Einstein's field theory is wrong and Nordström's correct

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Abstract

Gunnar Nordström published his second gravitation theory in 1913. This theory is today considered to be inconsistent with observations. At this time Einstein was working on his field theory, the General Relativity Theory. Einstein's theory has been accepted as the only theory of gravitation consistent with measurements. The article reconsiders Nordström's theory and proves the following claims. 1) If gravitation is caused by a scalar field, then the theory is Nordström's second gravitation, which in a vacuum outside a point mass reduces to his first gravitation theory. Nordström's scalar field theory gives proper time values that fully agree with gravitational redshift in the Pound-Rebka experiment and with the Shapiro time delay in Shapiro's radar bouncing experiment. Gravitation in Schwarzschild's solution is not a field but a deformed geometry. If proper time is calculated via the General Relativity formula, Schwarzschild's solution fails both the Pound-Rebka redshift and Shapiro time delay tests because the ball in Schwarzschild's solution is deformed and light as measured by an external clock can exceed c*. 2) The third classical tests of Einstein's theory is the movement of the perihelion of Mercury. Calculations from Schwarzschild's exact solution to Einstein's equations gave a correction that very well fitted the unexplained part of Mercury's movement. However, Schwarzschild's solution as a stationary solution it fails to explain why the orbit of Mercury, or any planet, is an ellipse. It is shown that the customary proof of Kepler's law stating that the orbit is an ellipse is incorrect: under a central stationary Newtonian force the orbit of a two mass system can only be a circle or (almost) a hyperbole because of conservation of energy. This observation invalidates the movement of Mercury as a test of General Relativity: Schwarzschild's solution cannot produce an elliptic orbit, therefore it is not the solution and that it gives a correct size modification to the movement of the perihelion is just a coincidence. Nordström's theory remains inconclusive in the Mercury test because calculating the orbit is difficult and cannot be done in this article. Nordström's theory, however, offers a possibility for explaining elliptic orbits: some energy is needed for waves in time-dependent solutions to Nordström's field equation and this* *loss of potential energy from the radial potential can lead to elliptic orbits. 3) The fourth classical test is the light bending test. Light bends in Nordström's theory as light behaves as a test mass in a gravitational field. Calculation of the amount of light bending in Norstrsöm's theory is similar to calculation of the orbit of planets and beyond the scope of this article. Theoretical consideration of bending of light leads to the conclusion that the stress-energy tensor in the General Relativity is incorrect: the diagonal entries should contain the energy of a stationary gravitational field in the vacuum outside a point mass and therefore diagonal Ricci tensor entries cannot be zeroes. Nordström's theory passes this theoretical consideration while Einstein's theory fails it.*

1. Introduction

Gunnar Nordstöm's two scalar theories of gravitation were published in the the set of articles [1]. A historical overview of the development and rejection of his theories explaining the arguments of that time is given in [2]. A fairly recent and very interesting scientific paper exploring the final for of Nordström's theory is in [3]. It explores the unique property that Nordström's theory shares with the General Relativity: in both theories the gravitational mass of the universe equals the inertial mass.

The field equation in Nordström's second gravitation theory is

$$\Phi\Box\Phi = -4\pi\rho \tag{1}$$

Φ is a continuous scalar field defined in the flat Minkowski four-space. ρ is scalar and defined in the four-space, but not necessarily continuous: in many cases mass can be replaced by a set of point masses, thus ρ is often best treated as a set of singularities. The gravitation constant G and the speed of light c are both set to one in this equation and $\Box$ is the D'Alembertian operator.

The field Φ in a flat Minkowski space gives the line element in Cartesian coordinates as (c is set to one.)

$$ds^2 = \Phi^2 dt^2 - \Phi^2 dx^2 - \Phi^2 dy^2 - \Phi^2 dz^2 \tag{2}$$

Einstein noticed (see [2][3][4]) that (2) can be written as

$$ds^2 = g^{ab} dx^a dx^b \tag{3}$$

for $g_{00} = \Phi^2$, $g_{ii} = -\Phi^2$ for $i > 0$ and $g_{ab} = 0$ if $a \neq b$. Thus, (2) can be interpreted as a line element of a curved Lorentz four-space. The Ricci scalar of the curved Lorentz space satisfies

$$R = -6\Phi^{-3}\Box\Phi \tag{4}$$

and (1) can be expressed as a geometric equation

$$\Phi^{-3}\Box\Phi = 24\pi GT \tag{5}$$

where T describes the mass-energy distribution. A bit later, in 1915-1916, Einstein formulated the General Relativity field equations

$$R_{ab} - \frac{1}{2}R = k_0 T_{ab} - \lambda g_{ab} \tag{6}$$

where $k_0 = 8\pi G/c^4$, T_{ab} is the stress tensor and λ is the cosmological constant. Nordström's field equation (1) can be obtained from Einstein's field equations by taking a trace provided that the metric tensor g_{ab} has the special form as in (3).

In the last form of Nordström's theory ρ in the field equation (1) was understood as T in (5) and as equal to the trace of the energy-stress tensor T_{ab} of the General Relativity, but in earlier forms of the theory ρ and T are not exactly related in this way. I will use the form (5) for Nordström's theory and will also adopt the interpretation of ρ in (1) as related to the stress-energy tensor T_{ab} as in (5), but will not consider the T_{ab} of Nordström's theory to be exactly the same as the T_{ab} in the General Relativity. The difference between the T_{ab} in the two theories is that at least in early forms of Nordström's theory the diagonal elements T_{aa} included the energy of the gravitational field and were not zero in a vacuum outside a point mass.

The line element in Nordström's theory has to be of the form (2) because if the field has the value $\Phi(x)$ at a point $x = (x^0, x^1, x^2, x^3)$ and it is continuous, then in a close distance from the point x in every direction the field has almost the same value $\Phi(x)$. It follows that the line element must be in Cartesian coordinates

$$ds^2 = \Phi^2 dt^2 - \Phi^2 dx^2 - \Phi^2 dy^2 - \Phi^2 dx^2$$

and in spherical coordinates

$$ds^2 = \Phi^2 dt^2 - \Phi^2 dx^2 - r^2\Phi^2 d\theta^2 - r^2 \sin^2(\theta)\Phi^2 d\psi^2$$

In Nordström's theory the field Φ is a continuous scalar in the Minkowski space and though the field equation can be calculated from geometry as in (5) the metric in (2) does not admit in a natural geometric interpretation: close to a mass the field Φ is stronger. It means that if ds^2 is a line element, the volume of the line element grows as Φ^3 when we go closer to a point mass. The space expands and the punctuated vacuum extends to the infinity when we approach the mass.

Schwarzschild's exact solution to Einstein's equations has the line element

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2(\theta) d\phi^2 \tag{7}$$

where

$$A(r) = \left(1 - \frac{2GM}{rc^2}\right)^{-1} \quad , \quad B(r) = 1 - \frac{2GM}{rc^2}$$

It is not of the type (2) and thus Schwarzschild's solution is not a field. In Schwarzschild's solution the space elements enlongate in the radial direction when we approach a mass and the solution can be imagined as (a four-dimensional version of) a membrane where a mass bends the geometry. In Schwarzschild's solution gravitation is curved geometry.

In orthogonal coordinates the nonzero Christoffel symbols are ($b \neq a$, no summation)

$$\Gamma^a_{aa} = \frac{1}{2} g^{aa} g_{aa,a} \qquad \Gamma^a_{ab} = \frac{1}{2} g^{aa} g_{aa,b} \qquad \Gamma^a_{bb} = -\frac{1}{2} g^{aa} g_{bb,a}$$

The Ricci curvature tensor is defined as $R_{bd} = R^a_{bad}$ where

$$R^a_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed}$$

is the Riemann curvature tensor. The Ricci scalar is $R = g^{ab} R_{ab}$. The Ricci tensor has the symmetry $R_{ab} = R_{ba}$ and consequently there are ten distinct entries.

Directly calculating from these definitions we get for orthogonal coordinates (no summing convention, $j \in \{0, 1, 2, \}$)

$$R_{jj} = f_j - \sum_{\substack{i=0 \\ i \neq j}}^{4} \left\{ \frac{1}{4} g^{ii} g_{jj,i} \left(\sum_{\substack{k=0 \\ k \neq j}}^{4} g^{kk} g_{kk,i} - g^{jj} g_{jj,i} \right) + \frac{1}{2} \partial_i (g^{ii} g_{jj,i}) \right\} \tag{8}$$

where

$$f_j = \sum_{\substack{i=0 \\ i \neq j}}^{4} \left\{ \frac{1}{4} g^{ii} g_{ii,j} (g^{jj} g_{jj,j} - g^{ii} g_{ii,j}) - \frac{1}{2} \partial_j (g^{ii} g_{ii,j}) \right\} \tag{9}$$

and the cross terms are

$$R_{ij} = \frac{1}{4} g^{jj} g_{jj,i} (g^{kk} g_{kk,j} + g^{mm} g_{mm,j}) + \frac{1}{4} g^{kk} g_{kk,i} (g^{ii} g_{ii,j} - g^{kk} g_{kk,j})$$

$$+ \frac{1}{4} g^{mm} g_{mm,i} (g^{ii} g_{ii,j} - g^{mm} g_{mm,j})$$

$$- \frac{1}{2} \partial_i (g^{ii} g_{ii,j}) - \frac{1}{2} \partial_j (g^{ii} g_{ii,i} + g^{kk} g_{kk,i} + g^{mm} g_{mm,i}) \tag{10}$$

where $j > i$, and $k, m \notin \{i, j\}$.

