A Poincaré - conformal matrix Lie algebra

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Abstract

The Poincaré group of spacetime rotations and spacetime translations has been fundamental for over a century. Also a century old are efforts to find alternatives, efforts that include invoking the larger symmetry group of Maxwell's electrodynamics, the conformal group. In this paper an 8×8 matrix representation of the Poincaré group is enhanced by defining a 4×4 matrix rep of the conformal group that acts on 4 of the 8 dimensions, a 4-spinor subset of 8-spinors. The matrix generators are described in detail and the commutation relations of the Lie algebra are displayed. There are additional generators needed to keep the enhanced algebra closed. The new generators add new transformations making a group larger than the direct product of the Poincaré and conformal groups.

Keywords: Lie Algebra; Poincaré group; conformal group; spin 1/2

1 Introduction

The Poincaré and conformal groups have been well-studied, with a wide range of applications in, for example, quantum field theory,[1] graphene,[2, 3] and theories of gravitation.[4, 5] Here, the two groups combine in something like a direct product, but with some mixing due to some of the transformations in one group not commuting with all the transformations in the other.

One can define a matrix Lie algebra by providing a suitable set of matrices. The commutators of the matrices must be expressible as linear combinations of the matrices. One

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standard example is the 4×4 matrix representations of the Lorentz group of spacetime rotations that, among other uses, describes 4-spinor Dirac reps of massive spin 1/2 fermions such as the electron in relativistic wave mechanics. The Lorentz group has 6 linearly independent generators in the set of sixteen angular momentum matrices $J^{\mu\nu} = -i\left(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}\right)/4$, made with Dirac γ-matrices and $\mu, \nu \in \{1,2,3,4\} = \{x,y,z,t\}$, Minkowski coordinates.

Just as angular momentum matrices generate a representation of the group of spacetime rotations, there are linear momentum, often shortened to just 'momentum', matrices that generate reps of spacetime translations. Naively, since both translations and rotations are fundamental spacetime symmetries, one expects that matrix reps of translations would be just as helpful in formulating quantum mechanics as the spin matrices that represent the group of spacetime rotations.

The matrix representations of the Poincaré group are found by, for example, seeking matrix solutions of the commutation relations of the Poincaré algebra.[6, 7] For spin 1/2, the process of representing translations increases the component count from 4-spinors, needed for rotations with parity invariance, to 8-spinors. By similarity transformations, one can arrange the eight components so there are two 4-spinors, each with its own standard 4-spinor representation of the Lorentz group. The two 4-spinors do not mix for spacetime rotations.

With translations, the first 4-spinor, the "donor", donates a linear combination of its components to the second 4-spinor, the "receiver." In general, the essence of translation generators, the momentum matrices, is to acquire linear combinations of one set of spinor components and deposit those combinations with a second set of components.

In this paper the receiver 4-spinor transforms also with the conformal group. There is just one matrix rep of the conformal group,[8, 9] the 4-spinor rep, within similarity transformations. That rep is applied to the receiver 4-spinor.

Since, for Lorentz rotation/boost transformations, the receiver 4-spinor is independent of the donor 4-spinor, the conformal group of the receiver mixes only with the donor via translations. The mixing introduces new transformations.

Counting generators, there are 10 generators needed for the 8-spinor Poincaré group and 15 generators of the 4-spinor conformal group. The 8-spinor Poincaré group representation has an additional 4 momentum generators since, in general, there are two linearly independent, commuting representations of translations. Closure, upon mixing the translations and conformal transformations, requires an additional 8 generators, making a group with 37 generators. This is well short of the limit of 64 generators allowed for 8×8 matrices.

For convenience, the Appendix has a Mathematica notebook[10, 11] that verifies the calculations. The Appendix will be jettisoned when the article is published.

The matrix representation is detailed in Sec. 2 and the commutation relations of the corresponding Lie algebra are given in Sec. 3. The algebra has three Poincaré subalgebras and a conformal subalgebra, as discussed in Sec. 4. Sec. 5 finishes with some concluding remarks.

2 Generators

Formulas to construct the 8×8 matrix generators are presented in this section. First, let us recall briefly some of the concepts.[12, 13, 14]

A matrix Lie algebra has a set of, say N, linearly independent matrices $\{X_1, X_2, ..., X_N\}$ called "generators." Two matrices M_a and M_b determine the commutator, $[M_a, M_b]$, which is the "product" operation in the algebra. In a Lie algebra, the matrix commutators of generators are expressible as linear combinations of the generators,

$$[X_a, X_b] \equiv X_a \cdot X_b - X_b \cdot X_a = \sum_{c=1}^{N} i s_{abc} X_c \ , \tag{1}$$

where the centerdot "$\cdot$" denotes matrix multiplication. The coefficients s_{abc} are called the "structure constants" and the indices range over generators $a, b, c \in \{1, 2, ..., N\}$. Each generator, say X, generates a transformation $T(\theta) = \exp(\pm i X \theta)$, where θ is a real valued parameter. The sign is conventionally negative when a momentum generates a translation and positive otherwise.

In this paper, the Lie algebra has $N = 37$ generators. Many of the generators are not Hermitian, $X \neq X^{\dagger}$ its complex transpose, so the group they generate is non-unitary, since $TT^{\dagger} \neq 1$ for some of the transformations T. The transformations are generated by 8×8 matrices with complex components and act on 8-component quantities called "8-spinors".

The generators can be sorted by the number of indices they have. Quantities with two indices are "tensors", one index indicates "vectors" and "scalars" have no index. The labels refer to the behavior of the quantities under the set of Lorentz transformations generated by the angular momentum matrices called "$J_8^{\mu\nu}$" below.

The tensor matrices are $J_8^{\mu\nu}$, $J_R^{\mu\nu}$, $J_{31}^{\mu\nu}$, and $J_{42}^{\mu\nu}$. The vector matrices are called P_{41}^{μ}, P_{32}^{μ}, P_{43}^{μ}, P_{34}^{μ}. Finally, there are the scalar matrices D, D_{31}, and D_{42}. Indices $\mu, \nu, ... \in \{x, y, z, t\}$ $= \{1, 2, 3, 4\}$, while $i, j, ... \in \{x, y, z\} = \{1, 2, 3\}$.

The $J_8^{\mu\nu}$ are given the subscript "8" because the generators act nontrivially on all 8 components of the 8-spinor. The generators $J_R^{\mu\nu}$, "R" for 'Receiver,' act nontrivially only on the second four components of the 8-spinor, which is the 'receiver' 4-spinor.

An 8×8 matrix generator can be divided into a 4×4 array of 2×2 matrix blocks. The subscript labels in, for example, the momentum matrices P_{41}^{μ} and P_{32}^{μ} designate 2×2 blocks of the 8×8 matrices with 41 the first 2×2 block in the fourth row and 32 the second block in the third row, respectively.

While the labels, such as "8," "41,", "34," of the generators are tied to the representation constructed here, the algebra may have other representations for which the labels do not have any such meaning.

It happens that each nonzero 2×2 block of the generators is a multiple of one of the four Pauli matrices σ^μ,

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \; \sigma^y = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \; ; \; \sigma^z = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \; ; \; \sigma^t = \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix} \quad , \tag{2}$$

The spacetime metric $\eta^{\mu\nu}$ has signature $+2$, $\eta^{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}\{+1,+1,+1,-1\}$. The metric is used to "raise" and "lower" indices. For example $\sigma^\mu = \eta^{\mu\nu}\sigma_\nu$. The summation convention for repeated indices is assumed.

