the dimensions of $L^{-1}$. Then (90) becomes

$$(\frac{X_1'}{2} + \frac{1}{2r^2}e^{\lambda}\partial_2\lambda X_2)(1 + 2u_0u^0) + (\frac{X_1'}{2} - \frac{1}{2r^2}e^{\lambda}\partial_2\lambda X_2)(1 + 2u_1u^1) = 0 \tag{92}$$

from which

$$\partial_2\lambda = a_1\lambda' r^2 \frac{u_2u^2}{(u_0u^0 - u_1u^1)X_2} \tag{93}$$

using $1 + u_0 u^0 + u_1 u^1 = -u_2 u^2$.

From (23) and (91) in the interval $0 < \theta < \pi$, $G_{22} + \Phi_{22} = 0$ gives

$$(-\lambda'' + \lambda'^2 - \frac{2}{r}\lambda') + \frac{e^\lambda}{r^2}(-\frac{1}{2}(\partial_2\lambda)^2 - \partial_2\partial_2\lambda - \partial_2\lambda\cot\theta) + \frac{2e^\lambda}{r^2}(\partial_2 X_2 + a_1 r)(1 + 2u_2 u^2) = 0 \tag{94}$$

and $G_{33} + \Phi_{33} = 0$ yields

$$-\lambda'' + \lambda'^2 - \frac{2}{r}\lambda' + \frac{e^\lambda}{r^2}(\frac{1}{2}(\partial_2\lambda)^2 - \partial_2\partial_2\lambda - \partial_2\lambda\cot\theta) + \frac{2e^\lambda}{r^2}(a_1 r + X_2\cot\theta) = 0. \tag{95}$$

Subtracting (94) from (95) requires

$$\cot\theta X_2 - \partial_2 X_2 + \frac{1}{2}(\partial_2\lambda)^2 - 2u_2 u^2(\partial_2 X_2 + a_1 r) = 0. \tag{96}$$

Choosing $\cot\theta X_2 = \partial_2 X_2$ gives

$$X_2 = a_2 \sin\theta \tag{97}$$

where $a_2 \neq 0$ is an otherwise arbitrary dimensionless constant, and demands

$$(\partial_2\lambda)^2 = 4u_2 u^2(a_2\cos\theta + a_1 r). \tag{98}$$

Equation (95) can now be expressed as

$$-\lambda'' + \lambda'^2 - \frac{2}{r}\lambda' + \frac{e^\lambda}{r^2}(\frac{1}{2}(\partial_2\lambda)^2 - \partial_2\partial_2\lambda - \partial_2\lambda\cot\theta) + \frac{2e^\lambda}{r^2}(a_1 r + a_2\cos\theta) = 0. \tag{99}$$

From (93) and (98), the derivative terms in $\partial_2\lambda$ can be neglected if $u_2 u^2$ is restricted to be very small but non-zero. Assuming $\partial_2\partial_2\lambda$ can also be neglected, equation (99) is then approximated by

$$-\lambda'' + \lambda'^2 - \frac{2}{r}\lambda' + \frac{2e^\lambda}{r^2}(a_1 r + a_2\cos\theta) = 0, \;\; 0 < \theta < \pi \tag{100}$$

which, for a fixed value of $\cos\theta$, has the exact solution

$$\lambda = -\ln(\frac{c_1}{r} + c_2 - a_1 r - 2a_2\cos\theta\ln r), \;\; 0 < \theta < \pi, \;\; 0 < r < \infty \tag{101}$$

where $c_1$ and $c_2$ are arbitrary constants.

## 7.2. The radial force and galactic rotation curves

The radial force on an object of mass m can now be calculated from (101). Using the conventional relationship of the Newtonian potential $\phi$ to $g_{00}$,

$$\phi = \frac{c^2}{2}(e^\nu - 1), \tag{102}$$

the radial force $F_r$ is

$$\begin{aligned} F_r &= -m\partial_1\phi \\ &= \frac{mc^2}{2}(\frac{c_1}{r^2} + \frac{2a_2\cos\theta}{r} + a_1). \end{aligned} \tag{103}$$

Choosing

$$c_1 = -\frac{2GM}{c^2}, \tag{104}$$

where M represents the total mass of the galaxy composed of mainly baryonic matter and no dark matter, we arrive at the modified Newtonian force

$$F_r = -\frac{GMm}{r^2} + \frac{mc^2 a_2\cos\theta}{r} + \frac{mc^2 a_1}{2}. \tag{105}$$

The correction terms to the Newtonian force come from the non-zero components of the line element field in the energy-momentum tensor $\Phi_{\alpha\beta}$. The components of the line element field can change their sign, which means $a_j$ can change to $-a_j$ with j=1,2 in this restricted metric. Thus the middle term is gravitationally attractive and represents the "dark matter" correction if $a_2 \cos\theta < 0$ in the interval $0 < \theta < \pi$. It is the term that gives rise to the flat rotation curves. The third constant term is positive and repulsive if $a_1 > 0$. This describes the repulsive dark energy force in the present epoch. However, during a part of the previous decelerating epoch observed by Riess et al. [29], $a_1 < 0$. They used the Hubble telescope to provide the first conclusive evidence for cosmic deceleration that preceded the current epoch of cosmic acceleration.

