Sharp estimates for the unique solution of the Hadamard-type two-point fractional boundary value problems

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Abstract. In this short note, we present the sharp estimate for the existence of a unique solution for a Hadamard-type fractional differential equations with two-point boundary conditions. The method of analysis is obtained by the Banach contraction principle. An exmple is presented to clarify the principle result.

Keywords: Hadamard fractional derivative; Unique solution; Green's function; Two-point boundary value problem

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1 Introduction and preliminaries

In the book [1] Kelley and Peterson considered the following classical two-point boundary value problems:

$$\begin{cases} u''(t) = f(t, u(t)), & a < t < b, \\ u(a) = A, \quad u(b) = B, & A, B \in \mathbb{R}, \end{cases} \tag{1.1}$$

and they included the following result:

Theorem 1.1. *Assume $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies an uniform Lipschitz condition with respect to x on $[a, b] \times \mathbb{R}$ with Lipschitz constant K, that is,*

$$|f(t, u) - f(t, v)| \leq K |u - v|$$

for all $(t, u), (t, v) \in [a, b] \times \mathbb{R}$. If

$$b - a < \frac{2\sqrt{2}}{\sqrt{K}}$$

Then the boundary value problem (1.1) has a unique solution.

Ferreira in 2016 [2] discussed the existence and uniqueness of solutions for the following fractional boundary value problems with Reimman-Liouville derivative:

$$\begin{cases} {}^{R}D_a^{\alpha}u(t) = -f(t,u(t)), & a < t < b, \quad 1 < \alpha \le 2, \\ u(a) = 0, \quad u(b) = B, & B \in \mathbb{R}, \end{cases} \tag{1.2}$$

so he included this result given by:

Theorem 1.2. *Assume $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies an uniform Lipschitz condition with respect to the second variable on $[a,b] \times \mathbb{R}$ with Lipschitz constant K, that is,*

$$|f(t,u) - f(t,v)| \le K\,|u - v|$$

for all $(t,u),(t,v) \in [a,b] \times \mathbb{R}$. If

$$b - a < \Gamma^{1/\alpha}(\alpha)\frac{\alpha^{(\alpha+1)/\alpha}}{K^{1/\alpha}(\alpha-1)^{(\alpha-1)/\alpha}}$$

Then the boundary value problem (1.2) has a unique solution.

In 2019, Ferreira [3] corrected a recent uniqueness result [4] for a two-point fractional boundary value problem with Caputo derivative:

$$\begin{cases} {}^{C}D_a^{\alpha}u(t) = -f(t,u(t)), & a < t < b, \quad 1 < \alpha \le 2, \\ u(a) = A, \quad u(b) = B, & A, B \in \mathbb{R}, \end{cases} \tag{1.3}$$

and he came to the following theorem:

Theorem 1.3. *Assume $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuaous and satisfies an uniform Lipschitz condition with respect to the second variable on $[a,b] \times \mathbb{R}$ with Lipschitz constant $l > 0$, that is,*

$$|f(t,u) - f(t,v)| \le l\,|u - v|$$

for all $(t,u),(t,v) \in [a,b] \times \mathbb{R}$. If

$$M(\alpha,a,b) < \frac{1}{l},$$

where

$$\begin{aligned} M(\alpha,a,b) &= \frac{1}{\Gamma(\alpha+1)}\max_{t\in[a,b]}\Big(-2(t-h(t))^{\alpha} + 2\frac{(t-a)(b-h(t))^{\alpha}}{b-a} \\ &\quad + (t-a)^{\alpha} - (t-a)(b-a)^{\alpha-1}\Big), \end{aligned}$$

with $h(t) = \frac{\left(\frac{t-a}{b-a}\right)^{\frac{1}{\alpha-1}}b-1}{\left(\frac{t-a}{b-a}\right)^{\frac{1}{\alpha-1}}-1}$. Then the boundary value problem (1.3) has a unique solution.

See [2, 3] and references therein for more details.