The equation (5) is a wave function, so all its solutions in polar coordinates can be constructed from product form solutions. The wave function in spherical coordinates is solved by separating all variables (r, θ, ψ, t). These product form solution are products of spherical Hankel functions and spherical harmonics. The metric tensor in Schwarzschild's solution also has the product form. Finally, it is very difficult to see what else but a product form could zero all six R_{ij}, $j > i$, in (10). For these reasons I assume that the solution of vacuum space outside a point mass, i.e., when $R_{jj} = 0$, $j > i$, and $R = 0$, has a product form. For a product form

$$g_{jj} = A_{j0}(x^0) A_{j1}(x^1) A_{j2}(x^2) A_{j3}(x^3) \tag{11}$$

the function

$$y_{ij} = g^{ii} g_{ii,j}$$

is a function of x^j only and f_j in (9) depends only on x^j. Inserting (11) to R_{jj} gives

$$R_{jj} = f_j(x^j) - \sum_{\substack{i=0 \\ i \neq j}}^{4} \frac{A_{jj}}{A_{ij}} G_{ji}$$

where G_{ji} is a function of other coordinates than x^j. In order for the solution to be separating variables, we must be able to separate x^j from this equation. It can be done in two ways, either A_{ij} is a constant times A_{jj}, or the terms G_{ji} disappear.

A stationary spherically symmetric solution has $A'_{j0} = A'_{j2} = A'_{j3} = 0$ for every j. This guarantees that $R_{ij} = 0$ for every case of $j > i$ and it also causes $G_{ji} = 0$ for all i and j for a product form solution. For this kind of solution holds:

$$g^{00}R_{00} = -\frac{1}{4}g^{11}g^{00}g_{00,1}(-g^{00}g_{00,1} + g^{11}g_{11,1} + g^{22}g_{22,1} + g^{33}g_{33,1}) \tag{12}$$

$$-\frac{1}{2}g^{00}\partial_1(g^{00}g_{00,1})$$

$$g^{22}R_{22} = -\frac{1}{4}g^{11}g^{22}g_{22,1}(g^{00}g_{00,1} + g^{11}g_{11,1} - g^{22}g_{22,1} + g^{33}g_{33,1})$$

$$-\frac{1}{2}g^{22}\partial_1(g^{22}g_{22,1})$$

$$g^{33}R_{33} = -\frac{1}{4}g^{11}g^{33}g_{33,1}(g^{00}g_{00,1} + g^{11}g_{11,1} + g^{22}g_{22,1} - g^{33}g_{33,1})$$

$$-\frac{1}{2}g^{33}\partial_1(g^{22}g_{22,1})$$

$$g^{11}R_{11} = \frac{1}{4}g^{11}g^{00}g_{00,1}(g^{11}g_{11,1} - g^{00}g_{00,1})$$

$$+\frac{1}{4}g^{11}g^{22}g_{22,1}(g^{11}g_{11,1} - g^{22}g_{22,1}) + \frac{1}{4}g^{11}g^{33}g_{33,1}(g^{11}g_{11,1} - g^{33}g_{33,1})$$

$$-\frac{1}{2}g^{11}\partial_1(g^{00}g_{00,1}) - \frac{1}{2}g^{11}\partial_1(g^{22}g_{22,1}) - \frac{1}{2}g^{11}\partial_1(g^{33}g_{33,1})$$

Summing these terms gives

$$R = -\frac{1}{2}g^{00}g_{00,1}g^{22}g_{22,1} \tag{13}$$

$$-\frac{1}{2}g^{00}g_{00,1}g^{33}g_{33,1} - \frac{1}{2}g^{22}g_{22,1}g^{33}g_{33,1}$$

$$-\frac{1}{2}(g^{00} + g^{11})\partial_1(g^{00}g_{00,1}) - \frac{1}{2}(g^{22} + g^{11})\partial_1(g^{22}g_{22,1}) - \frac{1}{2}(g^{33} + g^{11})\partial_1(g^{33}g_{33,1})$$

Taking the form for a gravitational field as in (2) and writing $A = \Phi^2$ the nonzero elements of the metric tensor are $g_{00} = A$, $g_{11} = -A$, $g_{22} = -r^2A$ and $g_{33} = -r^2\sin^2(\theta)A$. Assuming that $A = A(r) = A_{j1}(x^1)$ for every j the diagonal Ricci tensor entries from (12) are

$$R_{00} = \frac{1}{2}A''A^{-1} + \frac{1}{r}A'A^{-1} \tag{14}$$

$$R_{11}=-\frac{3}{2}A''A^{-1}+\frac{3}{2}(A')^{-2}+\frac{1}{r}A'A^{-1}$$

$$R_{22}=-\frac{1}{2}r^2A''A^{-1}-2rA'A'^{-1}$$

$$R_{33}=\sin^2(\theta)R_{22}$$

and the Ricci scalar from (13) as

$$R=3A''A^{-2}-\frac{3}{2}(A')^2A^{-3}+\frac{6}{r}A'A^{-2} \tag{15}$$

The equation $R=0$ gives

$$A''A^{-\frac{1}{2}}-\frac{1}{2}(A')^2A^{-\frac{3}{2}}+\frac{2}{r}A'A^{-\frac{1}{2}}=0$$

Denoting $y=A'A^{-1}$ the equation for Nordström's theory in the vacuum outside a point mass is

$$y'+\frac{1}{r}y=0$$

Thus $y=kr^{-2}$ for some k and

$$\Phi(r)^2=A=(b-\frac{k}{2r})^2$$

for some b. We get the field potential

$$\Phi(r)=b-\frac{k}{2r} \tag{16}$$

from which b and k can be identitied as $b=0$ and $k/2=GM$. Thus, $\Phi=\Phi(r)$ gives the Newtonian potential. This is what it should give since by (5) the equation $R=0$ in this vacuum reduces to

$$\Box\Phi=0$$

In spherical coordinates D'Alembertian is

$$\Box\Phi=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)$$

$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\psi^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

The solution for $\Phi = \Phi(r)$ is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right) = \frac{2}{r}\Phi' + \Phi'' = 0$$

Thus $\Phi' = kr^{-2}$ and $\Phi = -kr^{-1}$ giving the same result as (16).

The constant $k = GM$ but we for simplicity we set it to one and insert $A_{j1} = r^{-2}$ to (12). The result is $R_{00} = -1$, $R_{11} = 1$, $R_{22} = R_{33} = 0$ and $R_{ij} = 0$ if $i \neq j$. This shows that Einstein's equations are not satisfied by a metric tensor g_{ab} as in (3). The diagonal Ricci tensor entries are not all zeros for g_{ab} derived from a field Φ. However, they do satisfy $R = 0$ in a vacuum outside a point mass. This means that if gravitation is a field, Einstein's equations are not satisfied and Nordström's equations are.

2. Ricci tensor entries in Nordström's field equation

In the vacuum the field equation of Nordström's second gravitation theory is the wave equation. The wave equation is linear and therefore linear combinations of product form solutions also fill it, but these linear combinations are usually not solutions to time dependent Nordström's field equations in spherical coordinates because Nordström's field equation is not linear outside the vacuum and this restricts the set of acceptable solutions.

The product form solutions to (2) are found by separating variables. We can look for product form solutions of the form

$$g_{00} = A, \quad g_{11} = -A, \quad g_{22} = -r^2 A, \quad g_{33} = -r^2\sin^2(\theta)A$$

$$A = A_0(t)A_1(r)A_2(\theta)A_3(\psi)$$

From (10) we obtain

$$R_{0j} = \frac{1}{2}A_j' A_j^{-1} A_0' A_0^{-1}, \qquad j = 1, 2, 3$$

$$R_{12} = \frac{1}{2}A_2' A_2^{-1}\left(A_1' A_1^{-1} + \frac{2}{r}\right) \qquad R_{13} = \frac{1}{2}A_3' A_3^{-1} A_1' A_1^{-1}$$

$$R_{23} = \frac{1}{2}A_3' A_3^{-1}(A_2' A_2^{-1} + 2\cot\theta)$$

From these expressions we see that for the time-dependent case the cross entries $R_{0,j}$, $j > 0$, are not zero for g_{ab} derived from a field Φ. These entries indicate that there is a flow of mass-energy and momentum. A physical interpretation for such a flow can for instance be a planet moving in an orbit around the sun: the planet slightly disturbs the gravitational field of the sun.

