Stochastic space-time and quantum theory: Part A

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Much of quantum mechanics may be derived if one adopts a very strong form of Mach's Principle, requiring that in the absence of mass, space-time becomes not flat but stochastic. This is manifested in the metric tensor which is considered to be a collection of stochastic variables. The stochastic metric assumption is sufficient to generate the spread of the wave packet in empty space. If one further notes that all observations of dynamical variables in the laboratory frame are contravariant components of tensors, and if one assumes that a Lagrangian can be constructed, then one can derive the uncertainty principle. Finally, the superposition of stochastic metrics and the identification of $\sqrt{-g}$ (in the four-dimensional invariant volume element $\sqrt{-g}dV$) as the indicator of relative probability yields the phenomenon of interference, as will be described for the two-slit experiment.

PREFACE

A less developed version of this paper appeared years ago in Phys. Rev.[1] The paper was highly cited: e.g. [7–16][2, 17–20][3, 21–27].[50] Since then, there has been much activity in the stochastic approach, some of it spawned by the earlier version of this paper. However, that earlier version, being behind a pay-wall, was not easily accessible. Hence this newer and slightly modified version.

I. INTRODUCTION

When considering the quantum and relativity theories, it is clear that only one of them, namely relativity, can be considered, in the strict sense, a theory. Quantum mechanics, eminently successful as it is, is an operational description of physical phenomena. It is composed of several principles, equations, and a set of interpretive postulates[28]. These elements of quantum mechanics are justifiable only in that they work. Attempts[29, 30] to create a complete, self-contained theory for quantum mechanics are largely unconvincing. There are, in addition, a number of points where quantum mechanics yields troubling results. Problems arise when considering the collapse of the wave function, as in the Einstein-Podolsky-Rosen paradox[31]. Problems also arise when treating macroscopic systems, as in the Schrödinger cat paradox[32] and the Wigner paradox[33]. And quantum mechanics is not overly compatible with general relativity[34].

One way of imposing some quantum behavior on general relativity is the following: The uncertainty relation for time and energy implies that one can "borrow" any amount of energy from the vacuum if it is borrowed for a sufficiently short period of time. This energy fluctuation of the vacuum is equivalent to mass fluctuations which then gives rise to metric fluctuations via the general-relativity field equations.

An alternative approach is to impose, *ab initio,* an uncertainty on the metric tensor, and to see if by that, the results of quantum mechanics can be deduced. As this paper will shoe, with a few not particularly unreasonable assumptions, a large segment of the formalism of quantum theory can be derived and, more importantly, understood.
Mathematical spaces with stochastic metrics have been investigated earlier by Schweizer[35] for Euclidian spaces, and by March[36, 37] for Minkowski space. In a paper by Blokhintsev[38], the effects on the physics of a space with a small stochastic component are considered. It is our goal, however, not to show the effects on physical laws of a stochastic space, but to show that the body of quantum mechanics can be deduced from simply imposing stochasticity on the space-time. Our method will be to write down (in Section II) a number of statements (theorems, postulates, etc.). We will then (in Section III) describe the statements and indicate proofs where the statements are theorems rather than postulates. Finally (in Section IV) we will derive some physical results, namely, the spread of the free particle (in empty space), the uncertainty principle, and the phenomenon of interference. The paper concludes (Sec. V) with a general discussion of the approach and a summary of results.

II. THE STATEMENTS

Statement 1. Mach's principle (Frederick's version).

1.1. In the absence of mass, space-time becomes not flat, but stochastic.

1.2. The stochasticity is manifested in a stochastic metric $g_{\mu\nu}$.

1.3. The mass distribution determines not only the space-time geometry, but also the space-time stochasticity.

1.4. The more mass in the space-time, the less stochastic the space-time becomes.

1.5 At the position of a mass "point", the space-time is not stochastic.

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Statement 2, the contravariant observable theorem.

All measurements of dynamical variables correspond to contravariant components of tensors.

Statement 3. The metric probability postulate.

$P(x,t) = f\sqrt{-g}$, where for a one particle system $P(x,t)$ is the particle probability distribution. f is a real-valued function and g is the determinant of the metric. [But see III, Statement 3, for a more recent interpretation.]

Statement 4. the metric superposition postulate.

If at the position of a particle the metric due to a specific physical situation is $g_{\mu\nu}(1)$ and the metric due to a different physical situation is $g_{\mu\nu}(2)$ then the metric at the position of the particle due to the presence of both of the physical situations is $g_{\mu\nu}(3)$,

$g_{\mu\nu}(3) = \frac{1}{2}[g_{\mu\nu}(1) + g_{\mu\nu}(2)]$.

This is the case where the probabilities, $P_1$and P_2, of the two metrics are the same. In general though, Statement 4 becomes,

$g_{\mu\nu}(3) = P_1 g_{P\mu\nu}(1) + P_2 g_{\mu\nu}(2)$.

Statement 5. The metric Ψ postulate.

There exists a local complex diagonal coordinate system in which a component of the metric at the location of the particle is the wave function Ψ.

III. DESCRIPTION OF THE STATEMENTS

Statement 1, Mach's principle, is the basic postulate of the model. It should be noted that requirement 1.5, that at the position of a mass point the space-time be not stochastic, is to insure that an elementary mass particle (proton, quark, etc.) is bound.