Recently, many researchers are interested in studying the Hadamard-type fractional boundary value problems, and there are several results about the existence of solutions for the differential equations with Hadamard derivative, we refer the reader to the book [5] that contains the most important works that have been published in this domain.

Motivated by the papers of [2, 3] and [6], in this work, we investigated the sharp estimate for the unique solution of the following fractional differential equation with Hadamard derivative:

$$\begin{cases} {}^H D_a^\alpha u(t) = -f(t, u(t)), & 1 \le a < t < b, \quad 1 < \alpha \le 2, \\ u(a) = 0, \quad u(b) = B, & B \in \mathbb{R}, \end{cases} \tag{1.4}$$

where f is a continuous function, ${}^H D_a^\alpha$ denotes the Hadamard fractional derivative α, B is real constant.

Definition 1.4. [7] Let $a, b, \alpha \in \mathbb{R}^+$ where $a < b$ and $n-1 < \alpha \le n$ with $n \in \mathbb{N}^*$, the Hadamard fractional integral of ordre α for a function $g \in L^1[a,b]$ is defined by:

$$ {}^H I_a^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} g(s) \, \frac{ds}{s}. \tag{1.5}$$

Definition 1.5. [7] Let $\alpha > 0$ where $n-1 < \alpha \le n$ and $n \in \mathbb{N}^*$, and let $a < b$. The Hadamard fractional derivative of order α for a function $g \in L^1[a,b]$ is defined by:

$$ {}^H D_a^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt}\right)^n \int_a^t \left(\ln \frac{t}{s}\right)^{n-\alpha-1} g(s) \, \frac{ds}{s}. \tag{1.6}$$

Lemma 1.6. *[7] Let $0 < a < b$, and $\alpha > 0$ where $n-1 < \alpha \le n$, $n \in \mathbb{N}^*$, and let $u \in C[a,b] \cap L^1[a,b]$. The equation ${}^H D_a^\alpha u(t) = 0$ has the general solution:*

$$u(t) = \sum_{i=1}^{i=n} c_i \left(\ln \frac{t}{a}\right)^{\alpha-i}, \; t \in [a,b], \tag{1.7}$$

where c_i, $(i = 1, ..., n)$ are real constants. And moreover

$$ {}^H I_a^\alpha {}^H D_a^\alpha u(t) = u(t) + \sum_{i=1}^{i=n} c_i \left(\ln \frac{t}{a}\right)^{\alpha-i}. \tag{1.8}$$

Lemma 1.7. *Let $y \in C([a,b], \mathbb{R}) \cap L^1([a,b], \mathbb{R})$, the following problem*

$$\begin{cases} {}^H D_a^\alpha u(t) = -y(t), & 1 \le a < t < b, \quad 1 < \alpha \le 2, \\ u(a) = 0, \quad u(b) = B, & B \in \mathbb{R}, \end{cases} \tag{1.9}$$

is equivalent to the fractional integral equation

$$u(t) = \int_a^b G(t,s) y(s) \, ds + B \left(\frac{\ln \frac{t}{a}}{\ln \frac{b}{a}}\right)^{\alpha-1}, \tag{1.10}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} h_1(t,s) = h_2(t,s) - \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{1}{s}, & a \le s \le t \le b, \\ h_2(t,s) = \left(\frac{\ln \frac{t}{a}}{\ln \frac{b}{a}}\right)^{\alpha-1} \left(\ln \frac{b}{s}\right)^{\alpha-1} \frac{1}{s}, & a \le t \le s \le b. \end{cases} \tag{1.11}$$

