Dieudonné-type theorems for lattice group-valued k-triangular set functions

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Abstract

Some versions of Dieudonné-type convergence and uniform boundedness theorems are proved, for k-triangular and regular lattice group-valued set functions. We use sliding hump techniques and direct methods. We extend earlier results, proved in the real case. Furthermore, we pose some open problems.

1 Introduction

Dieudonné-type theorems (see [33]) are the object of several studies about convergence and uniform boundedness theorems for regular set functions and related topics about (weak) compactness of measures. A historical comprehensive survey can be found in [18]. Among the most important developments existing in the literature about these subjects, see for instance [2, 3, 29, 30, 31, 32, 37, 44], and in particular, concerning the setting of lattice group-valued measures, we quote [6, 9, 10, 12, 13]. In [14, 24] some Dieudonné-type theorems were proved for lattice group-valued finitely additive regular measures in the context of filter convergence, while some versions of uniform boundedness theorems in this setting are proved in [11, 25]. In [38, 39, 40, 46] some Dieudonné-type theorems were proved for k-triangular and non-additive regular set functions. Some examples of k-triangular set functions are the *M-measures*, that is monotone set functions m with $m(\emptyset) = 0$, continuous from above and from below and compatible with respect to supremum and infimum, which have several applications in several branches, among which intuitionistic fuzzy sets and observables (see also [1, 17, 27, 34, 41]). Some examples of non-monotone 1-triangular set functions are the Saeki measuroids (see [42]). In [17, 20, 21, 22, 23] some limit theorems were proved for lattice group-valued k-subadditive capacities and k-triangular set functions.

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In this paper we prove some Dieudonné convergence theorems and a version of Nikodým boundedness theorem for regular and k-triangular lattice group-valued set functions, extending earlier results proved in the real case in [38, 39, 40] using some diagonal matrix theorems. Our techniques are direct and inspired by sliding hump-type methods. We use the tool of (D)-convergence, because we can apply the powerful Fremlin lemma (see also [36, 41]), which replaces the $\frac{\varepsilon}{2^n}$-technique and allows to replace a sequence of regulators with a single (D)-sequence. Observe that, in the lattice group context, in the Nikodým boundedness theorem we assume the existence of a single increasing sequence of positive elements of the involved lattice group, with respect to which the set functions are supposed to be pointwise bounded on a suitable sublattice, playing a role similar to that of the class of all open subsets of a topological space. We see that in general this condition cannot be replaced by a simple setwise boundedness (see also [11, 25, 45]). Finally, some open problems are posed.

2 Preliminaries

We begin with recalling the following basic facts on lattice groups (see also [18, 28]).

Definitions 2.1 (a) A lattice group R is said to be *Dedekind complete* if every nonempty subset of R, bounded from above, has supremum in R.

(b) A Dedekind complete lattice group R is *super Dedekind complete* iff for every nonempty set $A \subset R$, bounded from above, there is a countable subset A', with $\bigvee A' = \bigvee A$.

(c) A nonempty subset S of a lattice group R is *bounded* iff there exists an element $u \in R$ with $|x| \leq u$ for each $x \in S$.

(d) Let $(t_n)_n$ be an increasing sequence of positive elements of R, and let $\emptyset \neq S \subset R$. We say that S is *bounded by* $(t_n)_n$ iff there is $n_* \in \mathbb{N}$ with $|x| \leq t_{n_*}$ whenever $x \in S$.

(e) A sequence $(\sigma_p)_p$ in a lattice group R is called an (O)*-sequence* iff it is decreasing and $\bigwedge_{p=1}^{\infty} \sigma_p = 0$.

(f) A bounded double sequence $(a_{t,l})_{t,l}$ in R is a (D)*-sequence* or a *regulator* iff $(a_{t,l})_l$ is an (O)-sequence for any $t \in \mathbb{N}$.

(g) A lattice group R is *weakly σ-distributive* iff $\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \right) = 0$ for every (D)-sequence $(a_{t,l})_{t,l}$ in R.

(h) A sequence $(x_n)_n$ in R is said to be *order convergent* (or (O)*-convergent*) to x iff there exists an (O)-sequence $(\sigma_p)_p$ in R such that for every $p \in \mathbb{N}$ there is a positive integer n_0 with $|x_n - x| \leq \sigma_p$ for each $n \geq n_0$, and in this case we write $(O) \lim_n x_n = x$.

(i) We say that $(x_n)_n$ is (O)*-Cauchy* iff there is an (O)-sequence $(\tau_p)_p$ in R such that for every $p \in \mathbb{N}$ there is a positive integer n_0 with $|x_n - x_q| \leq \tau_p$ for each n, $q \geq n_0$.

(j) A sequence $(x_n)_n$ in R is (D)*-convergent* to x iff there is a (D)-sequence $(a_{t,l})_{t,l}$ in R such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $n_0 \in \mathbb{N}$ with $|x_n - x| \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$ whenever $n \geq n_0$, and we write

$(D) \lim\limits_n x_n = x$.

(k) We say that $(x_n)_n$ is (D)-*Cauchy* iff there exists a (D)-sequence $(b_{t,l})_{t,l}$ in R such that for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $n_0 \in \mathbb{N}$ with $|x_n - x_q| \leq \bigvee\limits_{t=1}^{\infty} b_{t,\varphi(t)}$ whenever $n, q \geq n_0$.

(l) A lattice group R is said to be (O)-*complete* iff every (O)-Cauchy (resp. (D)-Cauchy) sequence is (O)-convergent (resp. (D)-convergent).

(m) We call *sum* of a series $\sum\limits_{n=1}^{\infty} x_n$ in R the limit $(O) \lim\limits_n \sum\limits_{r=1}^{n} x_r$, if it exists in R.

(n) If R is a vector lattice, then we say that $(x_n)_n$ (r)-*converges* to x iff there exists $u \in R$, $u \geq 0$, such that for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ with $|x_n - x| \leq \varepsilon\, u$ whenever $n \geq n_0$.

(o) A vector lattice R *satisfies property* (σ) iff for every sequence $(u_n)_n$ of positive elements of R there are a sequence $(a_n)_n$ of positive real numbers and an element $u \in R$ with $a_n\, u_n \leq u$ for each $n \in \mathbb{N}$.

(p) A lattice $\mathcal{E}$ of subsets of an infinite set G *satisfies property* (E) iff for each disjoint sequence $(C_h)_h$ in $\mathcal{E}$ there is a subsequence $(C_{h_r})_r$, such that $\mathcal{E}$ contains the σ-algebra generated by the sets C_{h_r}, $r \in \mathbb{N}$ (see also [43]).