For a stationary field we derived the Newtonian potential $\Phi(r) = k/r$. If a field Φ is time dependent, the radial dependency is different. This is because when r is separated from θ, ψ, t a constant must be added, i.e.,

$$f(r) + g(\theta, \psi, t) = 0 \quad \text{gives} \quad f(r) = C \quad \text{and} \quad g = -C$$

Looking at the forms of R_{ii} and R for the time dependent case of Nordström's theory given below make it obvious that the solution to the equation $R = 0$ does not usually satisfy the equations $R_{aa} = 0$ for all a. (They may satisfy all equations, for instance if $\Phi = 0$.) The most general $\Phi = \Phi(r, \theta, \psi, t)$ has the following nonzero diagonal Ricci tensor entries:

$$R_{00} = \frac{1}{2}A^{-1}\left\{\frac{\partial^2 A}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 A}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 A}{\partial \phi^2} - 3\frac{\partial^2 A}{\partial t^2}\right\} \tag{17}$$

$$+\frac{1}{2}A^{-1}\left\{+\frac{2}{r}\frac{\partial A}{\partial r} + \frac{1}{r^2}\cot\theta\frac{\partial A}{\partial \theta}\right\} + \frac{3}{2}A^{-2}\left(\frac{\partial A}{\partial t}\right)^2$$

$$R_{11} = \frac{1}{2}A^{-1}\left\{-3\frac{\partial^2 A}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2 A}{\partial \theta^2} - \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 A}{\partial \phi^2} + \frac{\partial^2 A}{\partial t^2}\right\}$$

$$+\frac{1}{2}A^{-1}\left\{-\frac{2}{r}\frac{\partial A}{\partial r} - \frac{1}{r^2}\cot\theta\frac{\partial A}{\partial \theta}\right\} + \frac{3}{2}A^{-2}\left(\frac{\partial A}{\partial r}\right)^2$$

$$R_{22} = \frac{1}{2}A^{-1}\left\{-r^2\frac{\partial^2 A}{\partial r^2} - 3\frac{\partial^2 A}{\partial \theta^2} - \frac{1}{sin^2\theta}\frac{\partial^2 A}{\partial \phi^2} + r^2\frac{\partial^2 A}{\partial t^2}\right\}$$

$$+\frac{1}{2}A^{-1}\left\{-4r\frac{\partial A}{\partial r} - \cot\theta\frac{\partial A}{\partial \theta}\right\} + \frac{3}{2}A^{-2}\left(\frac{\partial A}{\partial \theta}\right)^2 + 1$$

$$R_{33} = \frac{1}{2}A^{-1}\left\{-r^2\sin^2\theta\frac{\partial^2 A}{\partial r^2} - \sin^2\theta\frac{\partial^2 A}{\partial \theta^2} - 3\frac{\partial^2 A}{\partial \phi^2} + r^2\sin^2\theta\frac{\partial^2 A}{\partial t^2}\right\}$$

$$+\frac{1}{2}A^{-1}\left\{-4r\frac{\partial A}{\partial r} + 3\sin\theta\cos\theta\frac{\partial A}{\partial \theta}\right\} + \frac{3}{2}A^{-2}\left(\frac{\partial A}{\partial \phi}\right)^2 - \sin^2\theta$$

and the Ricci scalar is

$$R = 3A^{-3}\left\{\frac{\partial^2 A}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 A}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 A}{\partial \phi^2} - \frac{\partial^2 A}{\partial t^2} + \frac{2}{r}\frac{\partial A}{\partial r} + \frac{2}{r}cot\theta\frac{\partial A}{\partial \theta}\right\}$$

$$+3A^{-3}\left\{\frac{1}{2}A^{-1}\left((\frac{\partial A}{\partial t})^2 - (\frac{\partial A}{\partial r})^2 - \frac{1}{r^2}(\frac{\partial A}{\partial \theta})^2 - \frac{1}{r^2\sin^2\theta}(\frac{\partial A}{\partial \phi})^2\right)\right\}$$

In spherical coordinates D'Ambertian $\Box\Phi$ for $A = \Phi^2$ gives

$$\Box\Phi = -\frac{1}{2}A^{\frac{-1}{2}}\left\{\frac{2}{r}\frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial r^2} - \frac{1}{2}A^{-1}(\frac{\partial A}{\partial r})^2 - \frac{\partial^2 A}{\partial t^2} + \frac{1}{2}A^{-1}(\frac{\partial A}{\partial t})^2\right\}$$

$$-\frac{1}{2}A^{\frac{-1}{2}}\left\{+\frac{1}{r^2}\cot\theta\frac{\partial A}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 A}{\partial \theta^2} - \frac{1}{r^2}\frac{1}{2}A^{-1}(\frac{\partial A}{\partial \theta})^2\right\}$$

$$-\frac{1}{2}A^{\frac{-1}{2}}\left\{+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 A}{\partial \phi^2} - \frac{1}{r^2\sin^2\theta}\frac{1}{2}A^{-1}(\frac{\partial A}{\partial \phi})^2\right\}$$

Thus

$$R = -6A^{-\frac{3}{2}}\Box\Phi = -6\Phi^{-3}\Box\Phi$$

In Cartesian coordinates

$$g_{00} = \Phi^2 \quad , \quad g_{ii} = -\Phi^2 \quad , \quad g_{ab} = 0 \quad \text{if} \quad a \neq b$$

the diagonal elements of the Ricci curvature tensor are

$$R_{00} = -\Phi^{-1}\Box\Phi + \Phi^{-2}\sum_{j=1}^{3}(\frac{\partial\Phi}{\partial x^j})^2 + 3\Phi^{-2}(\frac{\partial\Phi}{\partial t})^2 - 2\Phi^{-1}\frac{\partial^2\Phi}{\partial t^2}$$

and for $i = 1,2,3$

$$R_{ii} = -\Phi^{-1}\Box\Phi - \Phi^{-2}\sum_{j=1}^{3}(\frac{\partial\Phi}{\partial x^j})^2 + \Phi^{-2}(\frac{\partial\Phi}{\partial t})^2$$

$$-2\Phi^{-1}\frac{\partial^2\Phi}{\partial(x^i)^2} + 4\Phi^{-2}(\frac{\partial\Phi}{\partial x^i})^2$$

As $g^{ab} = \Phi^{-2}\eta^{ab}$ in Cartesian coordinates, we get

$$R = g^{ab}R_{ab} = -4\Phi^{-3}\Box\Phi + 4\Phi^{-4}\sum_{j=1}^{3}(\frac{\partial\Phi}{\partial x^j})^2 + 0$$

$$-2\Phi^{-3}\Box\Phi - 4\Phi^{-4}\sum_{j=1}^{3}(\frac{\partial\Phi}{\partial x^j})^2$$

that is, we get Equation (4)

$$R = -6\Phi^{-3}\Box\Phi$$

but every R_{aa} is not zero in a vacuum outside a point mass.

3. Does Nordström's field theory fail the classical tests?

Evaluation which of the field theories is correct is today left to the empirical tests of the General Relativity. Does Nordström's theory fail three of the four tests, or is it Einstein's theory that fails three out of four? See the Wikipedia entry [4] for a calculation that concludes that Nordström's theory fails three of the four tests.

I see certain problems in the calculations of [4]. It is stated that Nordström's second field theory comes from the Langangian

$$L = \frac{1}{8\pi}\eta^{ab}\Phi_{,a}\Phi_{,b} - \rho\Phi$$

but calculating

$$\frac{\partial L}{\partial\Phi} = -\rho$$

$$\frac{\partial L}{\partial\Phi_{,a}} = \frac{1}{4\pi}\eta^{aa}\Phi_{,a}$$

gives the Euler-Langange equations

$$\frac{\partial L}{\partial\Phi} - \sum_{\mu}\partial_{\mu}\frac{\partial L}{\partial\Phi_{,\mu}}$$

$$= -\rho - \frac{1}{4\pi}\sum_{\mu}\eta^{\mu\mu}\frac{\partial^2\Phi}{\partial(x^{\mu})^2} = -\rho - \frac{1}{4\pi}\Box\Phi$$

and the Lagrangian produces the field equation of Nordström's first theory

$$\Box\Phi = -4\pi\rho$$

This is a minor issue since the second theory can be obtained from a very similar Lagrangian

$$L = \frac{1}{8\pi}\eta^{ab}\Phi_{,a}\Phi_{,b} - \frac{1}{4}\rho\Phi^4$$

but the calculations in [4] seem to use the geodesic Lagrangian, which is given as

$$L = \Phi^2 \eta_{ab} \dot{u}^a \dot{u}^b$$

and said to produce the equation of motion for Nordström's second theory

$$\Phi \dot{u}_a = -\Phi_{,a} - \dot{\Phi} u_a \tag{18}$$

However, the expression is not in a form of a Lagrangian. It can be obtained by inserting $\Phi \dot{u}_a = \Phi_{,a}$ into the Langangian

$$L = \Phi^2 \eta^{ab} \Phi_{,a} \Phi_{,b} = \eta^{ab} \Phi \dot{u}_a \Phi \dot{u}_b$$

which after raising and lowering indices comes to

$$= \Phi^2 \eta_{ab} \dot{u}^a \dot{u}^b$$

but if this is the way it is derived, it is for a stationary field, that is, $\dot{\Phi} u_a = 0$ in the equations of motion (18). Though is it correct to look at the stationary field in most of the tests of General Relativity (when Nordström's theory gives the Newtonian potential $\Phi = \Phi(r)$), the calculations in [4] take a time dependent solution for the wave function and derive properties from it, though the equations of motion seem to be for the stationary case.