The generators are grouped as tensors, vectors and scalars:

Nonzero 2×2 blocks of the tensors J:

$$\left(J_8^{ij}\right)_{11} = \left(J_8^{ij}\right)_{22} = \left(J_8^{ij}\right)_{33} = \left(J_8^{ij}\right)_{44} = \frac{1}{2}\epsilon^{ijk}\sigma^k \tag{3}$$

$$\left(J_8^{k4}\right)_{11} = -\left(J_8^{k4}\right)_{22} = \left(J_8^{k4}\right)_{33} = -\left(J_8^{k4}\right)_{44} = \frac{i}{2}\sigma^k$$

$$\left(J_R^{ij}\right)_{33} = \left(J_R^{ij}\right)_{44} = \frac{1}{2}\epsilon^{ijk}\sigma^k \tag{4}$$

$$\left(J_R^{k4}\right)_{33} = -\left(J_R^{k4}\right)_{44} = \frac{i}{2}\sigma^k$$

$$\left(J_{31}^{ij}\right)_{31} = -\frac{1}{2}\epsilon^{ijk}\sigma^k \quad ; \quad \left(J_{31}^{k4}\right)_{31} = -\frac{i}{2}\,\sigma^k \tag{5}$$

$$\left(J_{42}^{ij}\right)_{42} = -\frac{1}{2}\epsilon^{ijk}\sigma^k \quad ; \quad \left(J_{42}^{k4}\right)_{42} = +\frac{i}{2}\,\sigma^k \tag{6}$$

The Js are antisymmetric, meaning that, for $J^{\mu\nu} = J_8^{\mu\nu}$, $J_R^{\mu\nu}$, $J_{31}^{\mu\nu}$, and $J_{42}^{\mu\nu}$, we have $J^{\nu\mu} = -J^{\mu\nu}$. Thus $J_8^{\mu\nu}$ and $J_R^{\mu\nu}$ each has 6 linearly independent generators, one for each of the 6 pairs $\mu\nu \in \{12, 13, 14, 23, 24, 34\}$ of nonrepeated integers from 1 to 4.

But $J_{31}^{\mu\nu}$ and $J_{42}^{\mu\nu}$ have nonzero components only in one 2×2 matrix, the 31 and 42 block, respectively. At most four can be linearly independent. Inspection of (5) and (6) shows that the generator with space-space indices ij and the generator with indices $k4$, $k \neq i,j$, are proportional to the same matrix σ^k, $k \in \{1,2,3\}$. Thus just three of the $J_{31}^{\mu\nu}$ and three of the $J_{42}^{\mu\nu}$ are linearly independent.

The total is $6+6+3+3 = 18$. There are 18 linearly independent tensor generators J.

Nonzero 2×2 blocks of the vectors P:

$$(P^{\mu}_{41})_{41} = ik_b\,\sigma_\mu \tag{7}$$

$$(P^{\mu}_{32})_{32} = -ik_c\,\sigma^\mu \tag{8}$$

$$(P^{\mu}_{43})_{43} = -ik_d\,\sigma_\mu \tag{9}$$

$$(P^{\mu}_{34})_{34} = +ik_e\,\sigma^\mu \tag{10}$$

Since the four Pauli spin matrices σ^μ are linearly independent and the four vectors P^μ are nonzero in different blocks, there are $4+4+4+4 = 16$ linearly independent vector generators.

Nonzero 2×2 blocks of the scalars D:

$$(D)_{33} = -(D)_{44} = +\frac{i}{2}\,\sigma^4 \tag{11}$$

$$(D_{31})_{31} = +\frac{i}{2}\,\sigma^4 \tag{12}$$

$$(D_{42})_{42} = -\frac{i}{2}\,\sigma^4 \tag{13}$$

The nonzero components of D, D_{31}, D_{42} occupy different 2×2 blocks, so these three generators are linearly independent.

Therefore, the total number of linearly independent generators is $18 + 16 + 3 = 37$, which can be verified directly by showing that the only null linear combination of generators has vanishing coefficients. Finding the commutators of these 37 matrices is straightforward arithmetic. One can then unravel each commutator into a linear combination of generators. The commutation relations of the Lie algebra they form is presented next.

3 Lie algebra

The generators $\{J^{\mu\nu}_8, J^{\mu\nu}_R, J^{\mu\nu}_{31}, J^{\mu\nu}_{42}, P^{\mu}_{41}, P^{\mu}_{32}, P^{\mu}_{43}, P^{\mu}_{34}, D, D_{31}, D_{42}\}$ together with the operation of matrix commutator determine a Lie algebra because the commutators can be written as linear combinations of generators. The nonzero commutation relations are displayed in this section and may be verified directly with the matrices defined in Sec. 2. The Lie algebra is, of course, more general than the representation used to derive it.

The commutation relations for J_8 with itself and other generators are the following,

$$[J^{\mu\nu}_8, J^{\rho\sigma}_8] = -i\,(\eta^{\nu\rho} J^{\mu\sigma}_8 + \eta^{\mu\sigma} J^{\nu\rho}_8 - \eta^{\mu\rho} J^{\nu\sigma}_8 - \eta^{\nu\sigma} J^{\mu\rho}_8) \tag{14}$$

$$[J^{\mu\nu}_8, J^{\rho\sigma}_R] = -i\,(\eta^{\nu\rho} J^{\mu\sigma}_R + \eta^{\mu\sigma} J^{\nu\rho}_R - \eta^{\mu\rho} J^{\nu\sigma}_R - \eta^{\nu\sigma} J^{\mu\rho}_R) \tag{15}$$

$$[J_8^{\mu\nu}, J_{31}^{\rho\sigma}] = -i\left(\eta^{\nu\rho}J_{31}^{\mu\sigma} + \eta^{\mu\sigma}J_{31}^{\nu\rho} - \eta^{\mu\rho}J_{31}^{\nu\sigma} - \eta^{\nu\sigma}J_{31}^{\mu\rho}\right) \tag{16}$$

$$[J_8^{\mu\nu}, J_{42}^{\rho\sigma}] = -i\left(\eta^{\nu\rho}J_{42}^{\mu\sigma} + \eta^{\mu\sigma}J_{42}^{\nu\rho} - \eta^{\mu\rho}J_{42}^{\nu\sigma} - \eta^{\nu\sigma}J_{42}^{\mu\rho}\right) \tag{17}$$

$$[J_8^{\mu\nu}, P_{41}^{\rho}] = -i\left(\eta^{\nu\rho}P_{41}^{\mu} - \eta^{\mu\rho}P_{41}^{\nu}\right) \tag{18}$$

$$[J_8^{\mu\nu}, P_{32}^{\rho}] = -i\left(\eta^{\nu\rho}P_{32}^{\mu} - \eta^{\mu\rho}P_{32}^{\nu}\right) \tag{19}$$

$$[J_8^{\mu\nu}, P_{43}^{\rho}] = -i\left(\eta^{\nu\rho}P_{43}^{\mu} - \eta^{\mu\rho}P_{43}^{\nu}\right) \tag{20}$$

$$[J_8^{\mu\nu}, P_{34}^{\rho}] = -i\left(\eta^{\nu\rho}P_{34}^{\mu} - \eta^{\mu\rho}P_{34}^{\nu}\right) \tag{21}$$

For J_R, we get

$$[J_R^{\mu\nu}, J_R^{\rho\sigma}] = -i\left(\eta^{\nu\rho}J_R^{\mu\sigma} + \eta^{\mu\sigma}J_R^{\nu\rho} - \eta^{\mu\rho}J_R^{\nu\sigma} - \eta^{\nu\sigma}J_R^{\mu\rho}\right) \tag{22}$$

$$\begin{aligned}[J_R^{\mu\nu}, J_{31}^{\rho\sigma}] = &-\frac{i}{2}\left(\eta^{\nu\rho}J_{31}^{\mu\sigma} + \eta^{\mu\sigma}J_{31}^{\nu\rho} - \eta^{\mu\rho}J_{31}^{\nu\sigma} - \eta^{\nu\sigma}J_{31}^{\mu\rho}\right) \\ &+\frac{i}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\nu\rho}\eta^{\mu\sigma}\right)\left(1 - \eta^{\mu\nu}\eta^{\rho\sigma}\right)D_{31} - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}D_{31}\end{aligned} \tag{23}$$

$$\begin{aligned}[J_R^{\mu\nu}, J_{42}^{\rho\sigma}] = &-\frac{i}{2}\left(\eta^{\nu\rho}J_{42}^{\mu\sigma} + \eta^{\mu\sigma}J_{42}^{\nu\rho} - \eta^{\mu\rho}J_{42}^{\nu\sigma} - \eta^{\nu\sigma}J_{42}^{\mu\rho}\right) \\ &-\frac{i}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\nu\rho}\eta^{\mu\sigma}\right)\left(1 - \eta^{\mu\nu}\eta^{\rho\sigma}\right)D_{42} - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}D_{42}\end{aligned} \tag{24}$$