Proof. Using Lemma 1.6, we get

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} y(s) \, \frac{ds}{s} + c_1 \left(\ln \frac{t}{a}\right)^{\alpha-1} + c_2 \left(\ln \frac{t}{a}\right)^{\alpha-2}, \tag{1.12}$$

where $c_1, c_2 \in \mathbb{R}$.
Using the boundary condition $u(a) = 0$ and $u(b) = B$, we get $c_2 = 0$ and

$$c_1 = \frac{1}{\Gamma(\alpha)} \left(\ln \frac{b}{a}\right)^{1-\alpha} \int_a^b \left(\ln \frac{b}{s}\right)^{\alpha-1} y(s) \frac{ds}{s} + B \left(\ln \frac{b}{a}\right)^{1-\alpha}. \tag{1.13}$$

Substituting the values of c_1 and c_2 in (1.12), we obtain

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \left(\frac{\ln \frac{t}{a}}{\ln \frac{b}{a}}\right)^{\alpha-1} \int_a^b \left(\ln \frac{b}{s}\right)^{\alpha-1} y(s) \frac{ds}{s} \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} y(s) \frac{ds}{s} + B \left(\frac{\ln \frac{t}{a}}{\ln \frac{b}{a}}\right)^{\alpha-1} \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t \left[\left(\frac{\ln \frac{t}{a}}{\ln \frac{b}{a}}\right)^{\alpha-1} \left(\ln \frac{b}{s}\right)^{\alpha-1} - \left(\ln \frac{t}{s}\right)^{\alpha-1}\right] y(s) \frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_t^b \left(\frac{\ln \frac{t}{a}}{\ln \frac{b}{a}}\right)^{\alpha-1} \left(\ln \frac{b}{s}\right)^{\alpha-1} y(s) \frac{ds}{s} + B \left(\frac{\ln \frac{t}{a}}{\ln \frac{b}{a}}\right)^{\alpha-1} \\ &= \int_a^b G(t,s) y(s)\, ds + B \left(\frac{\ln \frac{t}{a}}{\ln \frac{b}{a}}\right)^{\alpha-1}. \end{aligned}$$

Hence, the proof is completed. □

2 Main results

This section is devoted to prove the main results of the problem.

Lemma 2.1. *The Green's function G defined in Lemma 1.7, has the following properties:*
i) For any $(t,s) \in [a,b] \times [a,b]$, we have

$$G(t,s) \geq 0. \tag{2.1}$$

ii)

$$\max_{t\in[a,b]} \int_a^b G(t,s)\, ds = \frac{1}{(\alpha-1)\,\Gamma(\alpha+1)} \left(\frac{\alpha-1}{\alpha} \ln \frac{b}{a}\right)^{\alpha}. \tag{2.2}$$

Proof. For $1 \leq a \leq t \leq s \leq b$, observe that

$$h_2(t,s) \geq 0. \tag{2.3}$$

While for $1 \leq a \leq s \leq t \leq b$. Differentiating the function h_1 with respect to t, we have

$$\begin{aligned} \partial_t h_1(t,s) &= \frac{(\alpha-1)(\ln \frac{t}{a})^{\alpha-2}(\ln \frac{b}{s})^{\alpha-1}}{\Gamma(\alpha)(\ln \frac{b}{a})^{\alpha-1} st} - \frac{(\alpha-1)}{\Gamma(\alpha)st} \left(\ln \frac{t}{s}\right)^{\alpha-2} \\ &= \frac{(\alpha-1)\left(\ln \frac{t}{a}\right)^{\alpha-2}}{\Gamma(\alpha)st} \left[\left(\frac{\ln \frac{b}{s}}{\ln \frac{b}{a}}\right)^{\alpha-1} - \left(\frac{\ln \frac{t}{a}}{\ln \frac{t}{s}}\right)^{2-\alpha}\right]. \end{aligned} \tag{2.4}$$

Using the inequalities $1 \leq a \leq s \leq t \leq b$, we get $\left(\frac{\ln \frac{t}{a}}{\ln \frac{t}{s}}\right)^{2-\alpha} \geq 1$, and $\left(\frac{\ln \frac{b}{s}}{\ln \frac{b}{a}}\right)^{\alpha-1} \leq 1$.