I make different calculations and the conclusions in the subsections are different form those in [4]. I hope my calculations are more transparent than the ones in [4].

3.1 Frequency shift in gravitational fields

Gravitational redshift has been demonstrated in the Pound-Rebka experiment. This redshift is caused by the equivalence principle, which Nordström's theory satisfies, see [2]. The equations of motion of Nordström's second theory in a vacuum outside a point mass are Lorentz invariant forms of equations of motion in classical Newtonian theory as the potential is the Newtonian potential and the field equation is the wave equation. Thus (18) reduces in this case to

$$\dot{u}_a = -\Phi_{,a}$$

where

$$-\Phi_{,1} = -\frac{\partial \Phi}{\partial r} = -\frac{GM}{r^2}$$

The stationary case $\Phi = \Phi(r)$ describes the gravitational field in the Pound-Rebka experiment and the radial direction is the only relevant one. The radial acceleration is thus

$$\dot{u} = -\frac{GM}{r^2}$$

where $u = u_r$ is the radial velocity of a photon. The derivation is with respect to the proper time τ. In the special relativity

$$\tau = \sqrt{1 - \frac{u^2}{c^2}} dt \qquad (19)$$

We can use this definition of proper time in this test as the proper time is actually not needed: the final result is wave lengths in the external time. From the metric tensor in Nordström's field theory follows that

$$\frac{dr}{dt} = \frac{cAdr}{Adt} = c$$

where the speed of light c is shown explicitly for clarity. Then

$$\frac{du}{d\tau} = \frac{du}{dr}\frac{dr}{dt}\frac{dt}{d\tau} = c\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}\frac{du}{dr}$$

and

$$\int_{r_1}^{r_2} \left(-\frac{GM}{r^2}\right) dr = \int_{u_1}^{u_2} c\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} du = \int_{u_1/c}^{u_2/c} c\frac{1}{\sqrt{1 - y^2}} dy$$

giving

$$\frac{GM}{c^2}\frac{r_2 - r_1}{r_2 r_1} = \arcsin\frac{u_1}{c} - \arcsin\frac{u_2}{c} = \frac{u_2 - u_1}{c\sqrt{1 - \frac{u_1^2}{c^2}}} = \frac{\Delta u}{c\sqrt{1 - \frac{u_1^2}{c^2}}}$$

The change of the speed u is here expressed as $\Delta u/\tau$, that is, the time is the proper time. In the external time the equation is

$$\frac{GM}{c^2}\frac{r_2 - r_1}{r_2 r_1} = \frac{\Delta u}{c}$$

Setting $r_2 = R + h$, $r_1 = R$ and approximating

$$\frac{GM}{c^2}\frac{h}{(R + h)R} \approx \frac{GM}{c^2}\frac{h}{R^2}$$

and changing to wave lengths

$$\frac{\Delta u}{c} = \frac{\Delta\lambda f}{\lambda f} = \frac{\Delta\lambda}{\lambda}$$

gives the redshift in the Pound-Rebka experiment

$$\frac{GM}{c^2}\frac{h}{R^2} = \frac{\Delta\lambda}{\lambda}$$

Einstein's field theory also fills the equivalence principle and passes this test if the special relativity definition is used for the proper time. In this definition proper time does not run at all for photons which travel with the speed of light. In the frame moving with photons the photons do not move at all. Thus, in that frame no space is moved and no time is used: the speed of light in the proper time is still c. In General Relativity there is another formula given in (20) for the proper time difference caused by a gravitational field. I leave it to a comment in the next section to consider what implications it has to the Pound-Rebka redshift for Schwarzschild's solution.

3.2 Shapiro time delay in gravitational fields

In the Shapiro time delay test a radar signal is sent from the Earth to another planet, like Venus, and echoed back to the Earth. A longer delay is measured if the sun is close to the path than if the sun is far. The expression of the additional delay contains three distances: the straight line connecting the Earth and the planet is divided into two parts, one of length x_e between the Earth and the connection point and one of length x_p between the planet and the connection point. The sun is on the distance d on the straight line from connection point orthogonal to the line from the Earth to the planet. The delay is

$$\Delta t \approx \frac{2GM}{c^3} \ln \frac{4x_p x_e}{d^2}$$

This delay is mainly a result of the variable speed of light in a gravitational field. The delay formula was derived from Schwarzshild's solution.

Light bends in gravitational fields but the formula for the Shapiro time delay does not have any parameters for the orbit: the orbit is hyperbolic (or at least very close to hyperbolic) and we would expect to see the parameters a and b of a hyperbole

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

but they do not appear. This is because the delay caused by bending of the path is much smaller than the delay caused by the gravitational redshift as in the Pound-Rebka experiment. We can ignore the bending of the path in the first order approximation and think about the path as a straight horizontal line. The sun is at the point $(0,d)$, the Earth is at $(-x_e,0)$ and Venus at $(x_p,0)$. The distance d is approximately sun's radius $d \approx 0.6957 * 10^9 m$. The distance x_e is almost the same as the distance of the Earth from the sun and can be taken as $x_e \approx 148 * 10^9 m$. The distance x_p is almost the distance of the planet from the sun. For Venus it is about $x_p \approx 108 * 10^9 m$. The distances x_e and x_p are approximations as the distances vary in the orbits, but they are quite sufficient for this test.

In the General Relativity theory there is the following formula for the proper time difference caused by a gravitational field

$$\Delta\tau = \int_{t_1}^{t_2} \frac{1}{c}\sqrt{g_{00}}dt \qquad (20)$$

This formula is scaled in a specific way and the metric tensor must be rescaled in order to use the formula. Thus, for Nordström's theory in the vacuum

$$\sqrt{g_{00}} = \Phi(r) = -\frac{GM}{r}$$

and as it has the quality $\frac{m^2}{s^2}$, the expression for $\Delta\tau$ is

$$\Delta\tau = \int_{t_1}^{t_2} \frac{1}{c^2}\sqrt{g_{00}}dt$$

that is, otherwise we do not get seconds. In Schwarzschild's solution

$$\sqrt{g_{00}} = \sqrt{B(r)}$$

is a plain number and we have to use the formula as

$$\Delta\tau = \int_{t_1}^{t_2} \sqrt{g_{00}}dt$$

in order to get seconds.

We can calculate the proper time for Nordström's theory

$$\Delta\tau = \int_{t_1}^{t_2} \frac{1}{c^2}\Phi dt$$

and as from the metric tensor $Adt = cAds$ for a line element ds of any path, we get $dt = cds$ and

$$\Delta\tau = \int_{s_1}^{s_2} \frac{1}{c^3}\frac{GM}{r}ds$$

The integral is easily calculated:

$$\int_0^x \frac{1}{r}dx = \int_0^x \frac{1}{\sqrt{1+\frac{x^2}{d^2}}}\frac{dx}{d}$$

$$= \int_0^{x/d} \frac{1}{\sqrt{1+y^2}}dy = \ln\left(\sqrt{1+\frac{x^2}{d^2}} + \frac{x}{d}\right)$$

$$\approx \ln(2\frac{x}{d})$$

as $\frac{x}{d} >> 1$. For any smooth f changing $x = -y$ shows that

$$\int_{-x}^0 f(x^2)dx = -\int_y^0 f(y^2)dy = \int_0^y f(y^2)dy$$

Thus

$$\int_{-x_e}^{x_p} \frac{1}{r}dx \approx \ln(2\frac{x_p}{d}) + \ln(2\frac{x_e}{d})$$

$$= \ln(4\frac{x_e x_p}{d^2})$$

The result for Nordström's theory for the proper time difference is

$$\Delta\tau = -\frac{GM}{c^3}\ln\left(\frac{4x_e x_p}{d^2}\right)$$

This $\Delta\tau$ is negative: the proper time goes forward slower than the external time. It means that light moves slower than c. That is, if light travels with the speed pc, $0 < p < 1$, light travels the distance $-x_e + x_p$ in the time $t = \frac{-x_e+x_p}{pc}$. The Shapiro time delay measured by an external clock is

$$\Delta t = \frac{-x_e + x_p}{c}\left(\frac{1}{p} - 1\right)$$

In the proper time in Nordström's theory light travels with the speed c. Thus, the proper time is $\tau = \frac{-x_e+x_p}{c}$ and the difference between the external time and proper time is

$$\Delta\tau = \tau - t = -\frac{-x_e + x_p}{c}\left(1 - \frac{1}{p}\right) = -\Delta t$$

For a roundtrip path we get the delay

$$\Delta t = 2\frac{GM}{c^3}\ln\left(\frac{4x_e x_p}{d^2}\right)$$

which is exactly the expression in the Shapiro time delay. Inserting numbers we get the roundtrip delay as $240\mu s$, which agrees with observations.