$$[J_R^{\mu\nu}, P_{41}^{\rho}] = -\frac{i}{2}\left(\eta^{\nu\rho}P_{41}^{\mu} - \eta^{\mu\rho}P_{41}^{\nu}\right) + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_{41\sigma} \tag{25}$$

$$[J_R^{\mu\nu}, P_{32}^{\rho}] = -\frac{i}{2}\left(\eta^{\nu\rho}P_{32}^{\mu} - \eta^{\mu\rho}P_{32}^{\nu}\right) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_{32\sigma} \tag{26}$$

$$[J_R^{\mu\nu}, P_{43}^{\rho}] = -i\left(\eta^{\nu\rho}P_{43}^{\mu} - \eta^{\mu\rho}P_{43}^{\nu}\right) \tag{27}$$

$$[J_R^{\mu\nu}, P_{34}^{\rho}] = -i\left(\eta^{\nu\rho}P_{34}^{\mu} - \eta^{\mu\rho}P_{34}^{\nu}\right) \tag{28}$$

$$[J_R^{\mu\nu}, D_{31}] = -\frac{i}{2}J_{31}^{\mu\nu} \tag{29}$$

$$[J_R^{\mu\nu}, D_{42}] = +\frac{i}{2}J_{42}^{\mu\nu} \tag{30}$$

Nonvanishing commutation relations for J_{31} and J_{42} are as follows,

$$[J_{31}^{\mu\nu}, P_{43}^{\rho}] = -i\frac{k_d}{2k_b}\left(\eta^{\nu\rho}P_{41}^{\mu} - \eta^{\mu\rho}P_{41}^{\nu}\right) - \frac{k_d}{2k_b}\epsilon^{\mu\nu\rho\sigma}P_{41\sigma} \tag{31}$$

$$[J_{31}^{\mu\nu}, D] = -\frac{i}{2}J_{31}^{\mu\nu} \tag{32}$$

$$[J_{42}^{\mu\nu}, P_{34}^{\rho}] = -i\frac{k_e}{2k_c}\,(\eta^{\nu\rho}P_{32}^{\mu} - \eta^{\mu\rho}P_{32}^{\nu}) + \frac{k_e}{2k_c}\epsilon^{\mu\nu\rho\sigma}P_{32\sigma} \tag{33}$$

$$[J_{42}^{\mu\nu}, D] = -\frac{i}{2}J_{42}^{\mu\nu} \tag{34}$$

The commutators for P_{41}^{μ} and P_{32}^{μ} are found to obey

$$[P_{41}^{\mu}, P_{34}^{\nu}] = -2ik_bk_e\,(\eta^{\mu\nu}D_{31} - J_{31}^{\mu\nu}) \tag{35}$$

$$[P_{41}^{\mu}, D] = +\frac{i}{2}P_{41}^{\mu} \tag{36}$$

$$[P_{32}^{\mu}, P_{43}^{\nu}] = +2ik_ck_d\,(\eta^{\mu\nu}D_{42} + J_{42}^{\mu\nu}) \tag{37}$$

$$[P_{32}^{\mu}, D] = -\frac{i}{2}P_{32}^{\mu} \tag{38}$$

With P_{43} and P_{34}, one gets

$$[P_{43}^{\mu}, P_{34}^{\nu}] = +2ik_dk_e\,(\eta^{\mu\nu}D + J_{R}^{\mu\nu}) \tag{39}$$

$$[P_{43}^{\mu}, D] = +iP_{43}^{\mu} \tag{40}$$

$$[P_{43}^{\mu}, D_{31}] = -\frac{i}{2}\frac{k_d}{k_b}P_{41}^{\mu} \tag{41}$$

$$[P_{34}^{\mu}, D] = -iP_{34}^{\mu} \tag{42}$$

$$[P_{34}^{\mu}, D_{42}] = +\frac{i}{2}\frac{k_e}{k_c}P_{32}^{\mu} \tag{43}$$

Finally, the commutators for D give

$$[D, D_{31}] = +\frac{i}{2}D_{31} \tag{44}$$

$$[D, D_{42}] = -\frac{i}{2}D_{42} \tag{45}$$

Commutators that vanish are not displayed.

When nonzero, the four parameters k_b, k_c, k_d, k_e are free parameters since the commutation relations are homogeneous in them. For example, by the definition of P_{41}^{μ} in (7), one sees that k_b is a common factor in every term of (18), (25), (35), (36), and cancels in (31) and (41). When k_b vanishes, assume the limit $k_b \to 0$ is taken in (31) and (41), so the cancellation still occurs. Similarly for k_c. With these stipulations the ks are free parameters that may vanish.

By the definitions of the vectors P^{μ} in (7-10), each k is a scale factor for one of the linear momentum-like vector matrices P^{μ}_{41}, P^{μ}_{32}, P^{μ}_{43}, P^{μ}_{34}. Thus, when real-valued, the four ks determine the dimensional units such as meters, feet, or seconds for the position-like parameters x^{μ} that occur when constructing translations with the momentum-like generators, such as $\exp\left(-iP^{\mu}_{41}\, x_{41\,\mu}\right)$. Changing k is equivalent to changing the units for x.

The commutation relation for $[J^{\mu\nu}_{8}, J^{\rho\sigma}_{8}]$ in (14) shows that the $J^{\mu\nu}_{8}$ generators represent the Lorentz group of spacetime rotations. The notation for generators $J^{\mu\nu}$, P^{ρ}, and D is keyed to the $J^{\mu\nu}_{8}$ generators. By the commutation relations (15-21)) and the vanishing commutators of $J^{\mu\nu}_{8}$ with D, D_{31}, D_{42} which are not shown above since they vanish, all the generators transform properly as tensors, vectors and scalars under the spacetime rotations generated by the $J^{\mu\nu}_{8}$.

While the $J^{\mu\nu}_{R}$ generators also represent the Lorentz group of spacetime rotations by the commutation relation for $[J^{\mu\nu}_{R}, J^{\rho\sigma}_{R}]$ in (22), the generators $J^{\rho\sigma}_{31}$, $J^{\rho\sigma}_{42}$, P^{ρ}_{41}, P^{ρ}_{32}, do not have the proper commutation relations (23-26) with the $J^{\mu\nu}_{R}$ to transform as tensors and vectors under the rotations generated by the $J^{\mu\nu}_{R}$. The terms tensor, vector and scalar are keyed to the $J^{\mu\nu}_{8}$ not to the $J^{\mu\nu}_{R}$.

By design, there are Poincaré and conformal subalgebras embedded in the Lie algebra. As discussed in the next section, these can be found by selecting commutation relations from the list above.

4 Subalgebras

There are three Poincaré subalgebras and one conformal subalgebra. Along side these there are two Lorentz subalgebras and dozens of Abelian subalgebras. We focus on the Poincaré and conformal subalgebras.

By the commutation relation (14), the matrices $J^{\mu\nu}_{8}$ constitute a representation of spacetime rotations, the Lorentz group. To make a representation of the Poincaré group, one needs linear momentum matrices to generate translations. Both P^{μ}_{41} and P^{μ}_{32} qualify by their commutation relations (18), (19) with $J^{\mu\nu}_{8}$. One could take P^{μ}_{41} and P^{μ}_{32} separately, but they commute, so let us combine P^{μ}_{41} and P^{μ}_{32} and define a vector matrix P^{μ}_{8},

$$P^{\mu}_{8} = \alpha P^{\mu}_{41} + \beta P^{\mu}_{32} \quad . \tag{46}$$

The two parameters k_b and k_c scale the components of P^{μ}_{41} and P^{μ}_{32} in (7) and (8), respectively. And k_b and k_c are arbitrary, so the arbitrary coefficients α and β could be absorbed with new parameters $k^{\prime}_{b}$ and $k^{\prime}_{c}$. By adjusting the arbitrary factors k_b and k_c one can get many special momentum matrices. For example, to eliminate P^{μ}_{41} and have just P^{μ}_{32}, one can make k_b vanish. With αk_b and βk_c arbitrary, one has the most flexible momentum matrices P^{μ}_{8}

to combine with the angular momentum matrices $J_8^{\mu\nu}$ in a representation of the Poincaré algebra.