Statement 2, the contravariant observable theorem, is also basic. It is contended, and the contention will be weakly proved, that measurements of dynamical variables are contravariant components of tensors. By this we mean that whenever a measurement can be reduced to a displacement in a coordinate system, it can be related to contravariant components of the coordinate system. Of course, if the metric $g_{\mu\nu}$is well known, one can calculate both covariant and contravariant quantities. In our model however, the quantum uncertainties in the mass distribution imply that the metric cannot be accurately known, so that measurements can only be reduced to contravariant quantities. Also, in our picture, the metric is stochastic, so again we can only use contravariant quantities. We will verify the theorem for Minkowski space by considering an idealized measurement. Before we do, consider as an example the case of measuring the distance to a Schwarzschild singularity (a black hole) in the Galaxy. Let the astronomical distance to the object be $\bar{r}(\equiv \bar{\xi}^1)$. The covariant equivalent of the radial coordinate r is ξ_1, and

$\xi_1 = g_{1\nu}\xi^\nu = g_{11}\xi^1 = \frac{r}{1-2Gm/r}$,

so that the contravariant distance to the object is

distance$= \int_0^{\bar{r}} dr = \bar{r}$,

while the covariant distance is

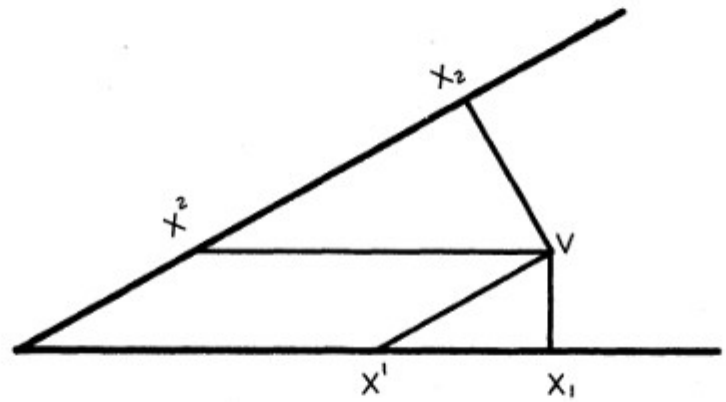

Figure 1. Covariant and contravariant components in oblique coordinates.

$\bar{\xi}_1 = \int_0^{\bar{r}} d(\frac{r}{1-2Gm/r}) = \infty$.

It is clear that only the contravariant distance is observable.

Returning to the theorem, note that when one makes an observation of a dynamical variable (e.g. position, momentum, etc.), the measurement is usually in the form of a reading of a meter (or meter-stick). It is only through a series of calculations that one can reduce the datum to, say, a displacement in a coordinate system. For this reduction to actually represent a measurement (in the sense of Margenau[39]) it must satisfy two requirements. It must be instantaneously repeatable with the same results, and it must be a quantity which can be used in expressions to derive physical results (i.e., it must be a physically "useful" quantity). It will be shown that for Minkowski space, the derived "useful" quantity is contravariant.

Note first (Fig. 1) that for an oblique coordinate system, the contravariant coordinates of a point V are given by the parallelogram law of vector addition, while the covariant components are obtained by orthogonal projection onto the axes[40].

We shall now consider an idealized measurement in special relativity, i.e., Minkowski space. Consider the space-time diagram of Fig. 2.

We are given that in the coordinate system $x^{'}$,$t^{'}$, an object (m,n) is at rest. If one considers the situation from a coordinate system x,t traveling with velocity v along the $x^{'}$axis, one has the usual Minkowski diagram[41] with coordinate axes Ox and Ot and velocity $v = tan(\alpha)$ (where the units are chosen such that the velocity of light is unity). OC is part of the light cone.

Noting that that the unprimed system is a suitable coordinate system in which to work, we now drop from consideration the original $x^{'}$,$t^{'}$coordinate system.

We wish to determine the "length" of the object in the x,t coordinate system. Let it be arranged that at time $t(0)$ a photon shall be emitted from each end of the object (i.e., from points F and B). The emitted

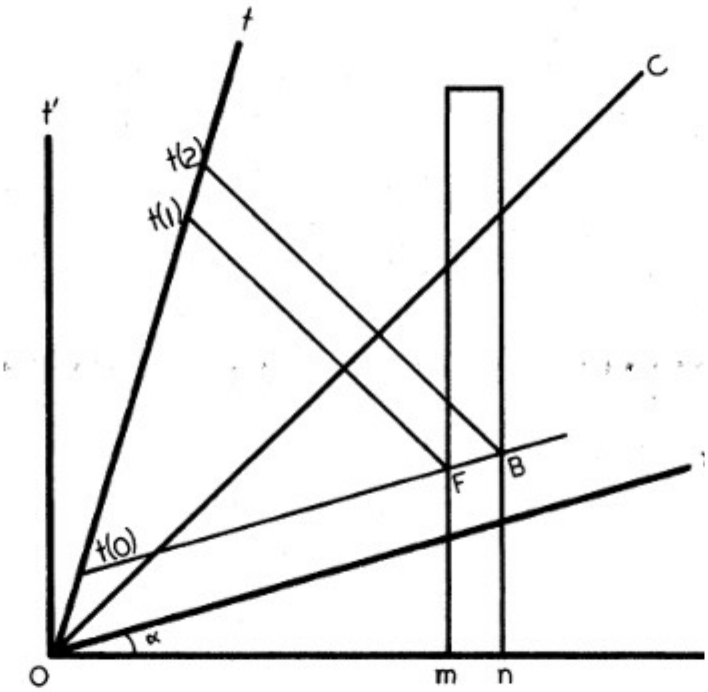

Figure 2. An idealized measurement.

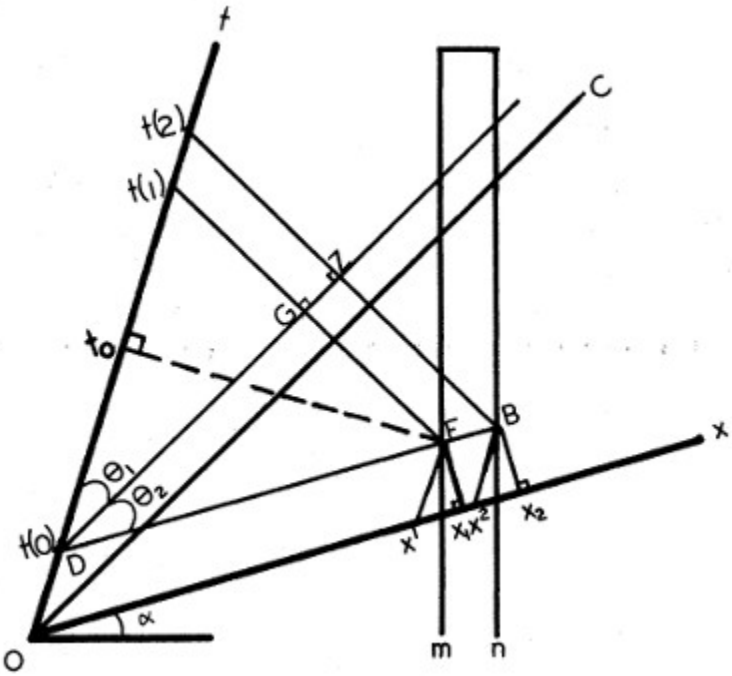

Figure 3. Analysis of the idealized measurement.