Let us calculate the Shapiro time delay for Schwarzschild's solution in a similar way. We start from

$$\Delta\tau = \int_{t_1}^{t_2} \sqrt{B}dt$$

Next we have to change the integration from time to a space variable. From the metric tensor of Schwarzschild's solution

$$\sqrt{B}dt = c\sqrt{A}dy \quad , \quad \sqrt{B}dt = dz$$

where the (y, z)-coordinates are Cartesian coordinates selected so that y is parallel to r and z is orthogonal to r. That is, the first expression we get by considering a move to a direction when $dr \neq 0$ and $d\theta = d\psi = 0$. The second expression corresponds to $d\theta \neq 0$ and $dr = d\psi = 0$. The reason for not using the polar coordinates is that $r^2 d\theta$ causes unnecessary complications in a simple calculation.

As we assume that the path is closely approximated by a horizontal line we can write dy and dz with the line element ds as

$$dy = \cos(\alpha)ds = \frac{x}{\sqrt{x^2+b^2}}ds$$

$$dz = \sin(\alpha)ds = \frac{b}{\sqrt{x^2+b^2}}ds$$

where α is the angle between the horizontal line and the line from (x, y) on the path to the sun. Thus

$$\Delta\tau = \frac{1}{c}\int_{s_1}^{s_2}\left(\frac{x}{\sqrt{x^2+b^2}}\frac{1}{\sqrt{1-\frac{2GM}{c^2 r}}} + \frac{b}{\sqrt{x^2+b^2}}\right)ds$$

Assuming that the path is a horizontal line and inserting

$$\frac{1}{\sqrt{1-\frac{2GM}{c^2r}}} \approx 1+\frac{GM}{c^2r}$$

the integral gives

$$\Delta\tau = \frac{1}{c}\int_{-x_e}^{x_p}\frac{x}{\sqrt{x^2+b^2}}dx$$

$$+\frac{1}{c}\frac{GM}{c^2}\int_{-x_e}^{x_p}\frac{x}{x^2+b^2}dx+\frac{b}{c}\int_{-x_e}^{x_p}\frac{1}{\sqrt{x^2+b^2}}dx$$

which is approximated by

$$\frac{b}{c}\int_{-x_e}^{x_p}\frac{\frac{x}{b}}{\sqrt{1+\frac{x^2}{b^2}}}\frac{dx}{b}$$

$$+\frac{GM}{c^3}\int_{-x_e}^{x_p}\frac{\frac{x}{b}}{1+\frac{x^2}{b^2}}\frac{dx}{b}+\frac{b}{c}\ln(4\frac{x_ex_p}{d^2})$$

$$\approx \frac{1}{c}(-x_e+x_p)+\frac{b}{c}\ln\left(\frac{4x_ex_p}{d^2}\right)+\frac{GM}{c^3}\ln\left(\frac{x_ex_p}{d^2}\right)$$

This $\Delta\tau$ does not agree with the Shapiro time delay and we notice that it is positive: the proper time goes forward faster than the external time. It means that light moves faster than c. The proper time takes into account the gravitational redshift also in the case of Schwarzschild's solution: it is the $\sqrt{g_{00}}$ term. Without this delay the speed of light in Scwarzschild's geometry would be even faster.

The reason for exceeding the speed of light is the geometry of the ball in Schwarzschild's geometry. The ratio of the space element to the time element gives the speed of light. In a flat Minkowski space in Cartesian coordinates the line element is

$$ds^2 = \frac{1}{c^2}dt^2 - dx^2 - dy^2 - dz^2$$

The speed c corresponds to moving the space distance dx in the time $\frac{1}{c}dt$. Likewise, in spherical coordinates moving dr in the time $\frac{1}{c}dt$ is moving with the speed c. In polar coordinates with θ we get the space element $r^2d\theta$ but this still means moving with the speed of light, only the tangential space element is longer in polar coordinates.

When the geometry is changed as in Schwarzschild's solution, moving the distance of a radial space element $\sqrt{A}dr$ in the time element $\frac{\sqrt{B}}{c}dt$ means the speed $c\sqrt{A/B}$. As in Schwarzschild's solution AB is a constant, it means that in a gravitational field light moves faster than c. In proper time it is less, because of the redshift the speed in the proper time is $c\sqrt{A}$. This is still above c. For most of the time the path is mostly to the direction of the polar angle. The speed of light is also faster in this direction since the polar angle element is r^2dr and the time element is Bdt. Instead of $r^2dr/dt = cr^2$, which means moving with the speed c, we get $r^2dr/\sqrt{B}dt$. In the proper time it is r^2dr/dt but this already includes the redshift. Therefore there is no gravitational redshift in the polar angle direction in Schwarzschild's solution.

A comment on the Pound-Rebka experiment was promised. It is that if we use the General Relativity definition of proper time (20) instead of (19), Schwarzschild's solution gives a blueshift in the Pound-Rebka experiment: the movement is radial and the integration is over $c\sqrt{A}$. So, in fact, Einstein's theory fails the gravitational redshift test.

Notice also that for Schwarzschild's solution the calculation does not give the proper time difference but $\Delta\tau$ includes the one-way delay from the Earth to Venus. It is because the proper time formula is understood differently in Schwarzschild's solution. The logic in Schwarzschild's solution follows the geometric paradigm: in a flat Minkowski space we can think of the functions A and B in Schwarzschild's solution as having the value one. When there is a gravitational field, the field is

$$\Phi = \sqrt{g_{00}} = \sqrt{1 - \frac{2GM}{r}} \approx 1 - \frac{GM}{r}$$

the Newtonian potential added to the potential of a Minkowski space. In the field paradigm we do not think in this way: there is no gravitational potential in an empty Minkowski space and the proper time formula gives only the proper time difference, not the one-way delay.

As a conclusion, Nordström's theory passes the Shapiro time delay test, but Einstein's theory fails it.

3.3 The motion of Mercury

Influence of other planets had been carefully studied with Newtonian physics long before Einstein's time, even if everything was not yet known such as that the sun creates a cloud or a field around itself. The known effects did not explain the precession of the perihelion of Mercury and Einstein proposed a relativistic explanation for it.

I did not name this test the precession of the perihelion of Mercury because there is a bigger problem in the movement of planets: if the central force is a stationary Newtonian gravitation force, an elliptic orbit is not possible because it violates conservation of energy. A direct calculation from equations of motion confirms this conclusion as will be shown in what follows. In reality, planets circulate around the sun on ellitic orbits or at least very close to elliptic orbits. This means that there must be some mechanism by which energy is lost so that the planes do not escape to the space. Failing to explain this isses should mean failing the test.

Let us consider the movement of planets around the sun in a fully classical way. By conservation of the angular momentum and momentum the orbits of two masses m_1 and m_2 attracted by a central force can only be circles, ellipses or hyberbolas around the center of mass in a system where the only force is a central force. The result does not require that the central force has the $\frac{1}{r^2}$ dependency from the distance. However, this result does not yet imply that all of these solutions satisfy other concervation laws when the central force has a particular form. Indeed, the elliptic orbit does not satisfy conservation of energy for a stationary Newtonian gravitational force.

To see the energy problem let us take two masses m_1 being Mercury and m_2 the sun. The center of mass is in the line connecting the masses. The distance from m_i to the center of mass being r_i. The center of mass is at the point where $r_2 = r_1 m_1 / m_2$.

