

# Deng Entropy in Thermodynamics

Fan Liu<sup>a</sup>, Yong Deng<sup>b,\*</sup>

<sup>a</sup>*Yingcai Honors of School & School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, 610054, China*

<sup>b</sup>*Institute of Fundamental and Frontier Science, University of Electronic Science and Technology of China, Chengdu, 610054, China*

---

## Abstract

Entropy as a measure of disorder can be widely used in many fields. In this paper, based on the localized system, a new entropy Deng entropy is derived assuming that the particles have superposition states.

*Keywords:* Deng entropy

---

## 1. Introduction

The entropy in chemistry and thermodynamics is a measure of the total amount of energy that does not work in terms of kinetics. When the overall entropy increases, its ability to do work also decreases, and the measure of entropy is the indicator of energy degradation. Entropy can also be used to measure the degree of system chaos. It is a function that describes the state of the system. It has important applications in probability theory, astrophysics, life sciences, etc.

The concept of entropy was proposed by the German physicist Rudolf Clausius [1]. Later, Boltzmann discovered that the entropy in a single system is more related to the number of thermodynamic microstates. According to Boltzmann's definition, entropy is a function of state.

In this paper, based on the localized system, a new entropy Deng entropy [2] is derived assuming that the particles have superposition states.

---

\*Institute of Fundamental and Frontier Science, University of Electronic Science and Technology of China, Chengdu, 610054, China

*Email address:* dengentropy@uestc.edu.cn, prof.deng@hotmail.com (Yong Deng)

This paper can be divided into the following sections: some simple preliminaries are introduced in section 2, section 3 shows the whole process of derivation, and section 4 simply discusses some properties of Deng entropy, and the final conclusion is given.

## 2. Preliminaries

In this section, some preliminaries are briefly introduced, including: evidence theory [3, 4], Deng Entropy [5] Entropy energy level .

Dempster-shafer evidence theory was firstly introduced by Dempster [3] and had been developed by shafer [4]. D-S theory is abroad applied in many fields of data fusion. Here we give some basic definitions.

Let  $\Omega$  be a countable set of  $n$  elements called the frame of discernment which denote a finite nonempty set of mutually exclusive and exhaustive hypotheses. Assume that it contains  $\Omega = \{A_1, A_2, A_3, \dots, A_n\}$ . The power set of  $\Omega$  which we denote with  $P(\Omega)$  contains all the possible subsets of it, which is composed of  $2^n$  elements of  $\Omega$ . Each element of  $2^n$  represents a proposition.

**Definition 1** A basic probability assignment (BPA) is a function. The range is from  $P(\Omega)$  to  $[0, 1]$ , which is defined by [3]

$$m : P(\Omega) \mapsto [0, 1], A \mapsto m(A) \quad (1)$$

and it should satisfy the following conditions:

$$\sum_{A \in P(\Omega)} m(A) = 1, m(\Phi) = 0 \quad (2)$$

The mass  $m(A)$  indicates the strength of the evidences support for  $A$ , while  $m(\Omega)$  is represented as the uncertainty of evidence. If  $m(\Phi) = 0$ , any useful information can't be got from the evidence.

**Definition 2** the Deng entropy [2] is defined as follows:

$$E_d = \sum_i m(A_i) \log \frac{m(A_i)}{2^{|A_i|} - 1} \quad (3)$$

where  $A_i$  is a proposition of BPA, and  $|A_i|$  is the cordiality of  $A_i$ . the Deng entropy is the generalization of Shannon entropy.

**Definition 3** Entropy was first proposed by Rudolf Clausius. It can be expressed as follows:

$$S = \frac{dQ}{T} \quad (4)$$

Where  $S$  is the entropy,  $dQ$  is the change in heat in the system, and  $T$  is the temperature of the system.