Thus

$$\partial_t h_1(t,s) \leq 0, \tag{2.5}$$

we obtain

$$h_1(s,s) \geq h_1(t,s) \geq h_1(b,s) = 0. \tag{2.6}$$

Hence, by (2.3) and (2.6) we conclude $G(t,s) \geq 0$ for all $(t,s) \in [a,b] \times [a,b]$.

By Green's functions (1.11), we have

$$\begin{aligned}
\Gamma(\alpha) \int_a^b G(t,s)\, ds &= \int_a^t \left[\left(\frac{\ln \frac{t}{a}}{\ln \frac{b}{a}}\right)^{\alpha-1} \left(\ln \frac{b}{s}\right)^{\alpha-1} - \left(\ln \frac{t}{s}\right)^{\alpha-1} \right] \frac{ds}{s} \\
&\quad + \int_t^b \left(\frac{\ln \frac{t}{a}}{\ln \frac{b}{a}}\right)^{\alpha-1} \left(\ln \frac{b}{s}\right)^{\alpha-1} \frac{ds}{s} \\
&= \left(\frac{\ln \frac{t}{a}}{\ln \frac{b}{a}}\right)^{\alpha-1} \int_a^b \left(\ln \frac{b}{s}\right)^{\alpha-1} \frac{ds}{s} - \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\
&= \frac{1}{\alpha} \left(\ln \frac{b}{a}\right)^{1-\alpha} \left(\ln \frac{t}{a}\right)^{\alpha-1} \left(\ln \frac{b}{a}\right)^{\alpha} - \frac{1}{\alpha} \left(\ln \frac{t}{a}\right)^{\alpha},
\end{aligned}$$

which yields

$$\Gamma(\alpha+1) \int_a^b G(t,s)\, ds = \left(\ln \frac{b}{a}\right) \left(\ln \frac{t}{a}\right)^{\alpha-1} - \left(\ln \frac{t}{a}\right)^{\alpha}. \tag{2.7}$$

Consider

$$g(t) = \left(\ln \frac{b}{a}\right) \left(\ln \frac{t}{a}\right)^{\alpha-1} - \left(\ln \frac{t}{a}\right)^{\alpha}. \tag{2.8}$$

Observe that $g(t) \geq 0$, for all $t \in [a,b]$, and $g(a) = g(b) = 0$.
If $t \in (a,b)$, we differentiate $g(t)$ to get

$$g'(t) = \frac{(\alpha-1)}{t} \left(\ln \frac{b}{a}\right) \left(\ln \frac{t}{a}\right)^{\alpha-2} - \frac{\alpha}{t} \left(\ln \frac{t}{a}\right)^{\alpha-1},$$

from which follows that $g'(t^*) = 0$ if and only if

$$t^* = a \left(\frac{b}{a}\right)^{(\alpha-1)/\alpha}, \tag{2.9}$$

which yields

$$\begin{aligned}
\max_{t \in [a,b]} g(t) &= g(t^*) \\
&= \left(\ln \frac{b}{a}\right) \left(\ln \left(\frac{b}{a}\right)^{(\alpha-1)/\alpha}\right)^{\alpha-1} - \left(\ln \left(\frac{b}{a}\right)^{(\alpha-1)/\alpha}\right)^{\alpha} \\
&= \left(\ln \frac{b}{a}\right) \left(\frac{\alpha-1}{\alpha} \ln \frac{b}{a}\right)^{\alpha-1} - \left(\frac{\alpha-1}{\alpha} \ln \frac{b}{a}\right)^{\alpha} \\
&= \frac{1}{\alpha-1} \left(\frac{\alpha-1}{\alpha} \ln \frac{b}{a}\right)^{\alpha}.
\end{aligned}$$

Therefore

$$\max_{t\in[a,b]}\int_a^b G\left(t,s\right)ds=\frac{1}{\left(\alpha-1\right)\Gamma(\alpha+1)}\left(\frac{\alpha-1}{\alpha}\ln\frac{b}{a}\right)^{\alpha}.$$