Remark 2.2 Note that every Dedekind complete lattice group is both (O)- and (D)-complete. Moreover, observe that every (O)-convergent sequence is also (D)-convergent to the same limit in any lattice group, while the converse is true if and only if the involved (ℓ)-group is weakly σ-distributive. Furthermore, it is known that every (r)-convergent sequence in any vector lattice is (O)-convergent too (see also [28, 47]). The converse, in general, is not true. For example, let $\mathcal{B}$ be the σ-algebra of all Borel subsets of $[0,1]$, λ be the Lebesgue measure on $[0,1]$, $L^0 := L^0([0,1], \mathcal{B}, \lambda)$ be the space of all measurable real-valued functions defined on $[0,1]$, with the identification of λ-null sets, and $R := \{f \in L^0([0,1], \mathcal{B}, \lambda)\colon\ f \text{ is bounded}\}$. If $(u_n)_n$ is any sequence of positive elements of R, then there exists a sequence $(L_n)_n$ of positive real numbers such that $u_n \leq \underline{L_n}$ for every $n \in \mathbb{N}$, where $\underline{L_n}$ denotes the function which assumes the constant value L_n. Since $\mathbb{R}$ fulfils property (σ), there are a sequence $(a_n)_n$ of positive real numbers and a positive real number v with $a_n\, L_n \leq v$, and hence $a_n\, u_n \leq a_n \underline{L_n} \leq \underline{v}$, for every $n \in \mathbb{N}$. Hence, R satisfies property (σ). It is known that in L^0 order and (r)-convergence coincide with almost everywhere convergence, while in R, order convergence coincides with the almost everywhere convergence dominated by a constant function, and (r)-convergence coincides with uniform convergence (see also [47]). Moreover, since L^0 is weakly σ-distributive (see also [8]), then in L^0 (O)- and (D)-convergence coincide in L^0, and so they coincide also in R. Hence, R is weakly σ-distributive too. Finally, observe that, in the space L^0, order, (D)- and (r)-convergence are equivalent (see also [8, 47]).

We now recall the following property of convergence in lattice groups (see also [23, Proposition 3.1]).

Proposition 2.3 *Let R be a Dedekind complete lattice group, $x \in R$, and $(x_n)_n$ be a sequence in R, such that*

2.3.1) *for every subsequence* $(x_{n_q})_q$ *of* $(x_n)_n$ *there is a sub-subsequence* $(x_{n_{q_r}})_r$*, convergent to* x *with respect to a single* (D)*-sequence* $(a_{t,l})_{t,l}$*.*

Then $(D)\lim_n x_n = x$ *with respect to* $(a_{t,l})_{t,l}$*.*

Proof: Suppose by contradiction that there are $\varphi \in \mathbb{N}^{\mathbb{N}}$ and a strictly increasing sequence $(n_q)_q$ with $|x_{n_q} - x| \not\leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$ for each $q \in \mathbb{N}$. Thus any subsequence of $(x_{n_q})_q$ does not (D)-converge to x with respect to $(a_{t,l})_{t,l}$, obtaining a contradiction with 2.3.1). □

Remark 2.4 An analogous of Proposition 2.3 holds, if (D)-convergence is replaced by (O)-convergence.

We now recall the Fremlin lemma, by means of which it is possible to replace a sequence of regulators with a single (D)-sequence, and which will be fundamental in the sequel, to prove our main results, because it has the same role as the $\frac{\varepsilon}{2^n}$-argument. This is one of the reason for which we often prefer to deal with (D)-convergence rather than (O)-convergence.

Lemma 2.5 (see also [36, Lemma 1C], [41, Theorem 3.2.3]) Let R be any Dedekind complete (ℓ)-group and $(a_{t,l}^{(n)})_{t,l}$, $n \in \mathbb{N}$, be a sequence of regulators in R. Then for every $u \in R$, $u \geq 0$ there is a (D)-sequence $(a_{t,l})_{t,l}$ in R with

$$u \wedge \Big(\sum_{n=1}^{q}\Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}^{(n)}\Big)\Big) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \quad \text{for every } q \in \mathbb{N} \text{ and } \varphi \in \mathbb{N}^{\mathbb{N}}.$$

We now deal with the main properties of k-triangular lattice group-valued set functions. Let R be a Dedekind complete and weakly σ-distributive lattice group, G be an infinite set, $\mathcal{L} \subset \mathcal{P}(G)$ be an algebra, $m : \mathcal{L} \to R$ be a bounded set function and k be a fixed positive integer.

Definitions 2.6 (a) The *semivariation* of m is defined by setting

$$v(m)(A) = v_{\mathcal{L}}(m)(A) := \bigvee\{|m(B)| : B \in \mathcal{L},\ B \subset A\}, \quad A \in \mathcal{L}.$$

If $\mathcal{E} \subset \mathcal{L}$ is a lattice, then we put

$$v_{\mathcal{E}}(m)(A) := \bigvee\{|m(B)| : B \in \mathcal{E},\ B \subset A\}, \quad A \in \mathcal{L}.$$

The set function $v_{\mathcal{E}}(m)$ is called the *semivariation of* m *with respect to* $\mathcal{E}$.

(b) We say that m is *k-triangular* on $\mathcal{L}$ iff

$$m(A) - k\, m(B) \leq m(A \cup B) \leq m(A) + k\, m(B) \text{ whenever } A, B \in \Sigma,\ A \cap B = \emptyset \qquad (1)$$

and

$$0 = m(\emptyset) \leq m(A) \text{ for each } A \in \Sigma. \qquad (2)$$

(c) Let $\mathcal{E} \subset \mathcal{L}$ be a sublattice of $\mathcal{L}$. We say that a set function $m : \mathcal{L} \to R$ is $\mathcal{E}$-(s)-*bounded* iff there exists a (D)-sequence $(a_{t,l})_{t,l}$ such that, for every disjoint sequence $(C_h)_h$ in $\mathcal{E}$, $(D) \lim_h v_{\mathcal{E}}(m)(C_h) = 0$ with respect to $(a_{t,l})_{t,l}$. A set function m is (s)-*bounded* iff it is $\mathcal{L}$-(s)-bounded.

(d) We say that the set functions $m_j : \mathcal{L} \to R$ are $\mathcal{E}$-*uniformly* (s)-*bounded* iff there exists a (D)-sequence $(a_{t,l})_{t,l}$ such that, for every disjoint sequence $(C_h)_h$ in $\mathcal{E}$,

$$(D) \lim_h \Big(\bigvee_j v_{\mathcal{E}}(m_j)(C_h) \Big) = 0$$

with respect to $(a_{t,l})_{t,l}$. The m_j's are *uniformly* (s)-*bounded* iff they are $\mathcal{L}$-uniformly (s)-bounded.

(f) We say that the set functions $m_j : \mathcal{L} \to R$, $j \in \mathbb{N}$, are *equibounded* on $\mathcal{L}$ iff there is $u \in R$ with $|m_j(A)| \le u$ for every $j \in \mathbb{N}$ and $A \subset \mathcal{L}$.

Now we recall the following

Proposition 2.7 (see also [23, Proposition 2.6]) *If $m : \mathcal{L} \to R$ is k-triangular, then $v(m)$ is k-triangular too.*

Proposition 2.8 (see also [23, Proposition 2.7]) *Let $m : \mathcal{L} \to R$ be a k-triangular set function. Then for every $n \in \mathbb{N}$, $n \ge 2$, and for every pairwise disjoint sets $E_1, E_2, \dots, E_n \in \mathcal{L}$ we have*

$$m(E_1) - k \sum_{q=2}^{n} m(E_q) \le m\Big(\bigcup_{q=1}^{n} E_q \Big) \le m(E_1) + k \sum_{q=2}^{n} m(E_q), \tag{3}$$

and in particular

$$m(E_1) \le m\Big(\bigcup_{q=1}^{n} E_q \Big) + k \sum_{q=2}^{n} m(E_q). \tag{4}$$

We now turn to regular lattice group-valued set functions.