photons will intercept the t axis at times $t(1)$ and $t(2)$. The observer then deduces that the length of the object is $t(2)-t(1)$ (where $c=1$). The question is: What increment on the x axis is represented by the time interval $t(2)-t(1)$? One should note that the arrangement that the photons be emitted at time $t(0)$ is nontrivial, but that it can be done in principle. For the present, let it simply be assumed that there is a person on the object who knows special relativity and who knows how fast the object is moving with respect to the coordinate system. This person then calculates when to emit the photons so that they will be emitted simultaneously with respect to the x, t coordinate system.

Consider now Fig. 3, which is an analysis of the measurement. Figure 3 is just figure 2 with a few additions: the contravariant coordinates of F and B, $x^1$and x^2 respectively. We assert, and it is easily shown, that $t(2)-t(1) = x^2 - x^1$. This is seen by noticing that $x^2-x^1=$ line segment B, F, and that triangle $t(2), t(0), Z$ is congruent to triangle $B, t(0), Z$. However, if we consider the covariant coordinates, we notice that $x_2 - x_1 = x^2-x^1$. This is not surprising since coordinate differences (such as $x_2 - x_1$) are by definition (in flat space) contravariant quantities. To verify our hypothesis we must consider not coordinate differences which automatically satisfy the hypothesis, but the coordinates themselves. Consider a measurement not of the length of the object, but of the position (of the trailing edge m) of the object. Assume again that at time $t(0)$ a photon is emitted at F and is received at $t(1)$. The observer would then determine the position of m at $t(0)$ by simply measuring off the distance $t(1)-t(0)$ on the x axis. Notice that this would coincide with the contravariant quantity x^1. To determine the corresponding covariant quantity x_1, one would need to know the angle α (which is determined by the metric).

The metric $g_{\mu\nu}$ is defined as $\bar{e}_\mu \cdot \bar{e}_\nu$, where $\bar{e}_\mu$and $\bar{e}_\nu$ are the unit vectors in the directions of the coordinate axes x^μ and x^ν. Therefore, in order to consider an uncertain metric, we can simply consider that the angle α is uncertain. In this case measurement x^1 is is still well defined $[x^1 = t(1) - t(0)]$, but now there is no way to determine x_1 because it is a function of the angle α. In this case then, only the contravariant components of position are measurable. [It is also easy to see from the geometry that if one were to use the covariant representation of $t(0)$, t_0, one could not obtain a metric-free position measure of m.]

Statement 3, the metric probability postulate, can be justified by the following: Consider that there is given a sandy beach with one black grain among the white grains on the beach. If a number of observers on the beach had buckets of various sizes, and each of the observers filled one bucket with sand, one could ask the following: What is the probability that a particular bucket contained the black grain? The probability would be proportional to the volume of the bucket.

Consider now the invariant volume element dV_I in Riemann geometry. One has that[42]

$dV_I = \sqrt{-|g|}dx^1dx^2dx^3dx^4$.

It is reasonable then, to take $\sqrt{-|g|}$ as proportional to the probability density ($\Psi^*\Psi$) for free space.

But see section III-A for a major revision to Statement 3 (not in the original paper).

Note that the metric $g_{\mu\nu}$ is stochastic while the determinant of the metric is not. This implies that the metric components are not independent.

Consider again, the sandy beach. Let the black grain of sand be dropped onto the beach by an aircraft as it flies over the center of the beach. Now the location of the grain is not random. The probability of finding the grain increases as one proceeds toward the center, so that in addition to the volume of the bucket there is also a term in the probability function which depends on the distance to the beach center. In general then, we expect the probability function $P(x,t)$ to be $P(x,t) = A\sqrt{-|g|}$ where A is a function whose value is proportional to the distance from the center of the beach.

Statement 4, the metric superposition postulate, is adopted on the grounds of simplicity. Consider the metric (for a given set of coordinates) $g^{s1}_{\mu\nu}(x)$ due to a given physical situation $s1$ as a function of position x. Also let there be the metric $g^{s2}_{\mu\nu}(x)$ due to a different physical situation $s2$ (and let the probabilities of the two metrics be the same). What is the metric due to the simultaneous presence of situations $s1$ and $s2$? We are, of course, looking for a representation to correspond to quantum mechanical linear superposition. The most simple assumption is that

$g^{s3}_{\mu\nu}(x)= \frac{1}{2}[g^{s1}_{\mu\nu}(x) + g^{s2}_{\mu\nu}(x)]$.

However, this assumption is in contradiction with general relativity, a theory which is nonlinear in $g_{\mu\nu}$. The linearized theory is still applicable. Therefore, the metric superposition postulate is to be considered as an approximation to an as yet full theory, valid over small distances in empty or almost empty space. We expect, therefore, that the quantum-mechanical principle will break down at some range. (This may eventually be the solution to linear-superposition-type paradoxes in quantum mechanics.

Statement 5, the metric Ψ postulate, is not basic to the theory. It exists simply as an expression of the following: There are at present two separate concepts, the metric $g_{\mu\nu}$ and the wave function Ψ. It is the aim of this geometrical approach to be able to express one of these quantities in terms of the other. The statement that in some arbitrary coordinate transformation, the wave function is a component of the metric, is just a statement of this aim.

A. A major change from the original paper regarding Statement 3

Again, the metric probability postulate, can be justified by the buckets on a beach argument. And again, the probability that a particular bucket contained the black grain would be proportional to the volume of the bucket.

Consider the invariant volume element dV_I in Riemann geometry. One has that

$dV_I = \sqrt{-|g|}dx^1dx^2dx^3dx^4$.