The two-body system can be modelled in such a way that there is a stationary central Newtonian force in the center of mass. The coordinates can be so selected that the center of mass does not move. The movement is in two dimensions only and we need two coordinates: r and θ with the origin at the center of mass. The velocities v_i of the masses can be divided into radial and angular components $v_{i,r}$ and $v_{i,\theta}$. The masses move symmetrically around the center of mass as it stays fixed. The velocities in the θ direction must satisfy

$$v_{2,\theta} = \frac{m_i^2}{m_2^2} v_{1,\theta}$$

because the center of mass stays fixed. This is conservation of the angular momentum. The velocities in the r direction must also satisfy

$$v_{2,r} = \frac{m_i^2}{m_2^2} v_{1,r}$$

because the total momentum must be zero if the center of mass does not move. The total kinetic energy is

$$E_k = \frac{1}{2}\left(m_1 v_{1,r}^2 + m_2 v_{2,r}^2 + m_1 v_{1,\theta}^2 + m_2 v_{2,\theta}^2\right)$$

as r and θ are orthogonal. We get

$$E_k = \frac{1}{2}\frac{m_1}{m_2}(m_1 + m_2)(v_{1,r}^2 + v_{1,\theta}^2)$$

By Kepler's law of areas, which follows from the conservation of the angular momentum,

$$v_{1,\theta} = \frac{r_{1,min}}{r_1} v_{1,\theta,max}$$

where $r_{1,min}$ is the minimum distance of m_1 from the center of mass and $v_{1,\theta,max}$ is the tangential velocity m_1 has at this distance. The velocity is on the θ-direction as the radial velocity $v_{1,r,max}$ is zero at the minimum distance. We get

$$E_k(r) = \frac{1}{2}\frac{m_1}{m_2}(m_1 + m_2)v_{1,r}^2 + \frac{1}{2}\frac{m_1}{m_2}(m_1 + m_2)\left(\frac{r_{1,min}^2}{r_1^2} - 1\right) v_{1,\theta,max}^2$$

The difference between $E_k(r_1)$ and $E_k(r_{1,min})$ is

$$\Delta E_k = E_k(r_{1,min}) - E_k(r_1)$$

$$= \frac{1}{2}\frac{m_1}{m_2}(m_1 + m_2)v_{1,r}^2 + \frac{1}{2}\frac{m_1}{m_2}(m_1 + m_2)\left(\frac{r_{1,min}^2}{r_1^2} - 1\right) v_{1,\theta,max}^2$$

The masses are at the opposite sides of the center of mass. Their distance is $r_1 + r_2 = r_1(m_1 + m_2)/m_2$. The Newtonian gravitational force between them is

$$F = m_1 m_2 G \frac{1}{(r_1 + r_2)^2} = m_1 m_2 G \frac{m_2}{m_1 + m_2}\frac{1}{r_1^2}$$

where G is the gravitational constant. The difference in gravitational potential energy between the situations when the masses are at (r_1, r_2) and at $(r_{1,min}, r_{2,min})$ is

$$\Delta E_p = -\frac{m_1 m_2^2}{m_1 + m_2} G \left(\frac{1}{r_1} - \frac{1}{r_{1,min}}\right)$$

By conservation of energy $\Delta E_k + \Delta E_p = 0$, thus

$$\frac{1}{2}\frac{m_1}{m_2}(m_1+m_2)v_{1,r}^2 + \frac{1}{2}\frac{m_1}{m_2}(m_1+m_2)\left(\frac{r_{1,min}^2}{r_1^2}-1\right)v_{1,\theta,max}^2$$

$$+\frac{m_1 m_2^2}{m_1+m_2}G\left(\frac{1}{r_1}-\frac{1}{r_{1,min}}\right)$$

Thus

$$v_{1,r}^2 = -\left(\frac{r_{1,min}^2}{r_1^2}-1\right)v_{1,\theta,max}^2 + 2\frac{m_1 m_2^3}{(m_1^2+m_2)^2}G\left(\frac{1}{r_1}-\frac{1}{r_{1,min}}\right) \tag{21}$$

Derivating with respect to r_1 and setting $\frac{\partial v_{1,r}}{dr_1}=0$ at $r_1 = r_{1,min}$ gives

$$0 = 2\frac{r_{1,min}^2}{r_1^3}v_{1,\theta,max}^2 - 2\frac{m_1 m_2^3}{(m_1^2+m_2)^2}G\frac{1}{r_{1,min}^2}$$

as $v_{1,r}$ is zero at $r_{1,min}$. Thus

$$v_{1,\theta,max}^2 = \frac{m_2^3}{(m_1^2+m_2)^2}G\frac{1}{r_{1,min}}$$

Let us notice as a check-up that this equation gives the centrifugal force as the left side of

$$m_1 v_{1,\theta,max}^2\frac{1}{r_{1,min}} = m_1 m_2 G\frac{1}{(r_{1,min}+r_{2,min})^2}$$

which is a correct formula as the rotation is around the central point at the distance $r_{1,min}$ from m_1 and the gravitation force is between the masses having a distance $r_{1,min}+r_{2,min}$ between them.

Inserting this expression to the equation (21) of $v_{1,r}^2$ yields

$$v_{1,r}^2 = 2\frac{m_1 m_2^3}{(m_1^2+m_2)^2}G\frac{1}{-r_{1,min}r^2}(r_1-r_{1,min})r^2 \tag{22}$$

There is a double zero at $r_1 = r_{1,min}$. That means that m_1 approaches m_2, gets to the minimum and then distances from m_2. It does not have another zero at $r_1 = r_{1,max}$ and thus an elliptic orbit is not possible. The orbit is either a circle, and then $r_1 = r_{1,min}$ all the time and $v_{1,r} = 0$ for every r, or the orbit is a hyperbole. In order to get an elliptic orbit we need a different central force or other forces. Some mechanism is necessary for reducing energy so that the radial velocity can have two zeroes.

Let us check this result in another way. We can by a direct calculation see if the equation of motion for m_1 can be an ellipse if there is a central Newtonian force at the focal point of the ellipse.

Let us take an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

At (x_0,y_0) the ellipse has the tangent

$$\frac{x_0}{a^2}x+\frac{y_o}{b^2}y=1$$

The rotation

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \frac{1}{c}\begin{bmatrix} a^2y_0 & -b^2x_0 \\ b^2x_0 & a^2y_0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad c=\sqrt{a^4y_0+b^4x_0}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{c}\begin{bmatrix} a^2y_0 & b^2x_0 \\ -b^2x_0 & a^2y_0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

takes z_2 to a line parallel to the tangent of the ellipse and z_1 is orthogonal to this line. The rotation has the determinant one and therefore does not change the distances. In coordinates (z_1,z_2) the equation of ellipse is

$$z_2^2-2hz_1z_2+ez_1^2-m=0$$

where

$$h=x_0y_0\frac{a^2b^2x_p^2}{b^6x_0^2+a^6y_0^2}$$

$$e=\frac{a^4b^4}{b^6x_0^2+a^6y_0^2}$$

$$m=\frac{(a^4y_0^2+b^4x_0^2)a^2b^2}{b^6x_0^2+a^6y_0^2}$$

$$x_p=\sqrt{a^2-b^2}$$

The rotation takes (x_0,y_0), $x_0>0$, $y_0>0$, to (z_{10},z_{20}) where

$$z_{10}=\frac{1}{c}x_0y_0x_p^2$$

$$z_{20} = \frac{1}{c}a^2b^2$$

Solving the elliptic equation gives

$$z_2'(z_1) = \frac{dz_2}{dz_1} = h + (h^2 - e)z_1(z_2 - hz_1)^{-1}$$

$$z_2''(z_1) = \frac{h^2 - e}{z_2 - hz_1}\left(1 - \frac{(h^2 - e)z_1^2}{(z_2 - hz_1)^2}\right)$$

A calculation shows that

$$\frac{h^2 - e}{z_{20} - hz_{10}} = -\frac{ca^2b^2}{b^6x_0^2 + a^6y_0^2}$$

$$\frac{z_{10}^2}{z_{20} - hz_{10}} = \frac{1}{c}\frac{x_0^2y_0^2x_p^4}{a^2b^2}\frac{b^6x_0^2 + a^6y_0^2}{b^6x_0^2 + a^6y_0^2 - x_0^2y_0^2x_p^4}$$