**Definition 4** In quantum mechanics, the energy of a particle is quantized, and particles in different states have different energies. So the quantized energy is called the energy level :  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_l, \dots$

### 3. Physical derivation

In this session, it is assumed that particles of different states can coexist at the same time, and a new entropy is derived.

In the Boltzmann system, the number of microscopic states of a particle can be expressed as :

$$\Omega_{M.B} = \frac{N!}{\prod_l a_l!} \prod_l (\omega_l)^{a_l} \quad (5)$$

Where  $\Omega_{M.B}$  is the total number of microscopic states of the particle,  $a_l$  is the number of particles at the energy level  $\varepsilon_l$ , and  $\omega_l$  is the degeneracy at the energy level  $\varepsilon_l$ . When the particles have no coexisting state, the degeneracy is  $N_l$ , and in the coexisting state, the degeneracy of the particles is  $2^{N_l} - 1$ , so the microscopic state of the particles can be expressed as:

$$\Omega_{M.B} = \frac{N!}{\prod_l a_l!} \prod_l (2^{N_l} - 1)^{a_l} \quad (6)$$

For the Boltzmann distribution

$$\Omega_{M.B} = \frac{N!}{\prod_l a_l!} \prod_l (2^{N_l} - 1)^{a_l} \quad (7)$$

The logarithm of both sides has

$$\ln \Omega = \ln N! - \sum_l \ln a_l! + \sum_l a_l \ln(2^{N_l} - 1) \quad (8)$$

Assume that a particle beam is very much in the system. Eq. 8 can be simplified to

$$\begin{aligned}
\ln\Omega &= N(\ln N - 1) - \sum_l a_l(\ln a_l - 1) + \sum_l a_l \ln(2^{N_l} - 1) \\
&= N \ln N - \sum_l a_l \ln a_l + \sum_l a_l \ln(2^{N_l} - 1)
\end{aligned} \tag{9}$$

Suppose that  $a_l$  has  $\delta a_l$  Then  $\ln \Omega$  has the following changes

$$\delta \ln \Omega = - \sum_l \ln\left(\frac{a_l}{2^{N_l} - 1}\right) \delta a_l \tag{10}$$

In order to make  $\ln \Omega$  have large distribution, let  $\delta \ln \Omega$  be equal to 0, then there is

$$\delta \ln \Omega = - \sum_l \ln\left(\frac{a_l}{2^{N_l} - 1}\right) \delta a_l = 0 \tag{11}$$

But these  $\delta a_l$  are not independent, they must meet the following conditions

$$\delta N = \sum_l \delta a_l = 0 \quad \delta E = \sum_l \varepsilon_l \delta a_l = 0 \tag{12}$$

In the case where the constraint *Eq. 12* is satisfied, *Eq. 10* is equivalent to the following *Eq. 13* regardless of the values of the parameters  $\alpha$  and  $\beta$ .

$$- \sum_l \left( \ln\left(\frac{a_l}{2^{N_l} - 1}\right) + \alpha + \beta \varepsilon_l \right) \delta a_l = 0 \tag{13}$$

Under the conditions given by *Eq. 12*, it makes the two  $\delta a_l$  cannot take any value. It is possible to set the two to determine  $\delta a_1$  and  $\delta a_2$ . Then  $\alpha$  and  $\beta$  can be determined by the following equation.

$$\ln \frac{a_1}{2^{N_1} - 1} + \alpha + \beta \varepsilon_1 \quad \ln \frac{a_2}{2^{N_2} - 1} + \alpha + \beta \varepsilon_2 \tag{14}$$

Reduce *Eq. 13* to

$$\sum_{l=3} \left( \ln\left(\frac{a_l}{2^{N_l} - 1}\right) + \alpha + \beta \varepsilon_l \right) \delta a_l = 0 \tag{15}$$

Since each  $\delta a_l$  in *Eq. 15* can take values independently, the coefficient of  $\delta a_l$  in *Eq. 15* is required to be equal to 0, that is :

$$\sum_{l=3} \left( \ln\left(\frac{a_l}{2^{N_l} - 1}\right) + \alpha + \beta \varepsilon_l \right) = 0 \quad l = 1, 2, 3, \dots \tag{16}$$

