Refutation of the planar Eucleadian R-geometry of Tarski

Abstract: We evaluate the axioms of the title. The axiom of identity of betweenness and axiom Euclid are tautologous, but the others are not. The commonplace expression of the axiom of Euclid does not match its other two variations which is troubling. This effectively refutes the planar R-geometry.

We assume the method and apparatus of Meth8/VŁ4 with `T`autology as the designated *proof* value, **F** as contradiction, `N` as truthity (non-contingency), and `C` as falsity (contingency). The 16-valued truth table is row-major and horizontal. The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET p, q, r, s, u, v, w, x, y, z:
a, b, r, B, u, v, w, x, y, z;
~ Not, ¬ ; + Or, ∨; & And, ∧; > Imply, →;
= Equivalent, ≡; @ Not Equivalent, ≠; % possibility, for one or some, ∃.

From: en.wikipedia.org/wiki/Tarski's_axioms

Congruence axioms

Identity of congruence

x y ≡ z z → x = y (2.1)

((x&y)=(z&z))>(x=y) ; `TTTT TTTT TTTT TTTT` (x96),
FFFF FFFF FFFF FFFF (x32) (2.2)

Transivity of congruence

(x y ≡ z u ∧x y ≡ v w) → z u ≡ v w (3.1)

((x&y)=(((z&u)&(x&y))=(v&w)))>((z&u)=(u&w)) ;
`TTTT TTTT TTTT TTTT` (x124),
FFFF FFFF FFFF FFFF (x 4) (3.2)

Betweenness axioms

Identity of betweenness

B x y x → x = y (4.1)

((s&x)&(y&x))>(x=y) ; `TTTT TTTT TTTT TTTT` (x128) (4.2)

Axiom of Pasch

$$B\ x\ u\ z \wedge B\ y\ v\ z\ ) \rightarrow \exists a\ (\ B\ u\ a\ y \wedge B\ v\ a\ x\ ) \tag{5.1}$$

(((s&x)&(u&z))&((s&y)&(v&z))) >
(((s&u)&(%p&y))&((s&v)&(%p&x))) ;
TTTT TTTT TTTT TTTT (x128),
TTTT TTTT CTCT CTCT (x 4) (5.2)

Axiom schema of continuity

LET u, v: ϕ, ψ.

Let $\varphi(x)$ and $\psi(y)$ be first-order formulae containing no free instances of either *a* or *b*.t there also be no free instances of *x* in $\psi(y)$ or of *y* in $\varphi(x)$. Then all instances of the following schema are axioms:

$$\exists a\ \forall x\ \forall y\ [\ (\ \phi\ (\ x\ ) \wedge \psi\ (\ y\ )\ ) \rightarrow B\ a\ x\ y\ ] \rightarrow$$
$$\exists b\ \forall x\ \forall y\ [\ (\ \phi\ (\ x\ ) \wedge \psi\ (\ y\ )\ ) \rightarrow B\ x\ b\ y\ ] \tag{6.1}$$

(((u&#x)&(v&#y)) > (s&(%p&(#x&#y)))) >
(((u&#x)&(v&#y)) > (s&((#x&%q)&#y))) ;
TTTT TTTT TTTT TTTT (x120),
TTTT TTTT CTCT CTCT (x 8) (6.2)

Lower dimension

LET p, q, r, s: a, b, c, B.

$$\exists a\ \exists b\ \exists c\ [\ \neg\ B\ a\ b\ c \wedge \neg\ B\ b\ c\ a \wedge \neg\ B\ c\ a\ b\ ] \tag{7.1}$$

(((~s&%p)&(%q&%r)) & ((~s&%q)&(%r&%p))) &
((~s&%r)&(%p&%q)) ; CCCC CCCT TTTT TTTT (x128) (7.2)

Congruence and betweenness

Upper dimension

$$(\ x\ u \equiv x\ v \wedge y\ u \equiv y\ v \wedge z\ u \equiv z\ v \wedge u \neq v\ ) \rightarrow (\ B\ x\ y\ z \vee B\ y\ z\ x \vee B\ z\ x\ y\ ) \tag{8.1}$$

(((x&u)=((x&v)&(y&u)))=(((y&v)&(z&u))=((z&v)&(u@v)))) >
((((s&x)&(y&z))&((x&y)&(z&z)))&((s&z)&(x&y))) ;
FFFF FFFFF TTTT TTTT, TTTT TTTT TTTT TTTT,
FFFF FFFF FFFF FFFF (8.2)

Axiom of Euclid

Each of the three variants of this axiom, all equivalent over the remaining Tarski's axioms to Euclid's parallel postulate, has an advantage over the others:

A dispenses with existential quantifiers;
B has the fewest variables and atomic sentences;
C requires but one primitive notion, betweenness. This variant is the usual one given in the literature.

A: $$((B\,x\,y\,w \wedge x\,y \equiv y\,w) \wedge (B\,x\,u\,v \wedge x\,u \equiv u\,v) \wedge (B\,y\,u\,z \wedge y\,u \equiv z\,u)) \rightarrow y\,z \equiv v\,w \qquad (9.1)$$

((((s&(x&y))&(w&(x&y)))=(y&w))&((((s&x)&((u&v)&(x&u)))=(u&v))&(((s&y)&((u&z)&(y&u)))=(z&u))))>((y&z)=(v&w)) ;

TTTT TTTT TTTT TTTT, **FFFF FFFF FFFF FFFF**,
TTTT TTTT **FFFF FFFF** (9.2)

B: $$B\,x\,y\,z \vee B\,y\,z\,x \vee B\,z\,x\,y \vee \exists a\,(x\,a \equiv y\,a \wedge x\,a \equiv z\,a) \qquad (10.1)$$

((((s&x)&(y&z))+((s&y)&(z&x)))+((s&z)&(x&y))) + (((x&%p)=(y&%p))&((x&%p)=(z@%p))) ;

N**F**N**F** N**F**N**F** N**F**N**F** N**F**N**F**, TTTT TTTT TTTT TTTT,
CTCT CTCT CTCT CTCT, **FFFF FFFF FFFF FFFF**,
FFFF FFFF TTTT TTTT (10.2)

C: $$(B\,x\,u\,v \wedge B\,y\,u\,z \wedge x \neq u) \rightarrow \exists a\,\exists b\,(B\,x\,y\,a \wedge B\,x\,z\,b \wedge B\,a\,v\,b) \qquad (11.1)$$

((s&(x&(u&v)))&((s&(y&(u&z)))&(x@u))) > (((s&(x&y))&(%p&(s&(x&z))))&(%q&(s&((%p&v)&%q)))) ;

TTTT TTTT TTTT TTTT (x128) (11.2)

Eqs. 4.2 and 11.2 as rendered are tautologous, but the others are not. The commonplace expression of the axiom of Euclid is tautologous, but oddly the other two such expressions are not. This effectively refutes the planar Eucleadian R-geometry of Tarski.