Definition 2.9 Let $\mathcal{G}$, $\mathcal{H}$ be two sublattices of $\mathcal{L}$, such that $\mathcal{G}$ is closed under countable unions, and the complement of every element of $\mathcal{H}$ belongs to $\mathcal{G}$. A set function $m : \mathcal{L} \to R$ is said to be *regular* iff there exists a (D)-sequence $(a_{t,l})_{t,l}$ such that

2.9.1) for every $E \in \mathcal{L}$ there are two sequences $(V_n)_n$ in $\mathcal{G}$ and $(K_n)_n$ in $\mathcal{H}$ with $V_n \supset E \supset K_n$ for each $n \in \mathbb{N}$ and such that for any $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $n_0 \in \mathbb{N}$ with

$$v(m)(V_n \setminus K_n) \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$$

whenever $n \ge n_0$, and

2.9.2) for every $W \in \mathcal{H}$ there are two sequences $(G_n)_n$ in $\mathcal{G}$ and $(F_n)_n$ in $\mathcal{H}$ with $W \subset F_{n+1} \subset G_n \subset F_n$ for every $n \in \mathbb{N}$, and such that for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $n^* \in \mathbb{N}$ with

$$v(m)(G_n \setminus W) \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$$

whenever $n \ge n^*$.

We now prove the following property of regular set functions.

Proposition 2.10 (see also [17, Theorem 3.10]) *If G is a compact Hausdorff topological space, $\mathcal{L}$, $\mathcal{G}$, $\mathcal{H}$ are the classes of all Borel, open and compact subsets of G, respectively, and $m : \mathcal{L} \to R$ is a k-triangular, increasing and regular set function, then*

$$(O) \lim_n m(I_n) = 0 \tag{5}$$

whenever $(I_n)_n$ is a decreasing sequence in $\mathcal{L}$ with $\bigcap_{n=1}^{\infty} I_n = \emptyset$, with respect to a single regulator independent of the choice of $(I_n)_n$.

Proof: Let $(I_n)_n$ be as in (5). Let $(a_{t,l})_{t,l}$ be a (D)-sequence satisfying 2.9.1). For every $n \in \mathbb{N}$ there is $K_n \in \mathcal{H}$ with $K_n \subset I_n$ and $m(I_n \setminus K_n) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}$. By virtue of Lemma 2.5, there is a (D)-sequence $(\alpha_{t,l})_{t,l}$ with

$$m(G) \wedge \Big(\sum_{n=1}^{q} \Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}\Big)\Big) \leq \bigvee_{t=1}^{\infty} \alpha_{t,\varphi(t)} \text{ for each } q \in \mathbb{N} \text{ and } \varphi \in \mathbb{N}^{\mathbb{N}}.$$

Let $O_n := G \setminus K_n$, $n \in \mathbb{N}$. Note that $O_n \in \mathcal{G}$ for every n and $G = \bigcup_{n=1}^{\infty} O_n$, since $\bigcap_{n=1}^{\infty} K_n = \emptyset$. As G is compact, there is $n_0 \in \mathbb{N}$ with $G = \bigcup_{i=1}^{n} O_i$, and hence $\bigcap_{i=1}^{n} K_i = \emptyset$, whenever $n \geq n_0$. For such n's, taking into account (3), we have

$$\begin{aligned} m(I_n) &\leq m(G) \wedge \Big(m(I_n \setminus \Big(\bigcap_{i=1}^{n} K_i\Big)\Big) \leq \\ &\leq m(G) \wedge \Big(m\Big(\bigcup_{i=1}^{n} (I_i \setminus K_i)\Big)\Big) \leq \\ &\leq m(G) \wedge \Big(k \sum_{i=1}^{n} m(I_i \setminus K_i)\Big) \leq k \bigvee_{t=1}^{\infty} \alpha_{t,\varphi(t)} \end{aligned} \tag{6}$$

(see also [38, Lemma 1]). Thus the assertion follows. □

Remark 2.11 Observe that, if $\mathcal{L}$ is an algebra with property (E) and $m : \mathcal{L} \to R$ is positive, increasing and satisfies (5), then m is also (s)-bounded (with respect to a single regulator). To prove this, let $(A_n)_n$ be any disjoint sequence in $\mathcal{L}$ and $(B_n)_n$ be any subsequence of $(A_n)_n$. By property (E), there is a subsequence $(C_n)_n$ of $(B_n)_n$, such that $\bigcup_{n \in P} C_n \in \mathcal{L}$ for every $P \subset \mathbb{N}$. Since m is increasing and $m(\emptyset) = 0$, we get

$$0 \leq m(C_n) \leq m\Big(\bigcup_{i=n}^{\infty} C_i\Big)$$

From (5) and (7) we get $(O) \lim_n m(C_n) = 0$ with respect to a single regulator (independent of $(A_n)_n$, $(B_n)_n$ and $(C_n)_n$). By arbitrariness of the sequence $(B_n)_n$ and Proposition 2.3 it follows that $(D) \lim_n m(C_n) = 0$ with respect to a single regulator, and this proves the claim.

The converse, in general, is not true (see also [23, Remark 2.12]).

Proposition 2.12 (see also [17, Proposition 3.4]) *If $m : \mathcal{L} \to R$ is a k-triangular and increasing set function satisfying (5), then we get*

$$m\Big(\bigcup_{n=1}^{\infty} E_n\Big) \leq m(E_1) + k \sum_{n=2}^{\infty} m(E_n) \tag{7}$$

for every sequence $(E_n)_n$ in $\mathcal{L}$, such that $\bigcup_{n \in A} E_n \in \mathcal{L}$ whenever $A \subset \mathbb{N}$.

The following proposition will be useful in proving our Dieudonné convergence theorem (see also [10, Lemma 3.1]).