By the commutation relations (14), (18), (19) among the generators $J_8^{\mu\nu}, P_{41}^{\rho}, P_{32}^{\rho}$ one sees that $J_8^{\mu\nu}$ and $P_8^{\mu} = \alpha P_{41}^{\rho} + \beta P_{32}^{\rho}$ satisfy the commutation relations of the Poincaré algebra,

$$[J_8^{\mu\nu}, J_8^{\rho\sigma}] = -i\left(\eta^{\nu\rho} J_8^{\mu\sigma} + \eta^{\mu\sigma} J_8^{\nu\rho} - \eta^{\mu\rho} J_8^{\nu\sigma} - \eta^{\nu\sigma} J_8^{\mu\rho}\right) \tag{47}$$

$$[J_8^{\mu\nu}, P_8^{\rho}] = -i\left(\eta^{\nu\rho} P_8^{\mu} - \eta^{\mu\rho} P_8^{\nu}\right) \tag{48}$$

$$[P_8^{\mu}, P_8^{\nu}] = 0 \quad . \tag{49}$$

Therefore $J_8^{\mu\nu}$ and P_8^{μ} form a representation of the Poincaré group.

Similarly, by the commutation relations with $J_R^{\mu\nu}, P_{43}^{\rho}, P_{34}^{\rho}$ in (22), (27), (28), the $J_R^{\mu\nu}$ represent the Lorentz algebra with two qualifying momenta matrices P_{43}^{μ} and P_{34}^{μ}. Unlike the momenta P_{41}^{μ} and P_{32}^{μ} with $J_8^{\mu\nu}$, the matrices P_{43}^{μ} and P_{34}^{μ} do not commute, so they must be taken one at a time.

By the commutation relations (22) and (27) among the generators $J_R^{\mu\nu}$ and P_{43}^{ρ}, one has the Poincaré algebra,

$$[J_R^{\mu\nu}, J_R^{\rho\sigma}] = -i\left(\eta^{\nu\rho} J_R^{\mu\sigma} + \eta^{\mu\sigma} J_R^{\nu\rho} - \eta^{\mu\rho} J_R^{\nu\sigma} - \eta^{\nu\sigma} J_R^{\mu\rho}\right) \tag{50}$$

$$[J_R^{\mu\nu}, P_{43}^{\rho}] = -i\left(\eta^{\nu\rho} P_{43}^{\mu} - \eta^{\mu\rho} P_{43}^{\nu}\right) \tag{51}$$

$$[P_{43}^{\mu}, P_{43}^{\nu}] = 0 \quad . \tag{52}$$

Therefore $J_R^{\mu\nu}$ and P_{43}^{μ} form a representation of the Poincaré group.

By the commutation relations (22) and (28), one sees that $J_R^{\mu\nu}$ and P_{34}^{μ} satisfy the Poincaré algebra,

$$[J_R^{\mu\nu}, J_R^{\rho\sigma}] = -i\left(\eta^{\nu\rho} J_R^{\mu\sigma} + \eta^{\mu\sigma} J_R^{\nu\rho} - \eta^{\mu\rho} J_R^{\nu\sigma} - \eta^{\nu\sigma} J_R^{\mu\rho}\right) \tag{53}$$

$$[J_R^{\mu\nu}, P_{34}^{\rho}] = -i\left(\eta^{\nu\rho} P_{34}^{\mu} - \eta^{\mu\rho} P_{34}^{\nu}\right) \tag{54}$$

$$[P_{34}^{\mu}, P_{34}^{\nu}] = 0 \quad . \tag{55}$$

Therefore $J_R^{\mu\nu}$ and P_{34}^{μ} form a representation of the Poincaré group.

The generators in each of the three sets $\{J_8^{\mu\nu}, P_8^{\mu} = P_{41}^{\mu} + P_{32}^{\mu}\}$, $\{J_R^{\mu\nu}, P_{43}^{\mu}\}$, $\{J_R^{\mu\nu}, P_{34}^{\mu}\}$, form a Poincaré subalgebra of the Lie algebra in Sec. 3.

Next, consider the conformal subalgebra which motivated the construction and whose inclusion was required. The generators $J_R^{\mu\nu}$ together with P_{43}^{μ}, P_{34}^{μ} and D obey the Lie algebra of the conformal group. Unlike the Lorentz and Poincaré groups, which have infinitely many finite dimensional matrix reps, there is just one finite dimensional matrix representation of the conformal group, within the equivalence of similarity transformations. That representation is produced by the generators $J_R^{\mu\nu}$, P_{43}^{μ}, P_{34}^{μ} , and D which, in the rep of Sec. 2, are confined to the 4×4 diagonal block that acts nontrivially only on components 5 to 8 of an 8-spinor.

The conformal subalgebra collects the commutation relations (22), (27), (28), (39), (40), (42) among the generators $J_R^{\mu\nu}, P_{43}^{\mu}, P_{34}^{\mu}, D$ Note that the arbitrary scale factors k_d and k_e should be set so that their product is unity, $k_d k_e = 1$ in (39). With that understanding, the subalgebra of the generators $J_R^{\mu\nu}$, P_{43}^{μ}, P_{34}^{μ} , and D takes on a conventional form of the Lie algebra of the conformal group,

$$[J_R^{\mu\nu}, J_R^{\rho\sigma}] = -i\left(\eta^{\nu\rho} J_R^{\mu\sigma} + \eta^{\mu\sigma} J_R^{\nu\rho} - \eta^{\mu\rho} J_R^{\nu\sigma} - \eta^{\nu\sigma} J_R^{\mu\rho}\right) \tag{56}$$

$$[J_R^{\mu\nu}, P_{43}^{\rho}] = -i\left(\eta^{\nu\rho} P_{43}^{\mu} - \eta^{\mu\rho} P_{43}^{\nu}\right) \quad ; \quad [J_R^{\mu\nu}, P_{34}^{\rho}] = -i\left(\eta^{\nu\rho} P_{34}^{\mu} - \eta^{\mu\rho} P_{34}^{\nu}\right) \tag{57}$$

$$[P_{43}^{\mu}, P_{34}^{\nu}] = +2i\left(\eta^{\mu\nu} D + J_R^{\mu\nu}\right) \tag{58}$$

$$[P_{43}^{\mu}, D] = +iP_{43}^{\mu} \quad ; \quad [P_{34}^{\mu}, D] = -iP_{34}^{\mu} \tag{59}$$

$$[J_R^{\mu\nu}, D] = [P_{43}^{\mu}, P_{43}^{\nu}] = [P_{34}^{\mu}, P_{34}^{\nu}] = [D, D] = 0 \quad , \tag{60}$$

where the product of the arbitrary quantities k_d and k_e is unity, $k_d k_e = 1$ in (39) and (58). By their definitions, (4), (9), (10), (11), the generators $J_R^{\mu\nu}, P_{43}^{\rho}, P_{34}^{\rho}, D$ vanish outside one 4×4 block. Thus this conformal rep has transformations that act on a single 4-spinor.

5 Conclusions

The 37-generator Lie algebra has distinct Poincaré and conformal subalgebras. The larger algebra introduces new generators and transformations because there is nontrivial mixing of the Poincaré's translations with the conformal's translations. For example, note the need to introduce D_{31} and $J_{31}^{\mu\nu}$ in the commutation relation (35) between the Poincaré momentum P_{41}^{μ} and the conformal's P_{34}^{ν}. Likewise, the commutation relation (37) between P_{32}^{μ} and the conformal's P_{43}^{ν} require D_{42} and $J_{42}^{\mu\nu}$, which generate transformations previously not associated with the Poincaré and conformal groups.