(From here on, we'll represent the determinant of $g_{\mu\nu}$ by g rather then by $|g|$.)

At first sight then, it might seem reasonable to take $\sqrt{-|g|}$ as proportional to the probability density for free space.

The arguments above apply to the three-dimensional volume element. But we left out the other determinant of the probability density, the speed of the particle (the faster the particle moves in a venue, the less likely it is to be there.) And therefore, the larger the Δt the more likely the particle is to be found in the venue. So indeed (it seems as if) it is the four-dimensional volume element that should be used.

The metric probability statement above, as it stands, has problems:

First, if one considers the 'particle in a box' solution, one has places in the box where the particle has zero probability of being. And if $P(x,t) = k\sqrt{-g}= 0$, that means the determinant of the metric tensor is zero and there is a space-time singularity at that point. We address this problem by noting that the metric tensor is composed of the average, non-stochastic, background (Machian) metric $g^M_{\mu\nu}$ and the metric due to the Particle itself $g^P_{\mu\nu}$. We say then that the probability density is actually $P(x,t) = k(\sqrt{-g^T} - \sqrt{-g^M})$ where g^T is the determinant of the composite metric. In this case, $P(x,t)$ can be zero without either $g^T_{\mu\nu}$ or $g^P_{\mu\nu}$ being singular.

A second problem is that $P(x,t) = k\sqrt{-g}$ describes the probability density for a test particle placed in a space-time with a given (average) metric due to a mass, with determinant g. What we want, however, is the probability *of* the particle (not the test-particle) due to the metric contribution of the particle itself. Related to this is that $P(x,t) = k\sqrt{-g}$ doesn't seem to replicate the probability distributions in quantum mechanics in that the probability distribution, $\Psi^*\Psi$, is the square of a quantity (assuring that the distribution is always positive). But the differential volume element, $dV = \sqrt{-g}\, dxdydzdt$ is not the square of any obvious quantity. Further, $P(x,t) = k\sqrt{-g}$ is something of a dead end, as it gives $\Psi^*\Psi$ but no hint of what Ψ itself might represent. It would be nice if the probability density were proportional to the square of the volume element rather than to the volume element itself. With that in mind we'll again look at the probability density. (Multiple researchers[43, 44] have agreed with Part A's $P(x,t) = k\sqrt{-g}$ and it is therefore with some trepidation that we consider that the probability density might be subject to revision.)

The initial idea was that, given a single particle, if space-time were filled with 3-dimensional boxes (venues), then the probability of finding a particle in a box would be proportional to the relative volume of the box. That was extended to consider the case where the particle was in motion. The probability density would then also depend on the relative speed of the particle. We will however, now argue that $P(x,t) \neq k\sqrt{-g}$, but instead $P(x,t) = -kg$ (essentially the square of the previous). But this will apply only when the quantum particle is measured (a contravariant measurement) in the laboratory frame. If however, one considers the situation co-temporally (i.e. covariantly) with the quantum particle, then $P(x,t)$ does equal $k\sqrt{-g}$, which is to say that the probability density is [co- or contra-variant] frame dependent.

There is another argument, but it requires Part C (the third paper in this series, relating to a sometimes timelike fifth dimension). We put it forth here as there seems to be no logical place for it in Part C[45]:

Consider a quantum particle at a τ-time slice at, say, τ=now. And also consider a static quantum probability function (e.g. a particle in a well) at τ=now+1. (That function is a result of the quantum particle's migrations in time and space,) Then if we take a negligible mass test particle at τ=now, it will have a probability of being found at a particular location at tau=now+1 equal to that static probability function. And that function is proportional to the volume element (the square root of minus the determinant of the metric tensor). But what we're interested in is the probability function of the quantum particle as τ goes from now to now+1. We are considering the probability function at τ+1 as static. But it is the result of the migrations of the particle. At tau=now, it would then be the same probability function. So, as we go from now to now plus one, we would need to multiply the two (equal) probability functions. This results in the function being proportional to the determinant of the metric tensor (not its square root). This is rather nice as it allows us to suggest that the volume element is proportional to Ψ while the probability density is proportional to $\Psi^*\Psi$. Note that this result is due to a mass interacting with the gravitational field it itself has generated. (This is analogous to the quantum field theory case of a charge interacting with the electromagnetic field it itself has created.)

As yet another approach, consider the spread of probability due to the migration of venues. In the absence of a potential, the spread (due to Brownian-like motion) will be a binomial distribution in space (think of it at the moment, in a single dimension and time). But there is also the same binomial distribution in time. This, for example, expresses that the distant wings of the space distribution require a lot of time to get to them. The distribution then seems to require that we multiply the space distribution by the time distribution. The two distributions are the same so the result is the square of the binomial distribution. (The argument can be extended to the three spacial dimensions.) In the laboratory frame, time advances smoothly, which is to say that the time probability density distribution is a constant, so we do not get the square of the binomial distribution..

It seems then that there are both the distribution and its square in play. It might be that the covariant representation, i.e. the distribution 'at' the particle, is the binomial while a distant observer where time advances smoothly (not in the quantum system being observed) observes (i.e contravariant measurements) the square of the binomial distribution.

So now we have $P(x,t) = -kg$, which is to say that the probability density is proportional to the *square* of the volume element. This is rather nice as it allows us to suggest that the volume element is proportional to Ψ while the probability density is proportional to $\Psi^*\Psi$. (We will in section 3.B suggest that the imaginary component of Ψ represents an oscillation of space-time.)

But to keep this paper as a reference to the original version, we'll still use $P(x,t) = k\sqrt{-g}$ for Statement 3 with the understanding that $P(x,t) = -kg$ is more likely to be correct (and will be used in later papers).

IV. PHYSICAL RESULTS

We derive first the motion of a test particle in an otherwise empty space-time. The requirement that the space is empty implies that the points in this space are indistinguishable. Also, we expect that, on the average, the space (since it is mass-free) is (in the average) Minkowski space.