Proposition 2.13 *With the same notations and assumptions as above, let $m : \mathcal{L} \to R$ be a regular and k-triangular set function. Then for each $V \in \mathcal{G}$ we get*

$$v_{\mathcal{L}}(m)(V) = v_{\mathcal{G}}(m)(V). \tag{8}$$

Proof: Pick arbitrarily $V \in \mathcal{G}$, and let $(\gamma_{t,l})_{t,l}$ be a (D)-sequence related to regularity of m. Choose $B \in \mathcal{L}$ with $B \subset V$, and fix arbitrarily $\varphi \in \mathbb{N}^{\mathbb{N}}$. By regularity of m, there is $O \in \mathcal{G}$, $O \supset B$, with

$$v_{\mathcal{L}}(m)(O \setminus B) \leq \bigvee_{t=1}^{\infty} \gamma_{t,\varphi(t)}. \tag{9}$$

Let $U := O \cap V$, then $U \supset B$. From (9) and k-triangularity of m we get

$$\begin{aligned} m(B) &\leq m(U) + k\, m(U \setminus B) \leq \\ &\leq v_{\mathcal{G}}(m)(V) + k\, v_{\mathcal{L}}(m)(O \setminus B) \leq \\ &\leq v_{\mathcal{G}}(m)(V) + k \bigvee_{t=1}^{\infty} \gamma_{t,\varphi(t)}. \end{aligned} \tag{10}$$

Taking in (10) the supremum as $B \in \mathcal{L}$, $B \subset V$, we obtain

$$v_{\mathcal{L}}(m)(V) \leq v_{\mathcal{G}}(m)(V) + k \bigvee_{t=1}^{\infty} \gamma_{t,\varphi(t)}. \tag{11}$$

From (11) and weak σ-distributivity of R we deduce

$$v_{\mathcal{L}}(m)(V) \leq v_{\mathcal{G}}(m)(V) + k \bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \Big(\bigvee_{t=1}^{\infty} \gamma_{t,\varphi(t)}\Big) = v_{\mathcal{G}}(m)(V). \tag{12}$$

Since the converse inequality is straightforward, then (8) follows from (12). This ends the proof. $\square$

Definition 2.14 A sequence $m_j : \mathcal{L} \to R$, $j \in \mathbb{N}$, of set functions is said to be (RD)-*regular* on $\mathcal{L}$ iff there is a (D)-sequence $(a_{t,l})_{t,l}$ such that

2.14.1) for every $E \in \mathcal{L}$ there are two sequences $(V_n)_n$ in $\mathcal{G}$ and $(K_n)_n$ in $\mathcal{H}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $j \in \mathbb{N}$ there is $n_0 \in \mathbb{N}$ with $v(m_j)(V_n \setminus K_n) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$ for every $n \geq n_0$, and

2.14.2) for every disjoint sequence $(H_n)_n$ in $\mathcal{L}$ there is a sequence $(O_n)_n$ in $\mathcal{G}$ such that $O_n \supset H_n$ for each $n \in \mathbb{N}$ and $(D) \lim_n v(m_j)\Big(\bigcup_{i=n}^{\infty} O_i\Big) = 0$ for every $j \in \mathbb{N}$ with respect to $(a_{t,l})_{t,l}$.

We now recall the following

Proposition 2.15 (see also [10, Proposition 2.6]) *Let R be any Dedekind complete and weakly σ-distributive lattice group, and $m_j : \mathcal{L} \to R$, $j \in \mathbb{N}$, be a sequence of regular equibounded set functions. Then they satisfy 2.14.1) and the following property:*

2.15.1) *there exists a regulator $(\beta_{t,l})_{t,l}$ such that for every $W \in \mathcal{H}$ there are two sequences $(G_n)_n$ in $\mathcal{G}$ and $(F_n)_n$ in $\mathcal{H}$, with $W \subset F_{n+1} \subset G_n \subset F_n$ for every $n \in \mathbb{N}$ and such that for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $j \in \mathbb{N}$ there is $n^* \in \mathbb{N}$ with*

$$v_{\mathcal{L}}(m_j)(G_n \setminus W) \leq \bigvee_{t=1}^{\infty} \beta_{t,\varphi(t)}$$

for every $n \geq n^*$.

Definition 2.16 Let $\mathcal{L}$, $\mathcal{G}$, $\mathcal{H}$ be as in Definition 2.9. The set functions $m_j : \mathcal{L} \to R$, $j \in \mathbb{N}$, are *uniformly regular* iff there exists a (D)-sequence $(a_{t,l})_{t,l}$ such that

2.16.1) for each $E \in \mathcal{L}$ there exist two sequences $(V_n)_n$ in $\mathcal{G}$ and $(K_n)_n$ in $\mathcal{H}$ with $V_n \supset E \supset K_n$ for every $n \in \mathbb{N}$ and such that for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $n_0 \in \mathbb{N}$ with

$$\bigvee_j v(m_j)(V_n \setminus K_n) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$$

for all $n \geq n_0$, and

2.16.2) for any $W \in \mathcal{H}$ there are two sequences $(G_n)_n$ in $\mathcal{G}$ and $(F_n)_n$ in $\mathcal{H}$ with $W \subset F_{n+1} \subset G_n \subset F_n$ for each $n \in \mathbb{N}$, and such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $n^* \in \mathbb{N}$ with

$$\bigvee_j v(m_j)(G_n \setminus W) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$$

whenever $n \geq n^*$.

3 The main results

In this section we prove a Dieudonné convergence-type theorem and a Dieudonné-Nikodým boundedness theorem for regular and k-triangular lattice group-valued set functions. Let R be a Dedekind complete and weakly σ-distributive lattice group. We begin with recalling the following Brooks-Jewett-type theorem for k-triangular set functions.

Theorem 3.1 (see [23, Theorem 3.3]) *Let G be any infinite set, $\mathcal{L} \subset \mathcal{P}(G)$ be an algebra, $\mathcal{E} \subset \mathcal{L}$ be a lattice, satisfying property (E), $m_j : \mathcal{L} \to R$, $j \in \mathbb{N}$, be a sequence of equibounded, k-triangular and $\mathcal{E}$-(s)-bounded set functions. If the limit $m_0(E) := \lim_j m_j(E)$ exists in R for every $E \in \mathcal{E}$ with respect to a single regulator, then the m_j's are $\mathcal{E}$-uniformly (s)-bounded, and m_0 is k-triangular and (s)-bounded.*

The following technical lemma will be useful in the sequel.

Lemma 3.2 (see [23, Lemma 3.4]) *Let $\mathcal{L} \subset \mathcal{P}(G)$ be an algebra, $\mathcal{G}$ and $\mathcal{H}$ be two sublattices of $\mathcal{L}$, such that the complement of every element of $\mathcal{H}$ belongs to $\mathcal{G}$, $m_j : \mathcal{L} \to R$, $j \in \mathbb{N}$, be a sequence of k-triangular and $\mathcal{G}$-uniformly (s)-bounded set functions. Fix $W \in \mathcal{H}$ and a decreasing sequence $(H_n)_n$ in $\mathcal{G}$, with $W \subset H_n$ for each $n \in \mathbb{N}$. If*

$$(D)\lim_n \Big(\bigvee_{A\in\mathcal{G}, A\subset H_n\setminus W} m_j(A) \Big) = \bigwedge_n \Big(\bigvee_{A\in\mathcal{G}, A\subset H_n\setminus W} m_j(A) \Big) = 0 \text{ for every } j \in \mathbb{N} \tag{13}$$

with respect to a single (D)-sequence $(a_{t,l})_{t,l}$, then

$$(D)\lim_n \Big(\bigvee_j \Big(\bigvee_{A\in\mathcal{G}, A\subset H_n\setminus W} m_j(A) \Big) \Big) = \bigwedge_n \Big(\bigvee_j \Big(\bigvee_{A\in\mathcal{G}, A\subset H_n\setminus W} m_j(A) \Big) \Big) = 0$$

with respect to $(a_{t,l})_{t,l}$.

The next step is to prove a Dieudonné-type theorem for k-triangular lattice group-valued set functions, which extends [10, Lemma 3.2].