The 8×8 matrix rep of the Poincaré group happens to be the most general non-unitary finite dimensional spin 1/2 representation of that group. One of its two 4-spinors sits in an awkward secondary position. That 4-spinor merely accepts the linear combinations of the first 4-spinor's components deposited on it by translations. While it transforms with rotations and boosts like the primary 4-spinor, the secondary, the "receiver", 4-spinor does not mix with the primary under these Lorentz transformations. By transforming the secondary 4-spinor with the conformal group, balance between the two 4-spinors is achieved; each 4-spinor is special in its own way.

References

[1] See, for example, S. Weinberg, *The Quantum Theory of Fields Vol. I*, (Cambridge University Press, Cambridge, UK, 1995), Chapter 2.

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[9] Liu YuFen, Ma ZhongQi and Hou BoYuan, *Pure Spinor Formalism for Conformal Fermion and Conserved Currents*, Commun. Theor. Phys. (Beijing, China) **31** (1999), pp 481-490

[10] Wolfram Research, Inc., Mathematica, Version 11.3, Champaign, IL 2018.

[11] R. Shurtleff, *A Poincaré-conformal matrix Lie algebra, the notebook*, 2018. To download the executable Mathematica notebook, copy the URL address into your browser: https://www.dropbox.com/s/o73xzotg156uvz4/PoincareConformalAlgebra4.nb?dl=0

[12] L. S. Pontryagin, *Topological Groups*, (Princeton University Press, Princeton, 1946).

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In[2]:=
```
(*Content-type:application/mathematica*)(***Wolfram Notebook File***)
(*http://www.wolfram.com/nb*)(*CreatedBy='Mathematica 11.1'*)
```

Appendix

A Poincaré - conformal matrix Lie algebra, the notebook, by Richard Shurtleff

This Mathematica notebook[1,2] verifies the calculations in the article *A Poincaré - conformal matrix Lie algebra*.[3] The article is abbreviated herein as "PCMLA". Equations in PCMLA are called "PCMLA Eqn (N)" in the work below.

The article PCMLA reports a set of 8x8 matrices with complex components that produce a Lie algebra. In this notebook, the matrices are constructed in detail. The commutation relations of the Lie algebra are displayed and verified. The structure constants are obtained, along with the associated adjoint representation. The Cartan-Killing metric is shown to have a null determinant and therefore has no inverse. Various subalgebras, Lorentz, Poincare, conformal, and Abelian are found. Some transformations are found.

References:

1. Wolfram Research, Inc., Mathematica, Version 11.1, Champaign, IL 2018.

2. A link to the executable Mathematica notebook can be found in the References section of the article in Ref. 3 on ViXra. The notebook runs on Windows 10, Mathematica 11.1 platforms and, most likely, on other Mathematica platforms.

3. R. Shurtleff, *A Poincaré - conformal matrix Lie algebra*, to be submitted to ViXra, 2018. (Technical difficulties associated with the link in Ref. 2 prevent providing more complete bibliographic data.)

Preliminaries

Fundamentals:

$\eta^{\mu\nu}$ flat spacetime metric = diag {+1,+1,+1,-1} ; δ^{mn} Kronecker delta ; ϵ^{ijk} 3D antisymmetric symbol ; $\epsilon^{\lambda\mu\nu\sigma}$ 4-dimensional antisymmetric symbol;

σ^{μ} Pauli spin matrices

In[3]:=
```
zip[f_] := f == 0 (* zip[f] makes equations.*)
eqnS = Map[zip, {3 x + 2 y - 7, x - y + 1}];
Print["Example: Mapping 'zip' on {3x+2y - 7,x-y+1} gives ", eqnS]
Print["The equations yield ", Solve[eqnS, {x, y}][[1]] ]
```

```
Example: Mapping 'zip' on {3x+2y - 7,x-y+1} gives {-7 + 3 x + 2 y == 0, 1 + x - y == 0}

The equations yield {x → 1, y → 2}
```

$\eta^{\mu\nu}$, δ^{mn}, ϵ^{ijk}, $\epsilon^{\lambda\mu\nu\sigma}$, 0^{mn}

In[7]:=
```
(* η, δ, ϵ, ϵ, 0 *)
ημν = { {+1, 0, 0, 0}, {0, +1, 0, 0}, {0, 0, +1, 0}, {0, 0, 0, -1} };
δ[i_, j_] := δ[i, j] = IdentityMatrix[50][[i, j]] (* 50 = ∞ *)
ϵijk[i_, j_, k_] := (k - j) (k - i) (j - i) / 2 /; (1 ≤ i ≤ 3) && (1 ≤ j ≤ 3) && (1 ≤ k ≤ 3)
ϵλμνσ[λ_, μ_, ν_, σ_] := (λ - μ) (λ - ν) (λ - σ) (μ - ν) (μ - σ) (ν - σ) / 12 /;
  (1 ≤ λ ≤ 4) && (1 ≤ μ ≤ 4) && (1 ≤ ν ≤ 4) && (1 ≤ σ ≤ 4)
ZeroMatrix[n_] := ZeroMatrix[n] = IdentityMatrix[n] - IdentityMatrix[n]
```

In[12]:=
```
Print["        Spacetime metric  ημν = ", ημν // MatrixForm, " ."]
Print["        Antisymmetric 3-symbol   ϵijk[1,2,3] = ", ϵijk[1, 2, 3], " ."]
Print["        Antisymmetric 4-symbol   ϵλμνσ[1,2,3,4] = " , ϵλμνσ[1, 2, 3, 4] , " ."]
Print[ "Note: With the time index first, as in ϵ^4123 = -1, we get a negative result.
   Often the convention is to have the time index t = 0 and to set ϵ^0123 = +1."]
Print["The two conventions differ by a minus sign in their epsilons ϵ^λμνσ.   " ]
```

Spacetime metric $\eta\mu\nu$ = $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

Antisymmetric 3-symbol   ϵijk[1,2,3] = 1 .

Antisymmetric 4-symbol   ϵλμνσ[1,2,3,4] = 1 .

Note: With the time index first, as in $\epsilon^{4123} = -1$, we get a negative result.
Often the convention is to have the time index t = 0 and to set $\epsilon^{0123} = +1$.

The two conventions differ by a minus sign in their epsilons $\epsilon^{\lambda\mu\nu\sigma}$.

Note that when time is index 0, one has the opposite convention $\epsilon^{0123} = 1$, whereas we have time as index 4 and $\epsilon^{4123} = -1$

The projection matrix that selects the first four “donor” components is eDONOR. The projection matrix that selects the last four components, the “receivers”, is eRECEIVER.

In[17]:=
```
(*Projection matrices eDONOR for first
 4 indices and eRECEIVER for the last 4 indices. *)
eDONOR = ArrayFlatten[
   {{IdentityMatrix[4], ZeroMatrix[4]}, {ZeroMatrix[4], ZeroMatrix[4]}}] ;
eRECEIVER = ArrayFlatten[{{ZeroMatrix[4], ZeroMatrix[4]},
    {ZeroMatrix[4], IdentityMatrix[4]}}] ;
Print["projection matrix on Donor components 1-4: e_donor  = ", eDONOR // MatrixForm,
 "  and Receiver components 5-8: e_receiver = ", eRECEIVER // MatrixForm]
```

projection matrix on Donor components 1–4: e_{donor} = $\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$

and Receiver components 5–8: $e_{receiver}$ = $\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}$

In[20]:=
```
σμ = { {{0, 1}, {1, 0}}, {{0, -ⅈ}, {ⅈ, 0}}, {{1, 0}, {0, -1}}, {{1, 0}, {0, 1}}  };
Print["PCMLA Eqn(2). Pauli spin matrices {σ^x,σ^y,σ^z,σ^t}  = ",
 Table[σμ[[ν]] // MatrixForm, {ν, 4}]]
```

PCMLA Eqn(2). Pauli spin matrices $\{\sigma^x,\sigma^y,\sigma^z,\sigma^t\}$ = $\left\{\begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&-i\\i&0\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}, \begin{pmatrix}1&0\\0&1\end{pmatrix}\right\}$

In[22]:=
```
γμ = -ⅈ Table[ArrayFlatten[{{0, σμ[[μ]]}, {Sum_{ν}^{4} (-ημν[[μ, ν]]) σμ[[ν]], 0}}], {μ, 4}];
Print["4-component Dirac matrices: γ^μ = ", Table[γμ[[ν]] // MatrixForm, {ν, 4}]]
```