Consider the metric tensor at point Θ_1. Let the metric tensor at Θ_1 be $\tilde{g}_{\mu\nu}$ (a tilde over a symbol indicates that it is stochastic). Since $\tilde{g}_{\mu\nu}$ is stochastic, the metric components, do not have well-defined values. We cannot then know $\tilde{g}_{\mu\nu}$ but we can ask for $P\left(g_{\mu\nu}\right)$ which is the probability of a particular metric $g_{\mu\nu}$. Note then that for the case of empty space, we have $P_{\Theta 1}\left(g_{\mu\nu}\right) = P_{\Theta 2}\left(g_{\mu\nu}\right)$ where $P_{\Theta 1}\left(g_{\mu\nu}\right)$ is to be interpreted as the probability of metric $g_{\mu\nu}$ at point $\Theta 1$.

If one inserts a test particle into the space-time, with a definite position and (ignoring quantum mechanics for the moment) momentum, the particle motion is given by the Euler-Lagrange equations,

$$\ddot{x}^i + \left\{ {i \atop jk} \right\} \dot{x}^j \dot{x}^k = 0,$$

where $\left\{ {i \atop jk} \right\}$ are the Christoffel symbols of the second kind, and where $\dot{x}^j \equiv dx^j/ds$ where s can be either proper time or any single geodesic parameter. Since $\tilde{g}_{\mu\nu}$ is stochastic, these equations generate not a path, but an infinite collection of paths, each with a distinct probability of occurrence from $P(g_{\mu\nu})$. (That is to say that $\left\{ {i \atop jk} \right\}$ is stochastic; $\widetilde{\left\{ {i \atop jk} \right\}}$.)

In the absence of mass, the test particle motion is easily soluble. Let the particle initially be at (space) point Θ_0. After time dt, the Euler Lagrange equations yield some distribution of position $D_1(x)$. [$D_1(x)$ represents the probability of the particle being in the region bounded by x and $x + dx$.] After another interval dt, the resulting distribution is $D_{1+2}(x)$. From probability theory[46], this is the convolution,

$$D_{1+2}(x)=\int_{-\infty}^{\infty}D_1(y)D_1(x-y)dy.$$

but in this case, $D_1(x)=D_2(x)$. This is so because the Euler-Lagrange equation will give the same distribution $D_1(x)$ regardless of at which point one propagates the solution. This is because

$g_{\mu\nu}(x)\equiv\{g_{\mu\nu}(x_1),\, g_{\mu\nu}(x_2),\, g_{\mu\nu}(x_3)....\}$

are identically distributed random variables.

Thus,

$D_i(x)\equiv\{D_1(x),\, D_2(x),...\}$

are also identically distributed random variables. The motion of the test particle(the free particle wave functions) is the repeated convolution $D_{1+2+....}(x)$, which by the central limit theorem is a normal distribution. Thus the position spread of the test particle at any time $t>0$ is a Gaussian. The spreading velocity is found as follows: After N convolutions (N large), one obtains a normal distribution with variance σ^2 which, again by the central limit theorem, is N times the variance of $D_1(x)$. Call the variance of $D_1(x)$, a.

$Var(D_1)=a.$

The distribution $D_1$is obtained after time dt. After N convolutions then,

$$\Delta x=Var\left(D_{\sum_0^n i}\right)=Na.$$

This is obtained after N time intervals dt. One then has,

$$\frac{\Delta x}{\Delta t}=\frac{Na}{N},$$

which is to say that the initially localized test particle spreads with a constant velocity a. In order that the result be frame independent, $a=c$, and one has the results of quantum mechanics. At the beginning of this derivation it was given that the particle had an initial well-defined position and also momentum. If for the benefit of quantum mechanics we had specified a particle with a definite position, but with a momentum distribution, one would have obtained the same result but with the difference of having a different distribution D_1 due to the uncertainty of the direction of propagation of the particle.

In the preceding, we have made use of various equations. It is then appropriate to say a few words about what equations mean in a stochastic space-time.

Since in our model the actual points of the space-time are of a stochastic nature, these points cannot be used as a basis for a coordinate system, nor, for that reason, can derivatives be formed. However, the space-time of common experience (i.e., the laboratory frame) is non-stochastic in the large. It is only in the micro world that the stochasticity is manifested. One can then take this large-scale non-stochastic space-time and mathematically continue it into the micro region. This mathematical construct provides a non-stochastic space to which the stochastic physical space can be referred.

The (physical) stochastic coordinates $\tilde{x}^{\mu}$ then are stochastic only in that the equations transforming from the laboratory coordinates x^{μ} to the physical coordinates $\tilde{x}^{\mu}$ are stochastic.

For the derivation of the motion of a free particle we used Statement 1, Mach's principle. We will now use also Statement 2, the contravariant observable theorem, and derive the uncertainty principle for position and momentum. Similar arguments can be used to derive the uncertainty relations for other pairs of conjugate variables. It will also be shown that there is an isomorphism between a variable and its conjugate, and covariant and contravariant tensors.

We assume that we're able to define a Lagrangian, L. One defines a pair of conjugate variables as usual,

$$p_j=\frac{\partial L}{\partial\tilde{q}^j}.$$

Note that this defines p_j a covariant quantity. So that a pair of conjugate variables so defined contains a covariant and a contravariant member (e.g. $p_j$and q^j). But since p_j is covariant, it is not observable in the laboratory frame. The observable quantity is just,

$$\tilde{p}^j=\tilde{g}^{j\nu}p_{\nu},$$

but $\tilde{g}^{j\nu}$ is stochastic so that $\tilde{p}^j$ is a distribution. Thus if one member of an observable conjugate variable pair is well defined, the other member is stochastic. By observable conjugate variables we mean not, say, p_j, q^j derived from the Lagrangian, but the observable quantities $\tilde{p}^j, q^j$ where $\tilde{p}^j=\tilde{g}^{j\nu}p_{\nu}$; i.e. both members of the pair must be contravariant.