Theorem 3.3 *Let $\mathcal{L} \subset \mathcal{P}(G)$ be an algebra, $\mathcal{G}$ and $\mathcal{H}$ be two sublattices of $\mathcal{L}$, such that $\mathcal{G}$ is closed under countable unions and the complement of every element of $\mathcal{H}$ belongs to $\mathcal{G}$, $m_j : \mathcal{L} \to R$, $j \in \mathbb{N}$, be a sequence of equibounded, regular, k-triangular and $\mathcal{G}$-uniformly (s)-bounded set functions. Then the m_j's are $\mathcal{L}$-uniformly (s)-bounded and uniformly regular on $\mathcal{L}$.*

Proof: Let $(H_n)_n$ be a disjoint sequence of elements of $\mathcal{L}$, $(a_{t,l})_{t,l}$ be a (D)-sequence, satisfying 2.14.1), $u = \bigvee_{j\in\mathbb{N}, A\in\mathcal{L}} m_j(A)$, and according to Lemma 2.5, let $(b_{t,l})_{t,l}$ be a regulator in R, with

$$u \wedge \Big(\sum_{h=1}^{q} \Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+h)} \Big) \Big) \leq \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} \quad \text{for every } \varphi \in \mathbb{N}^{\mathbb{N}} \text{ and } q \in \mathbb{N}. \tag{14}$$

Let $(c_{t,l})_{t,l}$ be a (D)-sequence associated with $\mathcal{G}$-uniform (s)-boundedness, and set $d_{t,l} = (k+1)(b_{t,l} + c_{t,l})$, $e_{t,l} = (k+1)(a_{t,l} + d_{t,l})$, for every t, $l \in \mathbb{N}$. We prove that the m_j's are $\mathcal{L}$-uniformly (s)-bounded with respect to the regulator $(e_{t,l})_{t,l}$. Otherwise, there is $\varphi \in \mathbb{N}^{\mathbb{N}}$ with the property that for every $h \in \mathbb{N}$ there are j_h, $n_h \in \mathbb{N}$ with $n_h \geq h$ and $B_h \in \mathcal{L}$ with $B_h \subset H_{n_h}$ and

$$m_{j_h}(B_h) \not\leq \bigvee_{t=1} e_{t,\varphi(t)}. \tag{15}$$

By 2.14.1), for every $h \in \mathbb{N}$ there is $A_h \in \mathcal{H}$, $A_h \subset B_h$, with

$$m_{j_h}(B_h \setminus A_h) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}. \tag{16}$$

From (15) and (16) it follows that

$$m_{j_h}(A_h) \not\leq \bigvee_{t=1}^{\infty} d_{t,\varphi(t)} : \tag{17}$$

otherwise, thanks to k-triangularity of m_{j_h}, we should get

$$m_{j_h}(B_h) \leq m_{j_h}(A_h) + k\, m_{j_h}(B_h \setminus A_h) \leq \bigvee_{t=1}^{\infty} d_{t,\varphi(t)},$$

which contradicts (15). Moreover, observe that from 2.14.1), in correspondence with φ, for every h there are $G_h \in \mathcal{G}$ and $F_h \in \mathcal{H}$, with $A_h \subset G_h \subset F_h$ and

$$[v(m_1) \vee \ldots \vee v(m_{j_h})](F_h \setminus A_h) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t+h)}.$$

Set now $G_1^* = G_1$, $G_{h+1}^* = G_{h+1} \setminus \left(\bigcup_{r=1}^{h} F_r\right)$, $h \geq 2$. Since the G_h^*'s are disjoint elements of $\mathcal{G}$, then, thanks to $\mathcal{G}$-uniform (s)-boundedness and taking into account Proposition 2.13, we find a positive integer h_0 with

$$\bigvee_j v_{\mathcal{L}}(m_j)(G_h^*) = \bigvee_j v_{\mathcal{G}}(m_j)(G_h^*) \leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t)}$$

whenever $h \geq h_0$. Since for every h we get $A_{h+1} \setminus G_{h+1}^* \subset \bigcup_{r=1}^{h} (F_r \setminus A_r)$, then

$$\begin{aligned} m_{j_h}(A_h) &\leq m_{j_h}(A_h \cap G_h^*) + m_{j_h}(A_h \setminus G_h^*) \\ &\leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t)} + k \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} \leq \bigvee_{t=1}^{\infty} d_{t,\varphi(t)} \text{ for every } h \geq h_0, \end{aligned}$$

which contradicts (17), getting $\mathcal{L}$-uniform (s)-boundedness of the m_j's. Conditions 2.16.2) and 2.16.1) on uniform regularity of the m_j's follow easily from Proposition 2.15 and Lemma 3.2 used with $H_n = G_n \setminus W$, $n \in \mathbb{N}$, and $H_n = V_n \setminus K_n$, $\mathcal{G} = \mathcal{H} = \mathcal{L}$, $W = \emptyset$ respectively, where G_n is as in 2.15.1), V_n and K_n are as in 2.14.1). □

Now we are in position to prove the following theorem, which extends [10, Theorem 3.3].

Theorem 3.4 *Let G, R, $\mathcal{L}$, $\mathcal{G}$, $\mathcal{H}$ be as above, and suppose that $m_j : \mathcal{L} \to R$, $j \in \mathbb{N}$, is a sequence of equibounded, regular, k-triangular and (s)-bounded set functions, such that there exists*

$$m_0(E) := (D) \lim_j m_j(E) \text{ for every } E \in \mathcal{G}$$

with respect to a single regulator. Then,

3.4.1) *the measures m_j, $j \in \mathbb{N}$, are $\mathcal{L}$-uniformly (s)-bounded and uniformly regular;*

3.4.2) *there exists in R the limit $m_0(E) = (D) \lim_j m_j(E)$ for each $E \in \mathcal{L}$ with respect to a single regulator;*

3.4.3) *the set function m_0 is regular, k-triangular and (s)-bounded.*

Proof: 3.4.1) is a consequence of Theorems 3.1 and 3.3.

3.4.2). Choose arbitrarily $E \in \mathcal{L}$, and let $(y_{t,l})_{t,l}$ be a (D)-sequence associated with uniform regularity. For each $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $U \in \mathcal{G}$ with $U \supset E$ and $v_{\mathcal{L}}(m_j)(U \setminus E) \leq \bigvee_{t=1}^{\infty} y_{t,\varphi(t)}$ for every $j \in \mathbb{N}$. Moreover, in correspondence with U there is $j_0 \in \mathbb{N}$ with

$$|m_j(U) - m_{j+p}(U)| \leq \bigvee_{t=1}^{\infty} \alpha_{t,\varphi(t)}$$

for every $j \geq j_0$ and $p \in \mathbb{N}$, where $(\alpha_{t,l})_{t,l}$ is a regulator related to (D)-convergence on $\mathcal{G}$. By k-triangularity of m_j and m_{j+p} we get

$$\begin{aligned} m_j(E) - m_{j+p}(E) &\leq m_j(U) - m_{j+p}(U) + k\, m_j(U \setminus E) + k\, m_{j+p}(U \setminus E), \\ m_{j+p}(E) - m_j(E) &\leq m_{j+p}(U) - m_j(U) + k\, m_j(U \setminus E) + k\, m_{j+p}(U \setminus E), \end{aligned}$$

and hence

$$\begin{aligned} |m_j(E) - m_{j+p}(E)| &\leq |m_j(U) - m_{j+p}(U)| + k\, m_j(U \setminus E) + k\, m_{j+p}(U \setminus E) \leq \\ &\leq \bigvee_{i=1}^{\infty} (2\,k+1)(y_{i,\varphi(i)} + \alpha_{i,\varphi(i)}) \end{aligned} \tag{18}$$

for every $j \geq j_0$ and $p \in \mathbb{N}$. From (18) it follows that the sequence $(m_j(E))_j$ is (D)-Cauchy in R. Since R is a Dedekind complete lattice group, then the sequence $(m_j(E))_j$ is (D)-convergent, with respect to a regulator independent of E (see also [7, 28]). Thus 3.4.2) is proved.