4–component Dirac matrices: γ^μ =

$\left\{\begin{pmatrix}0&0&0&-i\\0&0&-i&0\\0&i&0&0\\i&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&-i&0\\0&0&0&i\\i&0&0&0\\0&-i&0&0\end{pmatrix}, \begin{pmatrix}0&0&-i&0\\0&0&0&-i\\-i&0&0&0\\0&-i&0&0\end{pmatrix}\right\}$

In[24]:=
```
(*PCMLA Eqn(3)*)
Jμν4[μ_, ν_] := Jμν4[μ, ν] = (-ⅈ/4) (γμ[[μ]].γμ[[ν]] - γμ[[ν]].γμ[[μ]])
(*Print["Sample 4-spinor Rotation generator for rots about x^3: J^12 = ",
  Jμν4[1,2]//MatrixForm]
 Print["Sample 4-spinor Boost generator for boost along x^1: J^14 = ",
  Jμν4[1,4]//MatrixForm]*)
(*Print["check  γ^4J^μνγ^4 = η_μκη_νλJ^κλ = J_μν : ",
 {0}==Union[Flatten[Table[γμ[[4]].Jμν4[μ,ν].γμ[[4]]-
      Sum[ημν[[μ,κ]]ημν[[ν,λ]]Jμν4[κ,λ],{κ,4},{λ,4}],{μ,4},{ν,4}]]]]*)
```

Start Generators Section

In[25]:=
```
(*PCMLA Eqn(3)  *)
J8[μ_, ν_] := J8[μ, ν] = ArrayFlatten[{{Jμν4[μ, ν], 0}, {0, Jμν4[μ, ν]}}]
Print["8-spinor Rotation matrix for rots about x3: J^12 = ", J8[1, 2] // MatrixForm];
Print["8-spinor Boost matrix for boost along x1: J^14 = ", J8[1, 4] // MatrixForm];
```

8-spinor Rotation matrix for rots about x3: J^{12} =
$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

8-spinor Boost matrix for boost along x1: J^{14} =
$$\begin{pmatrix} 0 & \frac{i}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{i}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{i}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{i}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i}{2} & 0 \end{pmatrix}$$

In[28]:=
```
(*PCMLA Eqn(4)*)
JR[μ_, ν_] := JR[μ, ν] = eRECEIVER.J8[μ, ν]
Print["Receiver Rotation matrix for rots about x3: J^12 = ", JR[1, 2] // MatrixForm];
Print["Receiver Boost matrix for boost along x1: J^14 = ", JR[1, 4] // MatrixForm];
```

Receiver Rotation matrix for rots about x3: J^{12} =
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

Receiver Boost matrix for boost along x1: J^{14} =
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{i}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i}{2} & 0 \end{pmatrix}$$

In[31]:=
```
(*PCMLA Eqn(5)*)
J31[μ_, ν_] :=
 ArrayFlatten[{{ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]},
   {ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]},
   {Table[-Jμν4[μ, ν][[i, j]], {i, 2}, {j, 2}], ZeroMatrix[2], ZeroMatrix[2],
    ZeroMatrix[2]}, {ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]}}]
(*PCMLA Eqn(6)*)
J42[μ_, ν_] :=
 ArrayFlatten[{{ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]},
   {ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]},
   { ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]}, {ZeroMatrix[2],
    Table[-Jμν4[μ, ν][[i, j]], {i, 3, 4}, {j, 3, 4}], ZeroMatrix[2], ZeroMatrix[2]}}]
```

In[33]:=
```
Print["31 2x2 block Rotation matrix for rots about x3: J^12 = ", J31[1, 2] // MatrixForm];
Print["31 2x2 block Boost matrix for boost along x1: J^14 = ", J31[1, 4] // MatrixForm];
```

31 2x2 block Rotation matrix for rots about x3: J^{12} = $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

31 2x2 block Boost matrix for boost along x1: J^{14} = $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{i}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{i}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

In[35]:=
```
Print["42 2x2 block Rotation matrix for rots about x3: J^12 = ", J42[1, 2] // MatrixForm];
Print["41 2x2 block Boost matrix for boost along x1: J^14 = ", J42[1, 4] // MatrixForm];
```

42 2x2 block Rotation matrix for rots about x3: J^{12} = $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{pmatrix}$

41 2x2 block Boost matrix for boost along x1: J^{14} = $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

In[37]:=
```
(*PCMLA Eqn(7)*)
P41[μ_, kb_] :=
 ArrayFlatten[{{ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]},
   {ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]},
   {ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]},
   {ⅈ kb Sum[ημν[[μ, ν]] σμ[[ν]], {ν, 4}] , ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]}}]
Print["8-spinor momentum matrices in 41 2x2 block: P41^μ  = ",
 Table[P41[μ, kb] // MatrixForm, {μ, 4}]]
```

8-spinor momentum matrices in 41 2x2 block: $P41^{\mu}$ =

$$\left\{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i\,kb & 0 & 0 & 0 & 0 & 0 & 0 \\ i\,kb & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & kb & 0 & 0 & 0 & 0 & 0 & 0 \\ -kb & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ i\,kb & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i\,kb & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -i\,kb & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i\,kb & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right\}$$

In[39]:=
```
(*PCMLA Eqn(8)*)
P32[μ_, kc_] :=
 ArrayFlatten[{{ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]},
   {ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]},
   {ZeroMatrix[2], -ⅈ kc σμ[[μ]], ZeroMatrix[2], ZeroMatrix[2]},
   { ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]}}]
Print["8-spinor momentum matrices in 32 2x2 block: P32^μ  = ",
 Table[P32[μ, kc] // MatrixForm, {μ, 4}]]
```

8-spinor momentum matrices in 32 2x2 block: $P32^{\mu}$ =

$$\left\{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\,kc & 0 & 0 & 0 & 0 \\ 0 & 0 & -i\,kc & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -kc & 0 & 0 & 0 & 0 \\ 0 & 0 & kc & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i\,kc & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\,kc & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i\,kc & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\,kc & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right\}$$

In[41]:=
```
(*PCMLA Eqn(9)*)
P43[μ_, kd_] :=
 ArrayFlatten[{{ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]},
   {ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]},
   {ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]}, { ZeroMatrix[2],
    ZeroMatrix[2], -ⅈ kd Sum[ημν[[μ, ν]] σμ[[ν]], {ν, 4}], ZeroMatrix[2]}}]
Print["Receiver conformal momentum matrices in 43 2x2 block: P43^μ  = ",
 Table[P43[μ, kd] // MatrixForm, {μ, 4}]]
```

Receiver conformal momentum matrices in 43 2x2 block: $P43^{\mu}$ = $\left\{\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&-i\,kd&0&0\\ 0&0&0&0&-i\,kd&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&-kd&0&0\\ 0&0&0&0&kd&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&-i\,kd&0&0&0\\ 0&0&0&0&0&i\,kd&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&i\,kd&0&0&0\\ 0&0&0&0&0&i\,kd&0&0 \end{pmatrix}\right\}$

In[43]:=
```
(*PCMLA Eqn(10)*)
P34[μ_, ke_] :=
 ArrayFlatten[{{ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]},
   {ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]},
   {ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ⅈ ke σμ[[μ]]},
   { ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]}}]
Print["Receiver conformal momentum matrices in 34 2x2 block: P34^μ  = ",
 Table[P34[μ, ke] // MatrixForm, {μ, 4}]]
```

Receiver conformal momentum matrices in 34 2x2 block: $P34^{\mu}$ = $\left\{\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&i\,ke\\ 0&0&0&0&0&0&i\,ke&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&ke\\ 0&0&0&0&0&0&-ke&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&i\,ke&0\\ 0&0&0&0&0&0&0&-i\,ke\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&i\,ke&0\\ 0&0&0&0&0&0&0&i\,ke\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}\right\}$