However, we can say more. Indeed, we can derive an uncertainty relation. Consider

$$\triangle q^1\triangle p^1=\triangle q^1\triangle\left(p_{\nu}\tilde{g}^{\nu 1}\right).$$

What is the minimum value of this product, given an initial uncertainty $\triangle q^1$? Since p_j is an independent variable, we may take $\triangle p_j=0$ so that

$$\triangle p^1=\triangle\left(p_{\nu}\tilde{g}^{\nu 1}\right)=p_{\nu}\triangle\tilde{g}^{\nu 1}.$$

In order to determine $\triangle\tilde{g}^{\nu 1}$ we will argue that the variance of the distribution of the average of the metric over a region of space-time is inversely proportional to the volume,

$$Var\left(\frac{1}{V}\int_v\tilde{g}_{\mu\nu}dv\right)=\frac{k}{V}$$

In other words, we wish to show that if we are given a volume and if we consider the average values of the metric components over this volume, then these average values, which of course are stochastic, are less stochastic than the metric component values at any given point in the volume. Further, we wish to show that the stochasticity, which we can represent by the variances of the distributions of the metric components, is inversely proportional to the volume. This allows that over macroscopic volumes, the metric tensor behaves classically (i.e. according to general relativity).

For simplicity, let the distribution of each metric component at any point Θ be normal.

$$f_{\tilde{g}_{\mu\nu}}(g_{\mu\nu}) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{g_{\mu\nu}}{\sigma}\right)^2}.$$

Note also that if $f(y)$ is normal, the scale transformation $y \longrightarrow y/m$ results in $f(y/m)$ which is normal with

$$\sigma^2_{(y/m)} = \frac{\sigma_y^2}{m^2}$$

Also,for convenience, let
$f_{g_{\mu\nu}}$ at $e_1(g_{\mu\nu}) \equiv f_{\Theta_1}(g_{\mu\nu})$.
We now require

$$Var(f_{((\Theta_1+\Theta_2+....+\Theta_m)/m)}) \equiv \sigma^2_{((\Theta_1+\Theta_2+....+\Theta_m)/m)},$$

where $f_{(\Theta)}$ is normally distributed. Now again, the convolute $f_{(\Theta_1+\Theta_2)}(g_{\mu\nu})$ is the distribution of the sum of $g_{\mu\nu}$ at Θ_1 and $g_{\mu\nu}$ at Θ_2,

$$f_{(\Theta_1+\Theta_2)} = \int_{-\infty}^{\infty} f_{\Theta_1}(g^1_{\mu\nu}) f_{\Theta_2}(g^1_{\mu\nu} - g^2_{\mu\nu}) dg^2_{\mu\nu},$$

where $g^1_{\mu\nu}$ is defined to be $g_{\mu\nu}$ at Θ_1. Here, of course, $f_{\Theta_1} = f_{\Theta_2}$ as the space is empty so that,

$$f_{(\Theta_1+\Theta_2)/2} = f_{(g_{\mu\nu}/2 at \Theta_1 + g_{\mu\nu}/2 at \Theta_2)}$$

is the distribution of the average of $g_{\mu\nu}$ at Θ_1 and $g_{\mu\nu}$ at Θ_2. $\sigma^2_{(\Theta_1+\Theta_2+....\Theta_m)}$ is easily shown from the theory of normal distributions to be,

$$\sigma^2_{(\Theta_1+\Theta_2+....+\Theta_m)} = m\sigma^2_\Theta.$$

Also, $f_{(\Theta_1+\Theta_2....+\Theta_m)}$ is normal. Hence,

$$\sigma^2_{((\Theta_1+\Theta_2+....+\Theta_m)/m)} = \frac{m\sigma^2_\Theta}{m^2} = \frac{\sigma^2_\Theta}{m},$$

or the variance is inversely proportional to the number of elements in the average, which in our case is proportional to the volume. For the case where the distribution $f_{(g_{\mu\nu})}$ is not normal, but also not 'pathological', the central limit theorem gives the same result as those obtained for the case where $f_{(g_{\mu\nu})}$ is normal. Further, if the function $f_{(g_{\mu\nu})}$ is not normal, the distribution $f_{((\Theta_1+\Theta_2+....+\Theta_m)/m)}$ in the limit of large m is normal,

$$f_{((\Theta_1+\Theta_2+....+\Theta_m)/m)} \longrightarrow f_{((\int_V g_{\mu\nu} d\tilde{V})/V)}.$$

In other words, over any finite (i.e. non-infinitesimal) region of space, the distribution of the average of the metric over the region is normal. Therefore, (anticipating Part B) in so far as we do not consider particles to be 'point' sources, we may take the metric fluctuations at the location of a particle as normally distributed for for each of the metric components $\tilde{g}_{\mu\nu}$. Note that this does not imply that the distributions for any of the metric tensor components are the same for there is no restriction on the value of the variances σ^2 (e.g., in general, $f_{(\tilde{g}_{11})} \neq f_{(\tilde{g}_{22})}$. Note also that the condition of normally distributed metric components does not restrict the possible particle probability distributions, save that they be single-valued and non-negative. This is equivalent to the easily proved statement that the functions

$$f_{(x,\alpha,\sigma)} = \frac{1}{\sqrt{2\pi}\sigma} e^{\left(-\frac{1}{2}\left(\frac{x-\alpha}{\sigma}\right)^2\right)}$$

are complete for non-negative functions.

Having established that,

$$Var\left(\frac{\Theta_1+\Theta_2+....+\Theta_m}{m}\right) = \frac{\sigma^2_\Theta}{m},$$

consider again the uncertainty product,
$\triangle q^1 \triangle p^1 = p_\nu \triangle q^1 \triangle \tilde{g}^{\nu 1}$.
$\triangle q^1$ goes as the volume [volume here is V^1 the one-dimensional volume]. $\triangle \tilde{g}^{\nu 1}$ goes inversely as the volume, so that $p_\nu \triangle q^1 \triangle \tilde{g}^{\nu 1}$ is independent of the volume; i.e. as one takes q^1 to be more localized, p^1 becomes less localized by the same amount, so that for a given covariant momentum p_j (which we will call the proper momentum),$p_\nu \triangle q^1 \triangle g^{\nu 1}$ = a constant k. If also p_ν is also uncertain,
$p_\nu \triangle q^1 \triangle g^{\nu 1} \geq k$.
The fact that we have earlier shown that a free particle spreads indicates the presence of a minimum proper momentum. If the covariant momentum were zero, then the observable contravariant momentum $p^\nu = g^{\nu\mu} p_\mu$would also be zero and the particle would not spread. Hence,
$p_{\nu min} \triangle q^1 \triangle g^{1\nu} = k_{min}$.
or in general,
$\triangle q^1 \triangle (p_{\nu min} g^{\nu 1}) = \triangle q^1 \triangle p^1 > k_{min}$,
which is the uncertainty principle.