3.4.3). Straightforward. □

The next step is to prove a uniform boundedness theorem for k-triangular regular lattice group-valued set functions. We begin with the following result, which extends [11, Proposition 4.5].

Proposition 3.5 *Let $m_h : \mathcal{L} \to R$, $h \in \mathbb{N}$, be a sequence of k-triangular set functions, and let $(t_n)_n$ be an increasing sequence of positive elements of R. Suppose also that*

3.5.1) *for every disjoint sequence* $(H_j)_j$ *in* $\mathcal{L}$*, the set* $\{m_h(H_j)\colon h, j \in \mathbb{N}\}$ *is bounded by* $(t_n)_n$*.*

Then the set $\{m_h(A) : h \in \mathbb{N}, A \in \mathcal{L}\}$ *is bounded in* R*.*

Proof: First of all observe that, thanks to 3.5.1), for every fixed element $A \in \mathcal{L}$ there is $n = n(A) \in \mathbb{N}$ with $0 \leq m_h(A) \leq t_{n(A)}$ for every $h \in \mathbb{N}$. We now prove that the set $\{m_h(A) : h \in \mathbb{N},\ A \in \mathcal{L}\}$ is bounded by the sequence $((k+1)t_n)_n$. Suppose, by contradiction, that this is not true. By hypothesis, there is $n_1 \in \mathbb{N}$ such that $m_h(G) \leq t_{n_1}$ for all h. Moreover, there exist $A_1 \in \mathcal{L}$ and $h_1 \in \mathbb{N}$ such that $m_{h_1}(A_1) \not\leq (k+1)t_{n_1}$. We have also $m_{h_1}(G \setminus A_1) \not\leq t_{n_1}$: otherwise, by k-triangularity of m_{h_1} and (4) used with $q = 2$, $E_1 = A_1$, $E_2 = G \setminus A_1$, we get

$$m_{h_1}(A_1) \leq m_{h_1}(G) + k\, m_{h_1}(G \setminus A_1) \leq t_{n_1} + k\, t_{n_1} = (k+1)t_{n_1}.$$

It is not difficult to check that either $\{m_h(A \cap A_1)\colon\ A \in \mathcal{L},\ h \in \mathbb{N}\}$, or $\{m_h(A \setminus A_1)\colon\ A \in \mathcal{L},\ h \in \mathbb{N}\}$ (or both, possibly) is not bounded in R: otherwise, if $u_1 = \bigvee\{m_h(A \cap A_1) : A \in \mathcal{L}, h \in \mathbb{N}\}$, $u_2 = \bigvee\{m_h(A \setminus A_1) : A \in \mathcal{L}, h \in \mathbb{N}\}$, then, thanks to triangularity of the m_h's, we have

$$0 \leq m_h(A) \leq m_h(A \cap A_1) + k\, m_h(A \setminus A_1) \leq u_1 + k\, u_2$$

for each $A \in \mathcal{L}$ and $h \in \mathbb{N}$, and hence the set $\{m_h(A)\colon\ A \in \mathcal{L},\ h \in \mathbb{N}\}$ is bounded in R, getting a contradiction. In the first case, set $C_1 := A_1$, otherwise put $C_1 := G \setminus A_1$. Then, set $D_1 := G \setminus C_1$. Now we use the same argument as above, by replacing G by C_1: so we find a set $A_2 \subset C_1$, $A_2 \in \mathcal{L}$ and two integers $n_2 > n_1$, $h_2 > h_1$, with $m_{h_2}(A_2) \not\leq (k+1)t_{n_2}$ and $m_{h_2}(C_1 \setminus A_2) \not\leq t_{n_2}$. Put $C_2 := A_2$ or $C_2 := C_1 \setminus A_2$ according as the $\{m_h(A \cap A_2) : A \in \mathcal{L}, h \in \mathbb{N}\}$ or $\{m_h(A \setminus A_2) : A \in \mathcal{L}, h \in \mathbb{N}\}$ is bounded, set $D_2 := C_1 \setminus C_2$, and let us repeat the same argument as above. Proceeding by induction, we find a disjoint sequence $(D_j)_j$ and two strictly increasing sequences $(n_j)_j$, $(h_j)_j$ in $\mathbb{N}$ with $m_{h_j}(D_j) \not\leq t_{n_j}$ for every $j \in \mathbb{N}$, obtaining a contradiction with 3.5.1). This ends the proof. □

We now turn to our main uniform boundedness theorem for regular and k-triangular lattice group-valued set functions, which extends [11, Theorem 4.6].

Theorem 3.6 *Let* $m_j : \mathcal{L} \to R$*,* $j \in \mathbb{N}$*, be a* (RD)*-regular sequence of* k*-triangular set functions, and suppose that there is an increasing sequence* $(t_n)_n$ *of positive elements of* R *such that for every* $U \in \mathcal{G}$ *the set* $\{m_j(U)\colon j \in \mathbb{N}\}$ *is bounded by* $(t_n)_n$*.*

Then the set $\{m_j(E) : j \in \mathbb{N},\ E \in \mathcal{L}\}$ *is bounded in* R*.*

Proof: Let $(a_{t,l})_{t,l}$ be a (D)-sequence, according to 2.14.1) and 2.14.2), and choose arbitrarily $E \in \mathcal{L}$. By 2.14.1), there is $U \in \mathcal{G}$, $U \supset E$, with $v(m_j)(U \setminus E) \leq \bigvee_{t,l=1}^{\infty} a_{t,l}$ for every $j \in \mathbb{N}$. For each $n \in \mathbb{N}$ put $w_n := t_n + \bigvee_{t,l=1}^{\infty} a_{t,l}$. Taking into account k-triangularity of m_j, in correspondence with U there is $\overline{n} \in \mathbb{N}$ with

$$m_j(E) \leq m_j(U) + k\, v(m_j)(U \setminus E) \leq w_{\overline{n}}, \quad -m_j(E) \leq -m_j(U) + k\, v(m_j)(U \setminus E) \leq w_{\overline{n}}$$

for every $j \in \mathbb{N}$. Thus the set $\{m_j(E) : j \in \mathbb{N}\}$ is bounded by the sequence $(w_n)_n$.