Thus

$$1 - \frac{(h^2 - e)z_{10}^2}{(z_{20} - hz_{10})^2} = \frac{b^6x_0^2 + a^6y_0^2}{b^6x_0^2 + a^6y_0^2 - x_0^2y_0^2x_p^4}$$

and

$$z_2''(z_{10}) = -\frac{ca^2b^2}{b^6x_0^2 + a^6y_0^2 - x_0^2y_0^2x_p^4}$$

The left focal point is in the point $(-x_p, 0)$ in (x, y)-coordinates. Let r be the distance from the focal point to (x_0, y_0)

$$r^2 = (x_0 + x_p)^2 + y_0^2$$

Thus, it is also the distance between the focal point in coordinates (z_1, z_2) and (z_{10}, z_{20}). We mark the focal point in coordinates (z_1, z_2) by (z_{1p}, z_{2p}). The focal point is at

$$z_{1p} = -\frac{1}{c}a^2y_0x_p$$

$$z_{2p} = -\frac{1}{c}b^2x_0x_p$$

We can express x_0 as a function of r by using $y_0^2 = b^{-2}(1 - x_0^2/a^2)$. The result is

$$x_0 = \frac{a}{x_p}(r - a)$$

As $x_0 > 0, y_0 > 0$ we have $r > a$. Likewise we solve

$$y_0 = b^2(1-(r-a)^2 x_p^{-2})$$

$$c = ab\sqrt{a^2-(r-a)^2}$$

Inserting these to the expression of $z_2''(z_{10})$ we get

$$z_2''(z_{20}) = -\frac{ab\sqrt{a^2-(r-a)^2}}{a^4+2b^2(r-a)^2+(r-a)^4}$$

Let there be a point mass in the focal point and let it exercise gravitation force to the mass m_1 moving on the elliptic orbit. The sun is not exactly in the focal point, but we can take the mass as the mass of the center of mass and that is in the focal point. The angle θ between the horizontal line ($z_2 = 0$) in the (z_1, z_2)-coordinates and the line connecting (z_{1p}, z_{2p}) to (z_{10}, z_{20}) has the tangent

$$\tan\theta = \frac{-z_{2p}+z_{20}}{-z_{1p}+z_{10}}$$

Notice that $z_{1p} < 0$ and $z_{2p} < 0$ because $x_0 > 0, y_0 > 0$. Inserting expressions of these points as functions of x_0 and y_0 we get

$$\tan\theta = \frac{b^2}{y_0 x_p} = \frac{b}{\sqrt{a^2-(r-a)^2}}$$

The gravitation force F between the planet with mass m_1 and the center of mass can be divided into two components: F_{tan} tangential to the orbit (that is, parallel to the z_1-axis) and F_{ort} orthogonal to the tangent (that is, parallel to the z_2-axis). The z_2-axis does not point from (z_{10}, z_{20}) to the focal point. We have to take a projection

$$F_{ort} = F\sin\theta \quad , \quad F_{tan} = F\sin\theta$$

The equation of motion in the orthogonal direction is that the acceleration along the z_2-axis causes the displacement

$$\frac{1}{2}z_2''(z_{10})(dz_1)^2 = \frac{1}{2}a_{ort}(dt)^2$$

that is

$$z_2''(z_{10})(dz_1)^2 = \frac{F_{ort}}{m_1}(dt)^2$$

$$z_2''(z_{10})\frac{dz_1}{dt}^2 \sin^{-1}\theta = \frac{F}{m_1} = \frac{mG}{r^2} \tag{23}$$

where m is the equivalent mass that we should use for the center of mass. As

$$F = \frac{m_1 m_2 G}{(r+r_2)^2}$$

and $r_2 = rm_1/m_2$ is the distance of the sun from the center of mass we get

$$F = \frac{m_1}{r^2}m_2\left(\frac{m_2}{m_1+m_2}\right)^2$$

Thus, $m = m_2\left(\frac{m_2}{m_1+m_2}\right)^2$. It is practically m_2 for the sun and Mercury. In case we believe that we know the velocity

$$v(r) = \frac{dz_1}{dt}$$

from conservation laws, we can insert the expression here. But this should not be done without considerations since the previous argument shows that the conservation of energy does not allow an elliptic orbit.

It is better to solve the velocity from tangential acceleration. The tangential velocity $v(z_1)$ is the velocity of the mass m_1 at z_{10} as the mass cannot have velocity orthogonal to the tangent of its orbit. In the tangential direction the equation of motion is

$$v(z_1) = v(z_{10}) + v'(z_{10})dz_1$$

$$v'(z_{10})dz_1 = \frac{F_{tan}}{m}dt = \cos(\theta)\frac{F}{m}dt$$

Since

$$v'(z_{10})\frac{dz_1}{dt} = v'(z_{10})v(z_{10})$$

and

$$v'(z_1)v(z_1) = \frac{1}{2}\frac{d}{dt}v(z_1)$$

we get an equation

$$v(z_{10})^2 = 2\int_{z_{1s}}^{z_{10}} \frac{F(z_1)}{m}\cos(\theta)dz_1$$

where the lower bound z_{1s} can be selected in a suitable way. The lower bound only changes the initial value of the tangential velocity and does not affect the coefficients of the Fourier series of the velocity.

The equation (23) for the orthogonal direction gave

$$v(z_{10})^2 = z''(z_{10})^{-1}\sin(\theta)\frac{F(z_1)}{m} \tag{24}$$

Let us write

$$B(r) = z''(z_{10})\sin(\theta)^{-1} = \frac{a(a^2-(r-a)^2)}{a^4+4b^2(r-a)^2+(r-a)^4}$$

and notice that as dz_1 is in the horizontal direction in the (z_1, z_2) coordinates and the focal point is down and left of (z_10, z_{20}) in angle θ with the horizontal level

$$dz_1 = \cos(\theta)dr$$

This gives the equation

$$B(r)^{-1}F(r) = 2\int_{z_{1s}}^{z_{10}} F(r)\cos^2(\theta(r))dr \tag{25}$$

Assuming that z_{1s} is sufficiently small, $z_{1s} > z_{10} > 0$, the value

$$\epsilon = \frac{r-a}{a}$$

is small and we can sufficiently well evaluate the sides of this equation as power series of ϵ to some chosen degree. Changing $r-a = a\epsilon$ gives

$$B(r) = \frac{1}{a}\frac{1-\epsilon^2}{1+2\frac{b^2}{a^2}\epsilon^2+\epsilon^4}$$

$$F(r) = mG\frac{1}{r^2} = mG\frac{1}{a^2(1+\epsilon)^2}$$

Thus

$$B(r)^{-1}F(r) = amG\frac{1+2\frac{b^2}{a^2}\epsilon^2+\epsilon^4}{1-\epsilon^4}$$

$$= amG(1+2\frac{b^2}{a^2}\epsilon^2+\epsilon^4)(1+\epsilon^4)+O(\epsilon^8)$$

$$= amG(1+2\frac{b^2}{a^2}\epsilon^2+2\epsilon^4+2\frac{b^2}{a^2}\epsilon^6)+O(\epsilon^8) \tag{26}$$

In the right side of the equation (25) we have

$$\cos(\theta) = \frac{\sqrt{x_p^2 - (r-a)^2}}{\sqrt{b^2 + x_p^2 - (r-a)^2}}$$

Therefore

$$\cos^2(\theta) = \frac{1 - \epsilon^2 - \frac{b^2}{a^2}}{1 - \epsilon^2}$$

and

$$F(r)\cos^2(\theta) = \frac{mG}{a^2}\frac{1 - \epsilon^2 - \frac{b^2}{a^2}}{1 - \epsilon^4}$$

$$= \frac{mG}{a^2}\frac{1 - \frac{b^2}{a^2} - \epsilon^2}{1 + \epsilon^4} + O(\epsilon^8)$$

$$= \frac{mG}{a^2}(1 - \frac{b^2}{a^2} - \epsilon^2 + (1 - \frac{b^2}{a^2})\epsilon^4 - \epsilon^6) + O(\epsilon^8)$$

There remains the integration

$$2\int_{z_{1s}}^{z_{10}} F(r)\cos^2(\theta(r))dr$$

$$= 2\frac{mG}{a^2}(C - \frac{b^2}{a^2}\epsilon - \frac{1}{3}\epsilon^3 + \frac{1}{5}(1 - \frac{b^2}{a^2})\epsilon^5 - \frac{1}{7}\epsilon^7) + O(\epsilon^8) \tag{27}$$

where C is some constant and it includes the initial value at z_{1s}.

Equation (25) is not filled: (26) and (27) do not match in powers of ϵ. The elliptic orbit is not a solution to a two body problem with a Newtonian gravitation force in this fully classical calculation. However, if we set $a = b = R$, which implies that $x_p = 0$, $r = a$, $\sin(\theta) = 1$, $\cos(\theta) = 0$, a solution is obtained: $v_{1,r}$ is constant, $\epsilon = 0$, (25) reduces to aMG= constant and (24) reduces to

$$v(z_{10})^2 = a\frac{mG}{r^2} = \frac{mG}{r}$$

which is the velocity of a mass on a circular orbit. Thus, a circle is a solution in this calculation. Also the energy calculation allows a circular orbit as for a circle $r_1 = r_{1,min}$ and the radial velocity is zero for all times. In both calculations the center of the circle is the center of mass.

An ellipse is changed into a hyperbole by replacing b by ib. The formulae for an ellipse give formulae for a hyperbole if b^2 is replaced by $-b^2$. The calculation for an ellipse shows that a stationary Newtonian central force does not produce a mathematical hyperbole, in the same way as it cannot produce a mathematical elliptic orbit. But there is a difference: an orbit very similar to a hyperbole is possible because it does not contradict energy conservation: only one double root for $v_{1,r}^2$ agrees well with the hyperbolic orbit.