Assuming a circular orbit about a point mass, it follows that the orbital velocity of a star rotating in the galaxy satisfies

$$v^2 = v_N^2 - a_2 c^2 \cos\theta - \frac{a_1 c^2}{2} r \tag{106}$$

where $v_N^2$ is the Newtonian term

$$v_N^2 = \frac{GM}{r}. \tag{107}$$

Equation (106) demands an upper limit to r describing a large but finite galaxy.

Because $a_1 \neq 0$, it is possible for the Newtonian force to balance the dark energy force,

$$v_N^2 - \frac{a_1 c^2}{2} r = 0 \tag{108}$$

in (106). Then,

$$v^2 = -a_2 c^2 \cos\theta, \ \ a_2 \cos\theta < 0 \tag{109}$$

describes a specific class of galaxies with a flat orbital rotation curve. From (107) and (108), we obtain the Tully-Fisher relation

$$v_N^4 = \frac{GMc^2 a_1}{2}, \ \ a_1 > 0. \tag{110}$$

This result holds for any finite r in contrast to EG which holds only for large r as determined by Lelli, McGaugh and Schombert [40]. With $\frac{c^2 a_1}{2} := a_0$, the Tully-Fisher relation in MOND is evident.

The importance of the radial acceleration relative to the rotation curves of galaxies was discussed by Lelli, McGaugh, Schombert, and Pawlowski in [41] where it was determined that late time galaxies (spirals and irregulars), early time galaxies (ellipticals and lenticulars), and the most luminous dwarf spheroidals follow the same baryonic Tully-Fisher relation. The observed acceleration correlates well with that expected from the distribution of baryons.

Equation (105), which does not include dark matter in this analysis, is general enough to describe the rotation curves of many types of galaxies. For example, galaxy NGC4261 has a relatively flat rotation curve but starts to rise at larger radii, reaching velocities of 700 km $s^{-1}$ at 100 kpc [29]. That requires $a_1$ in (106) to be negative which was interpreted above. As another example, both $c^2 a_1 r$ and $c^2 a_2$ could be small enough relative to $\frac{GM}{r}$ so that the Newtonian term is dominant. Galaxies with no flat rotation curves have recently been observed by van Dokkum et.al [42]. It should also be remembered that equation (106) came from an approximation to equation (99) which could be used to model galaxies in greater detail. Furthermore, equation (99) is a restricted version of the general equation (74) which provides additional variables that may explain even more aspects of cosmology now attributed to dark matter.

However, it is still possible that dark matter particles may exist. As a part of (15) in the total matter energy-momentum tensor $\tilde{T}_{\alpha\beta}$, they would contribute to the gravitational field outside of its source along with baryonic matter in equation (74) and therefore in (105). But any dark matter contribution to the gravitational field would play a much lesser role because of the existence of $\Phi_{\alpha\beta}$.

## 8. Conclusion

The results in this article stem from the association of a Lorentzian metric with a Riemannian metric using the line element field $(X^\beta, -X^\beta)$. This allowed a classical result of Riemannian geometry to be adapted to the geometry of spacetime. An orthogonal decomposition of symmetric tensors on a time oriented Lorentzian manifold could then be developed. This introduced a new tensor, $\Phi_{\alpha\beta}$, which represented the energy-momentum of the gravitational field. By requiring the field equations to be determined from an action functional, and adhering to Einstein's postulate requiring the totality of the matter energy-momentum and the energy-momentum of the gravitational field to be the source of the gravitational field, a modified equation of general relativity was obtained.

Thus, Einstein's equation of general relativity is complete with the addition of the geometrical tensor $\Phi_{\alpha\beta}$. This leads to the following conclusions:

1. The gravitational field has an intrinsic energy-momentum attributed to $\Phi_{\alpha\beta}$. It has the structure to describe local gravitational energy-momentum.

2. $\Phi > 0$ defines dark energy and $\Phi < 0$ the attractive energy of the gravitational field interacting with itself. It has the global property $\int g^{\alpha\beta}\Phi_{\alpha\beta}\sqrt{-g}d^4x = 0$. $\Phi$ tends to accelerate or decelerate the universe but has a net zero affect on it. $\Phi$ is measurable and therefore the energy of the gravitational field is localizable. Thus, $\Phi_{\alpha\beta}$, a symmetric tensor, provides a solution to the energy localization problem.