By virtue of Proposition 3.5, it will be enough to prove that, for every disjoint sequence $(H_n)_n$ in $\mathcal{L}$, the set $\{m_j(H_n)\colon j, n \in \mathbb{N}\}$ is bounded by the sequence $(y_n)_n$, where $y_n = k\,n\,w_n$, $n \in \mathbb{N}$.

Proceeding by contradiction, assume that there is a disjoint sequence $(H_n)_n$ in $\mathcal{L}$, such that the set $\{m_j(H_n)\colon j, n \in \mathbb{N}\}$ is not bounded by $(y_n)_n$. For each n there are $i(n)$, $h(n) \in \mathbb{N}$ with

$$m_{h(n)}(H_{i(n)}) \not\leq (k\,n\,+1)w_n. \tag{19}$$

By passing to suitable subsequences, we can assume that

$$m_n(H_n) \not\leq (k\,n\,+1)w_n \text{ for any } n \in \mathbb{N}. \tag{20}$$

By 2.14.2), for each $n \in \mathbb{N}$ there exists a set $O_n \in \mathcal{G}$ with

$$O_n \supset H_n \text{ for each } n \in \mathbb{N} \text{ and } (D) \lim_n v(m_j)\Big(\bigcup_{i=n}^{\infty} O_i\Big) = 0 \text{ for every } j \in \mathbb{N} \tag{21}$$

with respect to $(a_{t,l})_{t,l}$, and hence there is an integer $n_1 > 1$ with $m_1(E) \leq \bigvee_{t,l=1}^{\infty} a_{t,l}$ for every $E \in \mathcal{L}$, $E \subset \bigcup_{i=n_1}^{\infty} O_i$, and a fortiori for each $E \in \mathcal{L}$, $E \subset \bigcup_{i=n_1}^{\infty} H_i$. We get

$$m_1(E \cup H_1) \not\leq w_1 \text{ for each } E \in \mathcal{L}, E \subset \bigcup_{i=n_1}^{\infty} H_i :$$

otherwise, by k-triangularity of m_1 and (4) used with $q = 2$, $E_1 = H_1$, $E_2 = E$, we have

$$m_1(H_1) \leq m_1(E \cup H_1) + k\,m_1(E) \leq w_1 + k \bigvee_{t,l=1}^{\infty} a_{t,l} \leq (k+1)w_1,$$

which contradicts (20). Let $j_2 > n_1$ be an integer such that

$$\bigvee\{m_n(H_1) : n \in \mathbb{N}\} \leq t_{j_2}.$$

By 2.14.2) there is an integer $n_2 > j_2$ such that $m_{j_2}(E) \leq \bigvee_{t,l=1}^{\infty} a_{t,l}$ for any $E \in \mathcal{L}$, $E \subset \bigcup_{i=n_2}^{\infty} H_i$. For such E's we have

$$m_{j_2}(E \cup H_1 \cup H_{j_2}) \not\leq w_{j_2} :$$

otherwise, by k-triangularity of m_{j_2} and (4) used with $q = 3$, $E_1 = H_{j_2}$, $E_2 = E$, $E_3 = H_1$, we get

$$\begin{aligned} m_{j_2}(H_{j_2}) &\leq m_{j_2}(E \cup H_1 \cup H_{j_2}) + m_{j_2}(E) + m_{j_2}(H_1) + m_{j_2}(H_{j_2}) \leq \\ &\leq w_{j_2} + k \bigvee_{t,l=1}^{\infty} a_{t,l} + k\,w_{j_2} \leq 3\,k\,w_{j_2} \leq (k\,j_2 + 1)w_{j_2}, \end{aligned}$$

which contradicts (20). Let $j_3 > n_2$ be an integer such that

$$\bigvee\{m_n(H_{j_2}) : n \in \mathbb{N}\} \leq w_{j_3}.$$

By 2.14.2), in correspondence with m_{j_3} there is $n_3 > j_3$ with $m_{j_3}(E) \leq \bigvee_{t,l=1}^{\infty} a_{t,l}$ for every $E \in \mathcal{L}$, $E \subset \bigcup_{i=n_3}^{\infty} H_i$. For such E's we have

$$m_{j_3}(E \cup H_1 \cup H_{j_2} \cup H_{j_3}) \not\leq w_{j_3} :$$

otherwise, by k-triangularity of m_{j_3} and (4) used with $q = 4$, $E_1 = H_{j_3}$, $E_2 = E$, $E_3 = H_1$, $E_4 = H_{j_2}$, we get

$$\begin{aligned} m_{j_3}(H_{j_3}) &\leq m_{j_3}(E \cup H_1 \cup H_{j_2} \cup H_{j_3}) + m_{j_3}(E) + m_{j_3}(H_1) + \\ &+ m_{j_3}(H_{j_2}) + m_{j_3}(H_{j_3}) \leq w_{j_3} + k \bigvee_{t,l=1}^{\infty} a_{t,l} + k\, w_{j_2} + k\, w_{j_3} \leq \\ &\leq 4\, k\, w_{j_3} \leq (k\, j_3 + 1) w_{j_3}, \end{aligned}$$

which contradicts (20). Proceeding by induction, it is possible to construct two strictly increasing sequences $(j_h)_h$, $(n_h)_h$, such that $n_h > j_h \geq h$ for every $h \in \mathbb{N}$, and

$$m_{j_h}(E \cup H_1 \cup H_{j_2} \cup \ldots \cup H_{j_h}) \not\leq w_{j_h}$$

whenever $h \in \mathbb{N}$ and $E \in \mathcal{L}$ with $E \subset \bigcup_{i=n_h}^{\infty} H_i$.

Set $j_1 = 1$ and $H = \bigcup_{h=1}^{\infty} H_{j_h}$. Note that $H \in \mathcal{G}$ and $m_{j_h}(H) \not\leq w_{j_h}$ for every $h \in \mathbb{N}$. But the set $\{m_h(H) : h \in \mathbb{N}\}$ is bounded by the sequence $(w_n)_n$, and so we get a contradiction. This ends the proof. $\square$

We now give an example of (RD)-regular sequence.

Example 3.7 Let $R = L^0 = L^0([0,1], \mathcal{B}, \lambda)$ be as in Remark 2.2, G be a compact Hausdorff topological space, $\mathcal{L}$ be the σ-algebra of all Borel subsets of G, $\mathcal{G}$ and $\mathcal{H}$ be the classes of all open and of all compact subsets of G, respectively. First of all, observe that 2.9.2) is a consequence of 2.9.1). Indeed, pick arbitrarily $W \in \mathcal{H}$ and let $(V_n)_n$ be a sequence of elements of $\mathcal{G}$, satisfying 2.9.1). Since G is compact and Hausdorff, G is also normal (see also [35, Theorem XI.1.2]). As G is normal, thanks to [35, Proposition VII.3.2], in correspondence with W and V_1 there is a set $U_1 \in \mathcal{G}$ with $W \subset U_1 \subset \overline{U_1} \subset V_1$, where $\overline{U_1}$ denotes the topological closure of U_1 in G. Analogously, we can associate to W and $U_1 \cap V_2$ a set $U_2 \in \mathcal{G}$ with $W \subset U_2 \subset \overline{U_2} \subset U_1 \cap V_2$. Proceeding by induction, we construct a decreasing sequence $(U_n)_n$ in $\mathcal{G}$, with $W \subset U_{n+1} \subset \overline{U_{n+1}} \subset U_n \cap V_{n+1}$. Since the sequence $(V_n)_n$ satisfies 2.9.1), it is not difficult to see that the sequences $(U_n)_n$ and $(\overline{U_n})_n$ fulfil 2.9.2).