With the usual methods of quantum mechanics, one treats as fundamental, not the probability density $P(x,t)$, but the wave function Ψ [$\Psi^*\Psi = P(x,t)$ for the Schrödinger equation]. The utility of using Ψ is that Ψ contains phase information. Hence, by using Ψ the phenomenon of interference is possible. It might be thought that our stochastic space-time approach, as it works directly with $P(x,t)$, might have considerable difficulty in producing interference. In the following, it will be shown that Statements 3 and 4 can produce interference in a particularly simple way.

Consider again the free particle in empty space. By considering the metric only at the location of the particle, we can suppress the stochasticity by means of Statement 1.5. Let the metric at the location of the particle be $g_{\mu\nu}$. We assume, at present, no localization, so that the probability distribution $P(x,t)$=constant. $P(x,t) = A\sqrt{-g}$ by Statement 3. Here A is just a normalization constant so that $\sqrt{-g} = constant$. We can take the constant to be unity.

Once again, the condition of empty space implies that the average value of the metric over a region of space-time approaches the Minkowski metric as the volume of the region increases.

Now consider, for example, a two-slit experiment in this space-time. Let the situation $s1$ where only one slit is open result in a metric $g^{s1}_{\mu\nu}$. Let the situation $s2$ where only slit two is open result in a metric $g^{s2}_{\mu\nu}$. The case where both slits are open is then by statement 4,

$g^{s3}_{\mu\nu} = \frac{1}{2}\left(g^{s1}_{\mu\nu} + g^{s2}_{\mu\nu}\right)$.

Let us also assume that the screen in the experiment is placed far from the slits so that the individual probabilities $(-|g^{s1}|)^{1/2}$ and $(-|g^{s2}|)^{1/2}$ can be taken as constant over the screen.

Finally, let us assume that the presence of the two-slit experiment in the space-time does not appreciably alter the situation that the metrics $g^{s1}_{\mu\nu}$ and $g^{s2}_{\mu\nu}$ are in the average $\eta_{\mu\nu}$ (that is to say that the insertion of the two-slit experiment does not appreciably change the geometry of space-time).

Now we will introduce an unphysical situation, a 'toy' model, the utility of which will be seen shortly

It is of interest to ask what one can say about the metric $g^{s1}_{\mu\nu}$. Around any small region of space-time, one can always diagonalize the metric, so we'll consider a diagonal metric. If the particle is propagated in, say, the x^3 direction and, of course, the x^4 (time) direction. We might expect the metric to be equal to the Minkowski metric, $\eta_{\mu\nu}$ save for g_{33} and g_{44}. (Here, we'll suppress the metric stochasticity for the moment, by, for example, averaging the metric components over a small region of space-time.) We will then, for the moment, take the following:

$$\tilde{g}^{s1}_{\mu\nu} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & -t \end{Vmatrix}, \text{ and so } |\tilde{g}^{s1}_{\mu\nu}| = -st,$$

where s and t are as yet undefined functions of position. In order that $|\tilde{g}^{s1}_{\mu\nu}|$ be constant, let $s = t^{-1}$ so that $\left|g^{s1}_{\mu\nu}\right| = \left|\eta_{\mu\nu}\right| = -1$.

Now we will introduce an unphysicalminently successful as it is, is an operational description of physical phenomena. It is composed of several principles, e situation, the utility of which will be seen shortly.

Let $s = e^{i\alpha}$where α is some unspecified function of position. Consider the following metrics:

$$g^{s1}_{\mu\nu} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\alpha} & 0 \\ 0 & 0 & 0 & -e^{-i\alpha} \end{Vmatrix},$$

$$g^{s2}_{\mu\nu} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\beta} & 0 \\ 0 & 0 & 0 & -e^{-i\beta} \end{Vmatrix},$$

where β is again some unspecified functions of position;

$(-|\tilde{g}^{s1}_{\mu\nu}|)^{1/2} = (-|\tilde{g}^{s2}_{\mu\nu}|)^{1/2} = 1$

(Note $|\frac{1}{2}A_{\mu\nu}| = \frac{1}{16}|A_{\mu\nu}|$),

$$(-|\tilde{g}^{s3}_{\mu\nu}|)^{1/2} = (-\tfrac{1}{16}|\tilde{g}^{s1}_{\mu\nu} + \tilde{g}^{s2}_{\mu\nu}|)^{1/2}$$
$$= [\tfrac{-1}{16}(2 + e^{i(\alpha-\beta)} + e^{-i(\alpha-\beta)})]^{1/2}$$
$$(-|\tilde{g}^{s3}_{\mu\nu}|)^{1/2} = \tfrac{1}{2}abs[cos\tfrac{1}{2}(\alpha-\beta)].$$

This is, of course, the phenomenon of interference. The metrics $g^{s1}_{\mu\nu}$, $g^{s2}_{\mu\nu}$, and $g^{s3}_{\mu\nu}$ describe, for example, the two-slit experiment described previously. The analogy of the function $e^{i\alpha}$ and $e^{-i\alpha}$ with Ψ and Ψ^* (the free particle wave functions) is obvious. The use of complex functions in the metric, however, is unphysical. The resultant line element $ds^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu$ would be complex and hence unphysical. The following question arises: Can we reproduce the previous scheme, but with real functions? The answer is yes, but first we must briefly discuss quadratic-form matrix transformations[47].