3. The cosmological constant $\Lambda$ is an integration constant and must vanish. It is dynamically replaced by $\Phi$.

$\Phi_{\alpha\beta}$ adds $X^\beta$ from the line element field as a dynamical variable independent of the metric. Variation of the action functional with respect to $X^\beta$ introduced a scalar $F = \frac{1}{2\lambda}\Phi^2$ which was interpreted to be the entropy of the gravitational field.

Important features attributed to dark energy resulted from the investigation of the modified Einstein equation in the FLRW metric. The dark energy pressure explained the observed cyclic nature of the universe after the Big Bang. The dark energy density explained the initial inflation of the universe and provided a natural explanation of why the vacuum energy density is so small and why it now dominates the expansion and acceleration of the present universe.

The self-interactions of the energy-momentum of the gravitational field are important in the description of dark matter. An exact static solution was obtained from the modified Einstein equation in a restricted spheroidal metric describing the gravitational field outside of its source, which does not contain dark matter. The modified Newtonian force contained two additional terms: one represented the constant dark energy force which depends on the parameter $a_1$ of the radial component of the line element field $X_1$ in $\Phi_{\alpha\beta}$ but not the radial variable; the other represented the $\frac{1}{r}$ "dark matter" force which depends on the parameter $a_2$ of $X_2$ in $\Phi_{\alpha\beta}$, and $\cos\theta$. The baryonic Tully-Fisher relation was obtained by balancing the dark energy force with the Newtonian force. This condition described the class of galaxies associated with MOND. The Newtonian rotation curves for galaxies with no flat orbital curves, and those with rising rotation curves for large radii were described as examples of the flexibility of the orbital rotation curve equation. The results obtained from the *complete* Einstein equation thus far are able to substantially describe the missing mass problem attributed to dark matter. Further mathematical and detailed numerical analyses to explore the ability of the energy-momentum tensor of the gravitational field to replace dark matter in cosmology, are fully warranted. This rigorous analysis with comparison to astronomical data may still point to the existence of dark matter to some extent. But even if that is the case, the gravitational role of dark matter is substantially reduced by the impact of the energy-momentum tensor of the gravitational field.

Thus, the "dark side" of the universe appears to have arisen from the absence of a symmetric tensor in general relativity that describes the energy-momentum of the gravitational field itself.

# References


[1] A. Einstein, Die Feldgleichungen der Gravitation, Königlich Preußische Akademie der Wissenschaften (Berlin), Sitzungsberichte 844-847, (1915); Die Grundlage der allgemeinen Relativitätstheorie, An-

nalen der Physik **354** (7), 769-822, (1916); Note on E. Schrödinger's Paper: The energy components of the gravitational field. Phys. Z. 19, 115-116 (1918).

[2] S.S. Xulu, The Energy-Momentum Problem in General Relativity, arXiv:hep-th/0308070v1 (2003).

[3] Y. Baryshev, Foundation of Relativistic Astrophysics: Curvature of Riemannian Space versus Relativistic Quantum Field in Minkowski Space, arXiv:1702.02020v1 [physics.gen-ph] (2017).

[4] M.J. Dupré, The Fully Covariant Energy Momentum Stress Tensor For Gravity and the Einstein Equation in General Relativity, arXiv:0903.5225 [gr-qc] (2009).

[5] M. Berger and D. Ebin, Some Decompositions of the Space of Symmetric Tensors on a Riemannian Manifold,J. Differential Geometry **3**, 379-392, (1969).

[6] David Lovelock, The Einstein Tensor and Its Generalizations, J. Math. Phys. **12**, 498-501 (1971).

[7] C. Misner, K. Thorne, J. Wheeler, Gravitation, Freeman, San Francisco 487 (1973).

[8] A. Ashtekar, Loop Quantum Cosmology: An Overview, arXiv:0812.0177v1 [gr-qc], (2008).

[9] P. J. Steinhardt and N. Turok, Cosmic evolution in a cyclic universe, Phys. Rev. D **65**, 126003 (2002).

[10] P. J. Steinhardt and N. Turok, The Cyclic Model Simplified, arXiv:astro-ph/0404480v1, (2004).

[11] L. Baum and P. Frampton, Turnaround in Cyclic Cosmology, arXiv:hep-th/0610213v2, (2006).

[12] R. Penrose, Before the Big Bang: An Outrageous New Perspective and its Implications for Particle Physics, Proceedings of EPAC 2006, Edinburgh, Scotland, (2006).

[13] R. R. Caldwell, R. Dave, and P. J. Steinhardt, Cosmological Imprint of an Energy Component with General Equation-of-State, arXiv:astro-ph/9708069, (1998).