Let $m_j : \mathcal{L} \to R$, $j \in \mathbb{N}$, be a sequence of k-triangular and regular set functions. We will prove that $(m_j)_j$ satisfies 2.14.1) and 2.14.2). Since in L^0 the (r)-, (O)- and (D)-convergences coincide (see Remark 2.2), then for every $j \in \mathbb{N}$ there exists $u_j \in R$, $u_j \geq 0$, such that for every $E \in \mathcal{L}$ there are two sequences $(V_n^{(j)})_n$ in $\mathcal{G}$ and $(K_n^{(j)})_n$ in $\mathcal{H}$, with $V_n^{(j)} \supset E \supset K_n^{(j)}$ for each n and such that for every $\varepsilon > 0$ there is a positive integer $n_0 = n_0(\varepsilon, j, E)$ with

$$v(m_j)(V_n^{(j)} \setminus K_n^{(j)}) \leq \varepsilon\, u_j \quad \text{whenever } n \geq n_0. \tag{22}$$

For every $n \in \mathbb{N}$, set $V_n := \bigcap_{j=1}^{n} V_n^{(j)}$, $K_n := \bigcup_{j=1}^{n} K_n^{(j)}$: note that $V_n \in \mathcal{G}$, $K_n \in \mathcal{H}$ and $V_n \supset E \supset K_n$ for every n. Since R satisfies property (σ), in correspondence with the sequence $(u_j)_j$ there exist a sequence $(a_j)_j$ of positive real numbers and an element $u \in R$, $u \geq 0$, with $0 \leq a_j\, u_j \leq u$ for every $j \in \mathbb{N}$. Note that u does not depend on the choice of $E \in \mathcal{L}$. For every $\varepsilon > 0$, $j \in \mathbb{N}$ and $E \in \mathcal{L}$, let $n_* = n_*(\varepsilon, j, E) = n_0(\varepsilon\, a_j, j, E)$, where n_0 is as in (22). We get

$$v(m_j)(V_n \setminus K_n) \leq v(m_j)(V_n^{(j)} \setminus K_n^{(j)}) \leq \varepsilon\, a_j\, u_j \leq \varepsilon\, u \tag{23}$$

for each $n \geq n_*$. If we take $\sigma_p = \frac{1}{p} u$, $p \in \mathbb{N}$, then it is not difficult to check that 2.14.1) is satisfied.

We now prove 2.14.2). Choose any disjoint sequence $(H_n)_n$ in $\mathcal{L}$ and let u be as in (23). In correspondence with j, $n \in \mathbb{N}$ and $\frac{1}{k\, 2^{n+j+1}}$ set $O_n^{(j)} = O_n^{(j)}\Big(\frac{1}{k\, 2^{n+j+1}}\Big) = V_{n_*(\frac{1}{k\, 2^{n+j+1}}, j, H_n)}$ and $F_n^{(j)} = F_n^{(j)}\Big(\frac{1}{k\, 2^{n+j+1}}\Big) = K_{n_*(\frac{1}{k\, 2^{n+j+1}}, j, H_n)}$, where n_* is as in (23). For each $n \in \mathbb{N}$, put $O_n = \bigcap_{j=1}^{n} O_n^{(j)}$ and $F_n = \bigcup_{j=1}^{n} F_n^{(j)}$. Note that $O_n \in \mathcal{G}$, $F_n \in \mathcal{H}$ and $O_n \supset H_n \supset F_n$ for each n. Moreover, from (23) we get

$$v(m_j)(O_n \setminus F_n) \leq v(m_j)(O_n^{(j)} \setminus F_n^{(j)}) \leq \frac{1}{k\, 2^{n+j+1}}\, u \quad \text{for every } j, n \in \mathbb{N}. \tag{24}$$

Now, for each $n \in \mathbb{N}$ set $U_n := \bigcup_{i=n}^{\infty} O_i$, $C_n := \bigcap_{i=n}^{\infty} F_i$. Since the sequence $(H_n)_n$ is disjoint and $F_n \subset H_n$ for every $n \in \mathbb{N}$, then $C_n = \emptyset$ for every $n \in \mathbb{N}$. Taking into account (7), from (24) we get

$$\begin{aligned} v(m_j)(U_n) &= v(m_j)(U_n \setminus C_n) = v(m_j)\Big(\Big(\bigcup_{i=n}^{\infty} O_i\Big) \setminus \Big(\Big(\bigcap_{i=n}^{\infty} F_i\Big)\Big) = \\ &= v(m_j)\Big(\bigcup_{i=n}^{\infty} (O_i \setminus F_i)\Big) \leq k \sum_{i=n}^{\infty} v(m_j)(O_i \setminus F_i) \leq k \sum_{i=n}^{\infty} \frac{1}{k\, 2^{i+j+1}}\, u = \frac{1}{2^{n+j}}\, u \end{aligned} \tag{25}$$

(see also [38, Lemma 1]). Thus 2.14.2) is proved. □

The following example shows that, in Theorem 3.5, in general the condition 3.5.1) cannot be replaced by the boundedness of the set $\{m_j(U) : j \in \mathbb{N}\}$.

Example 3.8 (see also [45, Example 5]) Let R be the vector lattice c_0 of all real sequences convergent to 0, endowed with the usual ordering, $\mathcal{B}$ be the σ-algebra of all Borel subsets of $[0, 1]$. Note that c_0 is Dedekind complete and weakly σ-distributive, and that in c_0 order, (D)- and (r)-convergence coincide with coordinatewise convergence dominated by an element of c_0 (see also [28, 45, 47]). For every $n \in \mathbb{N}$ and $E \in \mathcal{B}$ set $m_n(E) = (\mu_1(E), \ldots, \mu_n(E), 0, \ldots, 0, \ldots)$, where $\mu_n(E) = \int_E \sin(n\,\pi\,x)\,dx$. It is known (see [45]) that every m_n is a σ-additive measure and the set $\{m_n(E) : n \in \mathbb{N}\}$ is bounded in c_0 for every $E \in \mathcal{B}$. However, it is not possible to find a positive increasing sequence $(t_n)_n$ satisfying the hypothesis of Theorem 3.6, since $\sup\{\mu_n(A) : A \in \mathcal{B}\} = 1$ for each n. Moreover, from this it follows that the set $\{m_n(E) : n \in \mathbb{N},\, E \in \mathcal{B}\}$ is not bounded in c_0.

Open problems: (a) Prove similar results with respect to other kinds of (s)-boundedness, boundedness and/or convergence, and relatively to different types of variations in the setting of non-additive lattice-group valued set functions (see also [22, 40]).

(b) Find some other conditions under which 2.14.1) and/or 2.14.2) hold.

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