In[45]:=
```
(*PCMLA Eqn(11)*)
mD = ArrayFlatten[{{ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]},
    {ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]},
    {ZeroMatrix[2], ZeroMatrix[2], i/2 IdentityMatrix[2], ZeroMatrix[2]},
    {ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], -i/2 IdentityMatrix[2]}}];
Print["   dilation matrix for Receiver conformal algebra: D  = ", mD // MatrixForm]
```

dilation matrix for Receiver conformal algebra: D =
$$\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&\frac{i}{2}&0&0&0\\ 0&0&0&0&0&\frac{i}{2}&0&0\\ 0&0&0&0&0&0&-\frac{i}{2}&0\\ 0&0&0&0&0&0&0&-\frac{i}{2} \end{pmatrix}$$

In[47]:=
```
(*PCMLA Eqn(12)*)
mD31 = ArrayFlatten[{{ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]},
    {ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]},
    {i/2 IdentityMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]},
    {ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]}}];
Print["Dilation-like matrix in 31 and 42 2x2 blocks, 31-block portion  D31  = ",
 mD31 // MatrixForm]
```

Dilation-like matrix in 31 and 42 2x2 blocks, 31-block portion D_{31} =
$$\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ \frac{i}{2}&0&0&0&0&0&0&0\\ 0&\frac{i}{2}&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

In[49]:=
```
(*PCMLA Eqn(13)*)
mD42 = ArrayFlatten[{{ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]},
    {ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]},
    { ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2], ZeroMatrix[2]},
    {ZeroMatrix[2], -i/2 IdentityMatrix[2], ZeroMatrix[2], ZeroMatrix[2]}}];
Print["Dilation-like matrix in 31 and 42 2x2 blocks, 42-block portion  D42  = ",
 mD42 // MatrixForm]
```

Dilation-like matrix in 31 and 42 2x2 blocks, 42-block portion D_{42} =
$$\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&-\frac{i}{2}&0&0&0&0&0\\ 0&0&0&-\frac{i}{2}&0&0&0&0 \end{pmatrix}$$

Generator matrices that are the basis for the Lie algebra

In[51]:=
```
(*List of generator names = genNAME*)
genNAME = {"J8", "JR", "J31", "J42", "P41", "P32", "P43", "P34", "D", "D31", "D42"};
Print["There are ", Length[genNAME], " generator families: ", genNAME]
```

```
There are 11 generator families: {J8, JR, J31, J42, P41, P32, P43, P34, D, D31, D42}
```

In[53]:=
```
(*Flatten[Table[{μ,ν,J31[μ,ν]//MatrixForm},{μ,3},{ν,μ+1,4}],1]*)
```

In[54]:=
```
(*Flatten[Table[{μ,ν,J42[μ,ν]//MatrixForm},{μ,3},{ν,μ+1,4}],1]*)
```

In[55]:=
```
(*List of generators = gen*)
gen = {Flatten[Table[J8[μ, ν], {μ, 3}, {ν, μ + 1, 4}], 1],
   Flatten[Table[JR[μ, ν], {μ, 3}, {ν, μ + 1, 4}], 1],
   Flatten[Table[J31[μ, ν], {μ, 3}, {ν, μ + 1, 4}], 1],
   Flatten[Table[J42[μ, ν], {μ, 3}, {ν, μ + 1, 4}], 1], Table[P41[μ, kb], {μ, 4}],
   Table[P32[μ, kc], {μ, 4}], Table[P43[μ, kd], {μ, 4}],
   Table[P34[μ, ke], {μ, 4}], {mD}, {mD31}, {mD42}};
Print["There are ", Sum[Length[ gen[[i]] ], {i, Length[gen]}],
 " generators. Just 37 of the generators in the list are linearly independent."]
```

```
There are 43 generators. Just 37 of the generators in the list are linearly independent.
```

In[57]:=
```
Print["Each J family has six matrices J^μν with indices in the order {μ,ν} = ",
 Flatten[ Table[{μ, ν}, {μ, 3}, {ν, μ + 1, 4}], 1]]
Print["Each P family has four matrices P^μ with indices in the order μ = ",
 Table[μ, {μ, 4}]]
Print["Each D family has one matrix D with no indices."]
```

```
Each J family has six matrices J^μν with indices in the order {μ,ν} =
 {{1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}}

Each P family has four matrices P^μ with indices in the order μ = {1, 2, 3, 4}

Each D family has one matrix D with no indices.
```

In[60]:=
```
Print["The 3rd generator family J_31 has ",
 Length[gen[[3]]], " generators. Just 3 are linearly independent. "]
Print["The 2nd generator is J_31^13 = ", gen[[3, 2]] // MatrixForm,
 " and the 5th generator is J_31^24 = ", gen[[3, 5]] // MatrixForm, " = iJ_31^13 ."]
Print["Clearly, the two generators are proportional."]
```

The 3rd generator family J_{31} has 6 generators. Just 3 are linearly independent.

$$\text{The 2nd generator is } J_{31}^{13} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{i}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{i}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{and the 5th generator is } J_{31}^{24} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = i J_{31}^{13} .$$

Clearly, the two generators are proportional.

One finds that J_{31}^{ij} and J_{31}^{k4} are proportional, $J_{31}^{k4} = i J_{31}^{ij}$, where i,j,k = 1,2,3 are distinct and in order, 123, 231, or 312.

Similar comments hold for the J_{42} family of generators gen[[4]]. We have $J_{42}^{k4} = -i J_{42}^{ij}$, which has a negative compared to the J_{31} family.

The total number of linearly independent generators is 43 - 3 - 3 = 37. This is confirmed in the next cell.

In[63]:=

```
gen37 = {Flatten[Table[J8[μ, ν], {μ, 3}, {ν, μ + 1, 4}], 1],
   Flatten[Table[JR[μ, ν], {μ, 3}, {ν, μ + 1, 4}], 1],
   Flatten[Table[J31[μ, ν], {μ, 3}, {ν, μ + 1, 3}], 1],
   Flatten[Table[J42[μ, ν], {μ, 3}, {ν, μ + 1, 3}], 1], Table[P41[μ, kb], {μ, 4}],
   Table[P32[μ, kc], {μ, 4}], Table[P43[μ, kd], {μ, 4}],
   Table[P34[μ, ke], {μ, 4}], {mD}, {mD31}, {mD42}};
Print["There are ", Sum[Length[ gen37[[i]] ], {i, Length[gen]}],
 " generators in the list gen37."]
```

There are 37 generators in the list gen37.

In[65]:=

```
linearIndependentCHECKcoeffs =
  Flatten[Table[ac[i, m], {i, Length[gen37]}, {m, Length[gen37[[i]] ]}]];
LHS37 = Complement[Union[Flatten[Sum[ ac[i, m] gen37[[i, m]],
      {i, Length[gen37]}, {m, Length[gen37[[i]] ]}  ]]], {0}];
LINEARindependentCHECKeqns37 = Map[zip, LHS37]; (*zip[f_]:= f==0*)
liSOLN37 = Solve[LINEARindependentCHECKeqns37, linearIndependentCHECKcoeffs];
Print["All coefficients vanish, which means the 37 matrices are linearly independent: ",
 {0} == Union[Flatten[linearIndependentCHECKcoeffs /. liSOLN37]] ]
```

All coefficients vanish, which means the 37 matrices are linearly independent: True

The 37 generators are linearly independent.

But, it is easier to work with tensors and vectors, so work with all 43 generators and remember that just 37 are linearly independent because $J^{k4}_{31} = iJ^{ij}_{31}$ and $J^{k4}_{42} = -iJ^{ij}_{42}$.

Comment: With Dirac 4-spinors, there are Left- and Right- 2-spinor Lorentz reps, one for each sign of $\sqrt{-1}$ = +/- i. It does not seem possible to combine J31 and J42 like one does with Dirac 4-spinors . If you do combine J31 with J42 making J3142 := J31 + J42, then the commutation relations with JR and D make you include also J31 - j42. So, no gain, its equivalent.

End Generators Section

Start Commutation Relations Section

NoCommute: pairs of generator families {i,j} that do not commute.
That means that {i,j} is in NoCommute if at least one matrix in gen[i,m], any m, does not commute with some matrix in gen[j,n], any n.