[14] P.J. Steinhardt, A quintessential introduction to dark energy. Philosophical Transactions: Mathematical, Physical and Engineering Sciences, 361, no. 1812, 2497-2513, (2003).

[15] Yi-Fu Cai, E. N. Saridakis, M. R. Setare and Jun-Qing Xia, Quintom Cosmology: theoretical implications and observations, arXiv:0909.2776v2 [hep-th], (2010).

[16] L. Markus, Line Element Fields and Lorentz Structures on Differentiable Manifolds, **62**, No. 3, 411-417, (1955).

[17] S.W. Hawking and G.F.R. Ellis, The large scale structure of space-time, Cambridge University Press, Cambridge, 39 (1973).

[18] S. Deser, Covariant decomposition of symmetric tensors and the gravitational Cauchy problem, Ann. Inst. H. Poincaré **7**, 149 (1967).

[19] J.W. York Jr. Covariant decomposition of symmetric tensors in the theory of gravitation, Ann. Inst. H. Poincaré **21**, 319 (1974).

[20] T. Ma and S. Wang, Mathematical Principles of Theoretical Physics, Science Press, Beijing, 136-144 (2015).

[21] Y. Choquet-Bruhat, General Relativity and Einstein's Equations, Oxford University Press, Oxford, 373 (2009).

[22] M. Spivak, A comprehensive introduction to differential geometry, Volume 2, Publish or Perish Inc., Houston Texas, 250 (1999).

[23] T. Ma and S. Wang, Gravitational Field Equations and Theory of Dark Matter and Dark Energy, Discrete and Continuous Dynamical Systems **34**, Number 2, 354 (2014).

[24] S. Weinberg, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity, John Wiley & Sons, 390-395, 409-415 (1972).

[25] E.P. Hubble, A relation between distance and radial Velocity among extra-galacticv nebulae, Proc. Nat. Acad. Sci., **15**, 168-173 (1929).

[26] S. Weinberg, The cosmological constant problem, Rev. Mod. Phys., **61**, Iss.1 (1989).

[27] S. Perlmutter et al. (The Supernova Cosmology Project), Measurements of Omega and Lambda from 42 High-Redshift Supernovae, Astrophys. J., **517**, 565-586 (1999).

[28] A.G. Riess et al., Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant, Astrophys. J., **116**, 1009-1038 (1998).

[29] A. G. Riess et al., Type Ia Supernova Discoveries at $z > 1$ From the Hubble Space Telescope: Evidence for Past Deceleration and Constraints on Dark Energy Evolution, arXiv:astro-ph/0402512v2, (2004).

[30] L. Anderson et al.,The clustering of galaxies in the SDSS-III Baryon Oscillation Spectroscopic Survey: Baryon Acoustic Oscillations in the Data Release 10 and 11 Galaxy Samples, arXiv:1312.4877v2 [astro-ph.CO], (2014).

[31] M. Milgrom, A modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis, Astrophysical Journal, Part 1 270, 365-370, (1983).

[32] S. McGaugh, A Tale of Two Paradigms: the Mutual Incommensurability of $\Lambda CDM$ and MOND, arXiv:1404.7525v2 [astro-ph.CO], (2014).

[33] P. Mannheim, Alternatives to Dark Matter and Dark Energy, arXiv:astro-ph/0505266v2, (2005).

[34] J. W. Moffat, Modified Gravitational Theory as an Alternative to Dark Energy and Dark Matter, arXiv:astro-ph/0403266v5, (2004).

[35] M. Milgrom, Bimetric MOND gravity, Phys. Rev. D 80, 123536, (2009).

[36] L. Bernard, L. Blanchet and L. Heisenberg, Bimetric Gravity and Dark Matter, arXiv:1507.02802v1 [gr-qc], (2015).

[37] E. Verlinde, Emergent Gravity and the Dark Universe, arXiv:1611.02269v2 [hep-th], (2016).

[38] M. C. Campigotto, A. Diaferio and L. Fatibenec, Conformal gravity: light deflection revisited and the galactic rotation curve failure, arXiv:1712.03969v1 [astro-ph.CO], (2017).

[39] Kroupa, P. The Dark Matter Crisis: Falsification of the Current Standard Model of Cosmology. arXiv:1204.2546v2 [astro-ph.CO], (2012).

[40] F. Lelli, S.S. McGaugh,J.M. Schombert, Testing Verlinde's Emergent Gravity with the Radial Acceleration Relation, arXiv:1702.04355v1 [astro-ph.GA] (2017).

[41] F. Lelli, S.S. McGaugh, J. M. Schombert, and M.S. Pawlowski, One Law to Rule Them All: The Radial Acceleration Relation of Galaxies, arXiv:1610.08981v2 [astro-ph.GA], (2017).

[42] P. van Dokkum et.al., A galaxy lacking dark matter, Nature volume 555, 629-632 (2018).