On the Pythagoras’ and De Gua’s theorems in geometric algebra

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This small article is intended to be a contribution to the LinkedIn group “Pre-University Geometric Algebra”. The main idea is to show that in geometric algebra we have the Pythagoras’ and De Gua’s theorems without a metric defined. This allows us to generalize these theorems to any dimension and any signature.

Keywords: the Pythagoras’ theorem, De Gua’s theorem, geometric algebra, metric, bivector

The geometric product

In geometric algebra, we define a non-commutative product of two vectors with the properties of associativity and distributivity, which can be decomposed into the symmetric and anti-symmetric parts

\[ ab = \frac{ab + ba}{2} + \frac{ab - ba}{2} = S + A , \]

where we can define that vectors are orthogonal if

\[ S = \frac{ab + ba}{2} = 0 \Rightarrow ab = -ba , \]

which means that orthogonal vectors anti-commute. Likewise, we can define that vectors are parallel if

\[ A = \frac{ab - ba}{2} = 0 \Rightarrow ab = ba , \]
which means that parallel vectors commute. These definitions are in accordance with the usual definitions in algebras. For example, we could define that two vectors $a$ and $b$ are parallel if $a = \lambda b$, where $\lambda$ is a real number, but it is obvious that these vectors commute in geometric algebra, since real numbers commute with vectors.

Now we can show that products $a^2 = aa$ commute with all vectors. One can say that this is obvious, since $a^2$ is a real (or a complex) number. However, we do not need such an interpretation (that is, we do not need to introduce a metric, yet). Obviously, $a^2$ commutes with the vector $a$. Consider a vector $b$, which is orthogonal to the vector $a$. Then we have

$$a^2 b = aab = -aba = baa = ba^2,$$

which means that the commutativity here follows from the geometric product properties. Now we can show that this means that $a^2$ commutes with all vectors, but the pleasure is left to the reader.

**Orthogonal vectors**

Consider two orthogonal vectors in any dimension and of any signature. We have

$$(a + b)^2 = a^2 + ab + ba + b^2 = a^2 + ab - ab + b^2 = a^2 + b^2,$$

which means that the Pythagoras’ theorem is valid. Let us look at two 2D examples

$${\mathbb R}^2: \quad e_1^2 = e_2^2 = 1 \Rightarrow (e_1 + e_2)^2 = 1 + 1 + e_1 e_2 + e_2 e_1 = 2 = e_1^2 + e_2^2,$$

$${\mathbb R}^{1,1}: \quad e_1^2 = -e_2^2 = 1 \Rightarrow (e_1 + e_2)^2 = 1 - 1 + e_1 e_2 + e_2 e_1 = 0 = e_1^2 + e_2^2.$$

Note that the commutativity properties of geometric product play a central role here. Simply stated, with the geometric product we have the Pythagoras’ theorem in any vector space we can imagine. Moreover, we have this important result without definition of a metric.

**De Gua's theorem**

Now we can show how to get De Gua’s theorem easily. First, note that the anti-symmetric part of geometric product of two vectors is a bivector, which we can write as

$$A = \frac{ab - ba}{2} \equiv a \wedge b,$$
where \( \wedge \) stands for the outer (wedge) product. It is not difficult to show that the magnitude of a bivector is proportional to the area of the parallelogram defined by the vectors \( a \) and \( b \). Namely, decomposing the vector \( b \) into the prats parallel and orthogonal to the vector \( a \), we can write

\[
A = a \wedge b = a \wedge (b_\parallel + b_\perp) = a \wedge b_\perp = ab_\perp,
\]

whence, using \( |b_\perp| = |b||\sin \alpha| \), we get the parallelogram area formula. Defining the reverse involution

\[
A^\dagger = b_\perp a,
\]

we have

\[
AA^\dagger = ab_\perp b_\perp a = a^2 b_\perp^2,
\]

which we can interpret as the square of the area of the parallelogram defined by the vectors \( a \) and \( b \), but we have to define the square of a vector to be a positive real number (metric) first. Here, we will proceed without a metric, in order to get formulae that are more general.

Consider three orthogonal vectors \( a, \) \( b, \) \( \) and \( c \) (F.1) with the initial point \( O \), whose end points span a triangle. We can write

\[
\begin{align*}
a + d_1 - b &= 0, \\
b + d_2 - c &= 0, \\
c + d_3 - a &= 0,
\end{align*}
\]

whence follows that \( d_1 + d_2 + d_3 = 0 \). Now we can define the bivector \( B = d_1 \wedge d_2 \) whose magnitude is double of the red triangle area. Therefore, \( BB^\dagger / 4 \) gives the squared area of the red triangle. Ignoring the factor 4, we can calculate

\[
B = d_1 \wedge d_2 = (b - a) \wedge (c - b) = b \wedge c + c \wedge a - b \wedge b + a \wedge b = bc + ca + ab,
\]

whence follows that

\[
BB^\dagger = (bc + ca + ab)(cb + ac + ba) = \cdots = a^2 b^2 + a^2 c^2 + b^2 c^2.
\]

The details of the calculation are left to the reader; however, note that the result follows from the fact that orthogonal vectors anti-commute.

Finally, there are two important facts that we should stress here. First, note that the result is independent of a signature. Second, generalizations to higher dimension are straightforward; however, we should formulate a problem in terms of hyper-volumes.
Literature


[2] Josipović, Miroslav: *Geometric Multiplication of Vectors - An Introduction to Geometric Algebra in Physics*, Birkhäuser, 2019 (it is to be printed soon)