*Eq. 17* can be derived from *Eq. 16* and *14*.

$$a_l = (2^N - 1)e^{-\alpha - \beta \varepsilon_l} \quad (17)$$

In summary, the total number of particles is:

$$N = \sum_l (2^{N_l} - 1)e^{-\alpha - \beta \varepsilon_l} \quad (18)$$

Internal energy is the statistical average of the total energy of the irregular motion of particles in the system. So internal energy can be expressed as

$$U = \sum_l a_l \varepsilon_l = \sum_l \varepsilon_l (2^N - 1)e^{-\alpha - \beta \varepsilon_l} \quad (19)$$

Introduce a partition function whose expression is as follows:

$$Z = \sum_l (2^N - 1)e^{-\beta \varepsilon_l} \quad (20)$$

*Eq. 21* can be derived from *Eq. 18* and *20*.

$$N = e^{-\alpha} Z \quad (21)$$

Substituting *Eq. 21* into *Eq. 19*, eliminating  $\alpha$ , *Eq. 22* can be obtained as follows:

$$U = -N \frac{\partial}{\partial \beta} \ln Z \quad (22)$$

The change of the system's internal energy  $dU$  in the process is equal to the sum of the work  $dW$  done by the outside world on the system and the heat  $dQ$  absorbed by the system from the outside.

$$dU = dQ + dW \quad (23)$$

The force applied to a particle at the entropy of the energy level  $\varepsilon_l$  is  $\frac{\partial \varepsilon_l}{\partial y}$ . Therefore, the generalized force exerted by the outside world on the system

is:

$$\begin{aligned}
Y &= \sum_l \frac{\partial \varepsilon_l}{\partial y} a_l \\
&= \sum_l \frac{\partial \varepsilon_l}{\partial y} (2^N - 1) e^{-\alpha - \beta \varepsilon_l} \\
&= e^{-\alpha} \left( -\frac{1}{\beta} \frac{\partial}{\partial y} \right) \sum_l \frac{\partial \varepsilon_l}{\partial y} (2^N - 1) e^{-\beta \varepsilon_l} \\
&= \frac{N}{Z} \left( -\frac{1}{\beta} \frac{\partial}{\partial y} \right) Z \\
&= -\frac{N}{\beta} \frac{\partial}{\partial y} \ln Z
\end{aligned} \tag{24}$$

Where  $Y$  is the generalized force of the external system corresponding to the external parameter  $y$ .

The heat absorbed by the system from the outside during the process is related to the process, so  $dQ$  is not a full differential but an infinitesimal amount. It is proved by the second law of thermodynamics that  $dQ$  has integral factor  $\frac{1}{T}$ , and multiplying  $\frac{1}{T}$  by  $dQ$  gives the complete differential  $dS$ :

$$\frac{1}{T} dQ = \frac{1}{T} (dU - Y dy) = dS \tag{25}$$

Eq. 26 can be derived from Eq. 22, Eq. 22 and Eq. 25.

$$\begin{aligned}
dQ &= dU - Y dy \\
&= -N d\left(\frac{\partial \ln Z}{\partial \beta}\right) + \frac{N}{\beta} \frac{\partial \ln Z}{\partial y} dy
\end{aligned} \tag{26}$$

Eq. 26 is simultaneously multiplied by  $\beta$  :

$$\beta(dU - Y dy) = -N\beta \left(\frac{\partial Z}{\partial \beta}\right) + N \frac{\partial \ln Z}{\partial y} dy \tag{27}$$

Because the partition function  $Z$  is a function of  $\beta$  and  $y$ , the full differential of  $\ln Z$  :

$$d \ln Z = \frac{\partial \ln Z}{\partial \beta} d\beta + \frac{\partial \ln Z}{\partial y} dy \tag{28}$$