Let,

$$X = \begin{Vmatrix} dx^1 \\ dx^2 \\ dx^3 \\ dx^4 \end{Vmatrix}, \text{ and again let } G = \| \, g_{\mu\nu} \, \|.$$

Then $X^tGX = ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, where X^t is the transpose of X. Consider transformations which leave the line element ds^2 invariant. Given a transformation matrix W,

$X' = WX$

and

$X^tGX = X'^tG'X' = (X^t(W^t)^{-1})G'(W^{-1}X)$.

[Note: $(WX')^t = X'^tW'^t$.] However, $X^tGX \equiv (X^t(W^t)^{-1})(W^tGW)(W^{-1}X)$

so that $G' = W^tGW$.

In other words, the transformation W takes G into W^tGW. Now in the transformed coordinates, a metric $\tilde{g}^{s1}_{\mu\nu} \equiv G^{s1}$ goes to $W^tG^{s1}W$. Therefore,

$$\Psi_1^*\Psi_1 = (-|W^tG^{s1}W|)^{1/2}$$
$$= (-|W^t| \; |g^{s1}_{\mu\nu}| \; |W|)^{1/2},$$
$$\Psi_3^*\Psi_3 = (-\tfrac{1}{16}|W^tG^{s1}W + W^tG^{s21}W)^{1/2}.$$
$$= [-\tfrac{1}{16}|W^t(G^{s1} + G^{s2}) \; W|]^{1/2}$$
$$= (-\tfrac{1}{16}|W^t| \; |G^{s1} + G^{s2}| \; |W|)^{1/2}.$$

If we can find a transformation matrix W with the properties,

(i) $|W| = 1$,

(ii) W is not a function of α or β,

(iii) $W^tGW$is a matrix with only real components,

then we will again have the interference phenomenon with $g'_{\mu\nu}$ real, and again $\Psi_1^*\Psi_1 = \Psi_2^*\Psi_2 = 1$, and

$\Psi_3^*\Psi_3 = \frac{1}{2}abs\left(cos\frac{\alpha-\beta}{2}\right)$. The appropriate matrix W is,

$$\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{Vmatrix} = W.$$

If, as previously,

$$\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\alpha} & 0 \\ 0 & 0 & 0 & -e^{-i\alpha} \end{Vmatrix} \equiv \| \tilde{g}^{s1}_{\mu\nu} \|,$$

then,

$$\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -cos\alpha & sin\alpha \\ 0 & 0 & sin\alpha & cos\alpha \end{Vmatrix} = \| \tilde{g}'^{s1}_{\mu\nu} \| = W^t \tilde{g}^{s1}_{\mu\nu} W,$$

so that in order to reproduce the phenomenon of interference, the stochastic metric $\tilde{g}_{\mu\nu}$ will have off-diagonal terms. Incidentally, coordinates appropriate to $\tilde{g}^{s1}_{\mu\nu}$are,

$x^{1'} = x^1$,
$x^{2'} = x^2$,
$x^{3'} = \frac{-i}{\sqrt{2}} x^3 + \frac{1}{\sqrt{2}} x^4$,
$x^{4'} = \frac{1}{\sqrt{2}} x^3 - \frac{i}{\sqrt{2}} x^4$,

which is to say that with an appropriate coordinate transformation (which is complex), we can treat the free space probability distribution $\Psi^*\Psi$ in a particularly simple way. Since the components x^μ do not appear in predictions (such as $\Psi^*\Psi$), we may simply, as an operational convenience, take $\tilde{g}_{\mu\nu}$ to be diagonal, but with complex components.

The above metric has a problem. $\sqrt{-|g|} \neq \Psi^*\Psi$ (as is required by Statement 3). We could though, enforce an equality by replacing s and t by s^2 and t^2 respectively. [The modification of statement 3 in this version suggests we remove the square root, thus avoiding the problem.]

V. DISCUSSION

Having recognized that quantum mechanics is merely an operational calculus, and also having observed that general relativity is a true theory of nature with both an operational calculus and a *Weltanschauung*, we have attempted to generate quantum mechanics from the structure of space-time. As a starting point we have used a version of Mach's principle where in the absence of mass, space-time is not flat, but undefined (or more exactly, not well defined) such that $P_\Theta(g_{\mu\nu}) = k\sqrt{-|g_{\mu\upsilon}|}$ (where k is a constant) is, at a given point Θ, the probability distribution for $g_{\mu\nu}$ (in the Copenhagen sense[48]) .

From this, the motion of a free (test) particle was derived. This is a global approach to quantum theory. It should be noted that there are two logically distinct approaches to conventional quantum mechanics: a local, and a global formulation. The local formalism relies on the existence of a differential equation (such as the Schrödinger equation) describing the physical situation (e.g. the wave function of the particle) at each point in space-time. The existence of this equation is operationally very convenient. On the other hand, the global formulation (or path formulation, if you will) is rather like the Feynman path formalism for quantum mechanics[49], which requires the enumeration of the "action" over these paths. This formalism is logically very simple, but operationally it is exceptionally complex. Our approach is a local formalism. Statement 3, $P(x,t) = A\sqrt{-g}$, is local and provides the basis for the further development of stochastic space-time quantum theory. Statements 1 and 3 are then logically related. The remaining Statements 2, 4, and 5 are secondary in importance.

The conclusion is that with the acceptance of the statements, the following can be deduced:

(i) the motion of a free particle, and the spread of the wave packet,

(ii) the uncertainty principle,

(iii) the nature of conjugate variables,

(iv) interference phenomena,

(v) an indication of where conventional quantum mechanics might break down (i.e. the limited validity of linear superposition).

This paper represents an early stage of a theory. What is ultimately required is a set of 'field' equations (analogous to the general relativity field equations) which relate the mass distributions in the space-time to the stochasticity so that one can calculate $P(x,t)$ for all instances.

ACKNOWLEDGMENTS

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[50] NOTE: Re: Part A: Called 'Profound' by Siser Roy[2], 'Remarkable' by Steven Miller[3], 'Pioneering Work' by Luis de la Pena & Ana Cetto[4], etc., was the subject of a Ph.D. thesis by G. Townsend[5], and even found its way into a book on nanofabrication[6].