The proof is completed. □

Theorem 2.2. *Assume that $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ is continuous and satisfies an uniform Lipschitz condition with respect to the second variable on $[a,b]\times\mathbb{R}$ with Lipschitz constant $K>0$, that is,*

$$|f(t,u)-f(t,v)|\leq K\,|u-v|$$

for all $(t,u),(t,v)\in[a,b]\times\mathbb{R}$. If

$$\frac{b}{a}<\exp\left(\frac{\alpha\Gamma^{1/\alpha}(\alpha+1)}{(\alpha-1)\left(\frac{K}{\alpha-1}\right)^{1/\alpha}}\right) \tag{2.10}$$

Then the fractional boundary value problem

$$\begin{cases} {}^{H}D_a^{\alpha}u(t)=-f(t,u(t)), & 1\leq a<t<b,\quad 1<\alpha\leq 2,\\ u(a)=0,\;\; u(b)=B, & B\in\mathbb{R},\end{cases} \tag{2.11}$$

has a unique solution.

Proof. Let $E=C\left([a,b],\mathbb{R}\right)$ be the Banach space endowed with the norm $\|u\|=\sup_{t\in[a,b]}|u(t)|$, and we define the operator $P:E\to E$ by

$$Pu(t)=\int_a^b G(t,s)f(s,u(s))ds+B\left(\frac{\ln\frac{t}{a}}{\ln\frac{b}{a}}\right)^{\alpha-1}, \tag{2.12}$$

where the function G is given by (1.11).
The prolem (2.11) has a solution u if only if u is fixed point of the operator P.
For all $(t,u),(t,v)\in[a,b]\times\mathbb{R}$, we have

$$\begin{aligned}|Pu\left(t\right)-Pv\left(t\right)| &\leq \int_a^b G(t,s)\left|f(s,u(s))-f(s,v(s))\right|ds\\ &\leq \int_a^b KG(t,s)\left|u\left(s\right)-v\left(s\right)\right|ds\\ &\leq K\int_a^b G\left(t,s\right)ds\,\|u-v\|,\end{aligned}$$

using the second property in Lemma 2.1, yields

$$\|Pu-Pv\|=\frac{K}{(\alpha-1)\Gamma(\alpha+1)}\left(\frac{\alpha-1}{\alpha}\ln\frac{b}{a}\right)^{\alpha}\|u-v\|.$$

Can be easily check that the assumption (2.10) leads to principle of contraction mapping. Hence, P is contraction mapping, we conclude that the problem (2.11) has unique solution. □

Example 2.3. We consider the following problem

$$\begin{cases} {}^{H}D^{3/2}u(t)=(t-1)^2+\sqrt{t-1+u^2(t)}, & 1<t<e,\\ u(1)=0,\;\; u(e)=1.\end{cases} \tag{2.13}$$

Here $\alpha = \frac{3}{2}$ and $f(t,u) = (t-1)^2 + \sqrt{t-1+u^2}$. For all $(t,u),(t,v) \in (1,e) \times \mathbb{R}$, we have

$$\begin{aligned} |f(t,u) - f(t,v)| &= \left| \sqrt{t-1+u^2} - \sqrt{t-1+v^2} \right| \\ &= \frac{u^2 - v^2}{\sqrt{t-1+u^2} + \sqrt{t-1+v^2}} \\ &\leq \frac{|u-v|\,|u+v|}{|u|+|v|} \\ &\leq \|u-v\|, \end{aligned}$$

choose $K = 1$. We have

$$\exp\left(\frac{\alpha\Gamma^{1/\alpha}(\alpha+1)}{(\alpha-1)\left(\frac{K}{\alpha-1}\right)^{1/\alpha}} \right) = \exp\left(\frac{3}{4}(9\pi)^{1/3} \right) > e.$$

Then the inequality (2.10) is satisfied. Hence, by Theorem 2.2, we conclude that the Hadamard fractional boundary value problem (2.13) has a unique solution in the interval $[1,e]$.

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