Eq. 29 can be derived from Eq. 27 and Eq. 28.

$$\beta(dU - Ydy) = Nd(\ln Z - \beta \frac{\partial}{\partial \beta} \ln Z) \quad (29)$$

Then the expression of  $S$  can be derived from Eq. 4 and 29. Let  $\beta = \frac{1}{kT}$

$$S = Nk(\ln Z - \beta \frac{\partial}{\partial \beta} \ln Z) \quad (30)$$

Suppose the probability that the particle is on the quantum state  $l$  is  $P_l$

$$P_l = \frac{(2^N - 1)e^{-\alpha - \beta \epsilon_l}}{N} \quad (31)$$

Obviously  $P_l$  satisfies the normalization condition.

$$\sum_l P_l = 1 \quad (32)$$

Eq. 33 can be derived from Eq. 30 :

$$\begin{aligned} S &= Nk(\ln Z - \beta \frac{\partial}{\partial \beta} \ln Z) \\ &= Nk(\ln Z + \beta \bar{\epsilon}) \end{aligned} \quad (33)$$

Where  $\bar{\epsilon}$  is the average energy of the localized system , which can be expressed as :

$$\bar{\epsilon} = \sum_l P_l \epsilon_l \quad (34)$$

Bring Eq. 34 into Eq. 33 :

$$\begin{aligned} S &= Nk(\ln Z + \beta \bar{\epsilon}) \\ &= NkP_l \sum_l (\ln Z + \beta \epsilon_l) \end{aligned} \quad (35)$$

Eq. 36 can be derived from Eq. 31.

$$- \ln\left(\frac{P_l}{2^N - 1}\right) = \beta \epsilon_l + \ln Z \quad (36)$$

Bring the Eq. 36 into the Eq. 35

$$S = -Nk \sum_l P_l \ln\left(\frac{P_l}{2^N - 1}\right) \quad (37)$$

In the Dempster-Shafer evidence theory,  $S$  is called entropy which can be regarded as the promotion of Shannon entropy.

#### 4. Discussion

In this section, Deng Entropy will be discussed briefly. Dempster-Shafer(D-S) evidence theory is widely used in many theories, Deng Entropy as a tool to measure basic probability assessment(BPA) in D-S evidence theory [3, 4]. Note that when BPA degenerates into probability, Deng entropy [2] degenerates into Shannon entropy [5].

In order to further demonstrate Deng Entropy, a concrete example is shown. Give a framework of discernment  $X = \{a, b\}$ , for a mass function  $m(a) = m(b) = \frac{1}{2}$ , so the corresponding Shannon entropy  $H$  and Deng entropy  $E_d$  are:

$$H = -\frac{1}{2} \times \log_2 \frac{1}{2} - \frac{1}{2} \times \log_2 \frac{1}{2} = 1$$

$$E_d = -\frac{1}{2} \times \log_2 \frac{1}{2} - \frac{1}{2} \times \log_2 \frac{1}{2} = 1$$

It can be clearly seen that when the belief is assigned to a single element, the results of Deng entropy and Shannon entropy are consistent.

#### 5. Conclusion

In this paper, based on the localized system, a new entropy Deng entropy is derived assuming that the particles have superposition states.

- [1] R. Clausius, On the motive power of heat, and on the laws which can be deduced from it for the theory of heat, Poggendorff's Annalen der Physick (1850).
- [2] Y. Deng, Deng entropy, Chaos, Solitons 91 (2016) 549 – 553.
- [3] A. P. Dempster, Upper and lower probabilities induced by a multivalued mapping, The annals of mathematical statistics (1967) 325–339.
- [4] G. Shafer, et al., A mathematical theory of evidence, volume 1, Princeton university press Princeton, 1976.
- [5] C. E. Shannon, A mathematical theory of communication, SIGMOBILE Mob. Comput. Commun. Rev. 5 (2001) 3–55.