Kepler said that planets follow an elliptical orbit with the sun (almost) at one focal point and that the area law holds. These statements are correct, but the mathematical explanation of Kepler's laws for the orbits of planets by the conservation of the angular momentum and the momentum ignores the problem there is with conservation of energy. The elliptic orbit cannot be produced by a single stationary Newtonian central force.

A friction force would make an elliptic orbit possible, but there is little friction in space. The double root (22) can be broken into two roots also by modifying the gravitational potential. Let us see if Schwarzschild's solution can solve the energy problem. The force in Schwarzschild's solution is slightly larger than Newtonian gravitation force. We can change the gravitational potential to

$$E_p = -\frac{k}{r} + \frac{\alpha}{r^2}$$

For one value of α this gives the gravitational potential in Schwarzschild's solution.

The term $(r_1 - r_{1,min})^2$ in Equation (22) changes to the form

$$r_1^2 - \frac{2r_{1,min}}{1 + 2\alpha r_{1,min}^{-1}} r_1 + \frac{r_{1,min}(r_{1,min} - 2\alpha)}{1 + 2\alpha r_{1,min}^{-1}} = 0$$

In order to get two roots, $r_{1,min}$ and $r_{1,max}$, the equation must equal

$$r_1^2 - (r_{1,min} + r_{1,max})r_1 + r_{1,min}r_{1,max} = 0$$

Matching the parameters yields

$$\alpha = -\frac{r_{1,min}(r_{1,max} - r_{1,min})}{2(r_{1,min} + r_{1,max})}$$

For the orbit of Mercury $r_{1,min} \approx 45.9 * 10^9 m$ and $r_{1,max} \approx 69.9 * 10^9 m$. Inserting these values we get

$$\alpha \approx -4.7$$

This is not the value proposed by Schwarzschild's solution, thus it cannot explain the elliptic orbit of Mercury.

I will not try to calculate an predictions for Mercury's orbit from Nordström's second theory in this article as I consider it too difficult. I will only briefly discuss the problematics quantitatively. The suggestion is that when Mercury is close to the sun it slightly disturbs the stationary field $\Phi(r)$ and forces a time dependent solution $\Phi(r,\theta,t)$ (or even $\Phi(r,\theta,\psi,t)$). The wave equation is still almost $R=0$ but the entries $R_{0,j}$, $j>0$, are not zeros. The potential Φ has a slightly different r dependency than kr^{-1} coming from the constant needed to separate r from the other coordinates in the solution of the wave equation. As some energy is used for waves, less energy is left to the gravitational potential and the orbits of planets can be ellipses. The problem in calculating this is that if one planet disturbs the sun's potential, they all do, and the multibody problem becomes difficult.

Nordström's theory does not fail this test: the result is inconclusive. Einstein's theory fails this test as Schwarzschild's solution does not give an elliptic orbit.

3.4 Bending of light from stars by the Sun

Light bends in Nordström's theory: Nordström accepted Einstein's special relativity and considered light to have mass and the speed of light c to be the maximal velocity. Consequently, light in Nordström's theory behaves as a test mass and is attracted by gravity. Nordström's theory gives the Newtonian potential for the vacuum, $\Phi=\Phi(r)$. This stationary gravitational field of the sun may be disturbed by planets as suggested in 3.3, but the path is at least very close to a hyperbole. I consider calculating the amount of the bending of the light too difficult to be done in this article. The result of this test remains inconclusive for Nordström's theory. What I can do is to discuss the problematics from a theoretical point of view.

The argument in [4] that that light does not bend in gravitational fields in Nordström's field theory is based on the following reasoning: Electro-magnetic fields in Einstein's theory are described by a stress-energy tensor which is traceless. Nordström's field equation can be expressed in the geometric form (5). In that form T on the right side is the trace of T_{ab}. Consequently, light does not bend in Nordström's theory.

The caveat in this argument is that T_{ab} in Nordström's theory was originally not the T_{ab} in Einstein's theory, though in the last version Nordström accepted Einstein's proposals. The right side in Nordström's theory was $4\pi\rho$ or $4\pi T_{\text{matter}}$ and this entity contains all mass-energy of the system, including mass-energy of electro-magnetic fields. We cannot assume that the T_{ab} in Nordström's theory are exactly the same as in Einstein's theory.

The argument in [4] can be turned against Einstein's theory: assuming that Nordström's theory correctly describes gravitation, the entity T must include the mass-energy of electro-magnetic fields. If T is the trace of T_{ab} from Einstein's theory, then T from does not include the mass-energy electro-magnetic fields as it should. Therefore T_{ab} in Einstein's theory must be wrong.

Nordström's theory does not consider the problem how electro-magnetic fields are included to the field equation. According to [2] Laue discarded Nordström's theory because he could not couple electro-magnetic fields into it, but maybe this should be reconsidered. A time dependent Φ in Nordström's theory causes R_{ij}, $j \neq i$, to differ from zero. These elements may couple electro-magnetic fields to Φ in Nordström's theory even though the field equation is only the trace. Nordström's theory did not say anything of R_{ij}, $j \neq i$, but the theory can be augmented by adding the requirements for these cross entries from Einstein's theory. Einstein's theory may be incorrect in requiring $R_{ii} = 0$, but it can pose correct requirements for the cross entries.

4. Conclusions

My arguments why to question the superiority of Einstein's field theory over Nordström's second gravitation theory are three:

1) Nordström's theory is an ordinary field theory, not a geometry. Gravitation is a field in this theory and in this respect similar to other interactions, potentially making unification of the interactions easier. Einstein's theory is a geometry and the geometry in Schwarzschild's solution is not even quasiregular to the Euclidian geometry. The result is that in a gravitational field the speed of light is exceeded even if only very slightly. Light can go slower than c in Nordström's theory as the gravitational redshift shows, but if light moves faster than c the theory contradits special relativity.

2) I do not see any strong reasons for Einstein's requirement that each R_{ab} be zero. In the Newtonian potential $\Phi = \Phi(r)$ in Nordström's theory for a stationary

spherically symmetric field the elements R_{aa} are not all zero. This issue can be solved by changing the definition of T_{ab}: in the vacuum around a point mass the gravitational field has energy and it should be reflected somewhere, such as in nonzero elements R_{aa}.

Consider a point mass and the vacuum outside it. Placing the point mass in the origin of spherical coordinates there is a (static) gravitational field in this vacuum and the field has energy. A spherically symmetric static field does not depend on the polar angle θ, the azimuthal angle ψ or on the time t. As a result the Ricci curvature tensor entries R_{ab}, $a \neq b$, are zero and we may conclude that T_{ab}, $a \neq b$, must be zero in such a vacuum. However, the diagonal elements R_{aa} do not disappear for a metric tensor g_{ab} as in (3). Therefore T_{aa} are not all zero in Nordström's theory. I suggest that in Nordström's theory in this vacuum case $T_{ab} = 0$ for $a \neq 0$ and T_{aa} fill the condition

$$g^{00}T_{00} = -g^{11}T_{11} - g^{22}T_{22} - g^{33}T_{33}$$

If there is mass-energy in the system, there is a similar difference between the stress-energy tensors of Nordström's and Einstein's theory: the diagonal elements T_{aa} are different as the energy of the gravitational field is included in Nordström's T_{ab} and is missing in Einstein's but the trace T is the same for both theories. This way of understanding T describes the ideas of Nordström's field theory as an extension of Newtonian gravitation.

3) Nordström's second field theory does not fail any of the four tests of General Relativity. It passes the redshift and Shapiro time delay tests and remains inconclusive in the Mercury and light bending tests. Einstein's theory fails the Shapiro time delay test and if the proper time is defined as in General Relativity it also fails the redshift test. Additionally, Einstein's theory fails to explain the motion of Mercury because Schwarzschild's solution does not give an elliptic orbit.

In the 1980ies I asked my supervisor for a mathematical topic with physical connections and I was given a geometric topic on quasiregular mappings between low-dimensional manifolds. The supervisor mentioned that Schwarzschild's solution is very odd: physical fields are images of conformal mappings but the ball in Schwarzschild's solution is not even quasiregular to our ball. I read at that time a book of black holes [5] and concluded that physics apparently can be consistently built on Schwarzschild's solution and did not look at the issue deeper, fortunately, as trying to pass a paper showing Einstein's field theory wrong would hardly have been accepted as a Ph.D. thesis. I made the thesis on quasiregular mappings between closed orientable 3-manifolds. But now, as retired, I can investigate the geometry of General Relativity and there does seem to be some problems associated with it.

If Nordström's second field theory turns out to be the correct theory for gravitation it has some implications. For instance, the discovery of gravitational waves has been recently questioned, see [6]. In Einstein's theory gravitational waves are caused by the Weyl tensor and the way of finding the waves used patterns for waves derived from Einstein's theory. It the correct theory is Nordström's second theory, then the patters are different: one should look for gravitational waves from the Ricci tensor elements R_{ab}, $a \neq b$.

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