In[70]:=
```
(* pairs of generators with nonzero commutators*)
NoCommute = {};
Table[{i, m, j, n,
   If[Length[Union[Flatten[gen[[i, m]].gen[[j, n]] - gen[[j, n]].gen[[i, m]]]]] ≥ 2,
    AppendTo[NoCommute, {i, j}]]}, {i, Length[gen]},
  {m, Length[gen[[i]] ]}, {j, i, Length[gen]}, {n, Length[gen[[j]] ]}];
Union[NoCommute];
Union[Table[{NoCommute[[i, 1]], NoCommute[[i, 2]], genNAME[[ NoCommute[[i, 1]] ]],
    genNAME[[ NoCommute[[i, 2]] ]]}, {i, Length[NoCommute]}]];
Length[
  %];
```

In[75]:=
```
Print["Generator families with members that do not commute
   with each other. Non-commuting families by family number and name: ",
 Union[Table[{NoCommute[[i, 1]], NoCommute[[i, 2]], genNAME[[ NoCommute[[i, 1]] ]],
    genNAME[[ NoCommute[[i, 2]] ]]}, {i, Length[NoCommute]}]]]
Print["Each case of noncommuting families gives a commutation
   relation with nonzero structure constants. There are ",
 Length[Union[Table[{NoCommute[[i, 1]], NoCommute[[i, 2]],
     genNAME[[ NoCommute[[i, 1]] ]], genNAME[[ NoCommute[[i, 2]] ]]},
    {i, Length[NoCommute]}]]], " nontrivial commutation relations."]
```

```
Generator families with members that do not commute
   with each other. Non-commuting families by family number and name:
 {{1, 1, J8, J8}, {1, 2, J8, JR}, {1, 3, J8, J31}, {1, 4, J8, J42}, {1, 5, J8, P41},
  {1, 6, J8, P32}, {1, 7, J8, P43}, {1, 8, J8, P34}, {2, 2, JR, JR}, {2, 3, JR, J31}, {2, 4, JR, J42},
  {2, 5, JR, P41}, {2, 6, JR, P32}, {2, 7, JR, P43}, {2, 8, JR, P34}, {2, 10, JR, D31},
  {2, 11, JR, D42}, {3, 7, J31, P43}, {3, 9, J31, D}, {4, 8, J42, P34}, {4, 9, J42, D},
  {5, 8, P41, P34}, {5, 9, P41, D}, {6, 7, P32, P43}, {6, 9, P32, D}, {7, 8, P43, P34}, {7, 9, P43, D},
  {7, 10, P43, D31}, {8, 9, P34, D}, {8, 11, P34, D42}, {9, 10, D, D31}, {9, 11, D, D42}}
```

```
Each case of noncommuting families gives a commutation relation with
   nonzero structure constants. There are 32 nontrivial commutation relations.
```

Commute: pairs of generator families {i,j} that do commute.

That means that {i,j} is in Commute if all the matrices in gen[i,m], any m, commute with all the matrices in gen[j,n], any n.

In[77]:=
```
(*Generators that commute with one another*)
commutingGEN = {};
Table[
  {i, j, If[Union[Flatten[Table[gen[[i, m]].gen[[j, n]] - gen[[j, n]].gen[[i, m]], {m,
         Length[gen[[i]] ]}, {n, Length[gen[[j]] ]}]]] == {0},
    AppendTo[commutingGEN, {i, j}]]}, {i, Length[gen]}, {j, i, Length[gen]}];
Union[commutingGEN];
Union[
  Table[{commutingGEN[[i, 1]], commutingGEN[[i, 2]], genNAME[[ commutingGEN[[i, 1]] ]],
    genNAME[[ commutingGEN[[i, 2]] ]]}, {i, Length[commutingGEN]}]];
Length[%];
Intersection[commutingGEN, NoCommute];
```

In[83]:=
```
Print["Generator families that commute with each
   other. Commuting families by family number and name: ", Union[
  Table[{commutingGEN[[i, 1]], commutingGEN[[i, 2]], genNAME[[ commutingGEN[[i, 1]] ]],
    genNAME[[ commutingGEN[[i, 2]] ]]}, {i, Length[commutingGEN]}]]]
Print["The commutation relations for commuting families are not
   displayed in the paper. There are ", Length[Union[
   Table[{commutingGEN[[i, 1]], commutingGEN[[i, 2]], genNAME[[ commutingGEN[[i, 1]] ]],
     genNAME[[ commutingGEN[[i, 2]] ]]}, {i, Length[commutingGEN]}]]],
 " commutation relations with commuting families."]
```

```
Generator families that commute with each other. Commuting families by family number and name:
 {{1, 9, J8, D}, {1, 10, J8, D31}, {1, 11, J8, D42}, {2, 9, JR, D}, {3, 3, J31, J31},
  {3, 4, J31, J42}, {3, 5, J31, P41}, {3, 6, J31, P32}, {3, 8, J31, P34}, {3, 10, J31, D31},
  {3, 11, J31, D42}, {4, 4, J42, J42}, {4, 5, J42, P41}, {4, 6, J42, P32}, {4, 7, J42, P43},
  {4, 10, J42, D31}, {4, 11, J42, D42}, {5, 5, P41, P41}, {5, 6, P41, P32}, {5, 7, P41, P43},
  {5, 10, P41, D31}, {5, 11, P41, D42}, {6, 6, P32, P32}, {6, 8, P32, P34}, {6, 10, P32, D31},
  {6, 11, P32, D42}, {7, 7, P43, P43}, {7, 11, P43, D42}, {8, 8, P34, P34}, {8, 10, P34, D31},
  {9, 9, D, D}, {10, 10, D31, D31}, {10, 11, D31, D42}, {11, 11, D42, D42}}

The commutation relations for commuting families are not displayed in the paper. There are
 34 commutation relations with commuting families.
```

The list "commutingGEN" has pairs of generator families (a,b) that commute: [a,b] = 0 and a <= b, ordered. "commutingGENbi" has both [a,b] and [b,a], ***bi***directional. For example, both (1,9) and (9,1) are in "commutingGENbi", while just (1,9) is in "commutingGEN".

In[85]:=
```
commutingGENbi =
 Union[Table[Reverse[commutingGEN[[i]]], {i, Length[commutingGEN]}], commutingGEN]
```

Out[85]=
```
{{1, 9}, {1, 10}, {1, 11}, {2, 9}, {3, 3}, {3, 4}, {3, 5}, {3, 6}, {3, 8}, {3, 10},
 {3, 11}, {4, 3}, {4, 4}, {4, 5}, {4, 6}, {4, 7}, {4, 10}, {4, 11}, {5, 3}, {5, 4},
 {5, 5}, {5, 6}, {5, 7}, {5, 10}, {5, 11}, {6, 3}, {6, 4}, {6, 5}, {6, 6}, {6, 8},
 {6, 10}, {6, 11}, {7, 4}, {7, 5}, {7, 7}, {7, 11}, {8, 3}, {8, 6}, {8, 8}, {8, 10},
 {9, 1}, {9, 2}, {9, 9}, {10, 1}, {10, 3}, {10, 4}, {10, 5}, {10, 6}, {10, 8}, {10, 10},
 {10, 11}, {11, 1}, {11, 3}, {11, 4}, {11, 5}, {11, 6}, {11, 7}, {11, 10}, {11, 11}}
```

There are 11 families of generators. That makes 11*12/2 = 66 commutation relations between families. There are 32 commutation relations with nonvanishing structure constants and 34 commutation relations for the families that commute with each other.

The next problem is to find the nonzero structure constants by expanding the nonvanishing commutators in terms of generators.

Commutation relations

For J8 with general generator G,

[J8,G]:

In[86]:=

```
Print[" PCMLA Eqn(14). {1,1,J8,J8} [J8^μν,J8^ρσ] = -ⅈ(η^νρJ8^μσ+η^μσJ8^νρ-η^μρJ8^νσ-η^νσJ8^μρ) : ",
  {0} == Union[Flatten[Table[((J8[μ, ν].J8[ρ, σ] - J8[ρ, σ].J8[μ, ν]) -
       (-ⅈ (ημν[[ν, ρ]] J8[μ, σ] + ημν[[μ, σ]] J8[ν, ρ] - ημν[[μ, ρ]] J8[ν, σ] -
            ημν[[ν, σ]] J8[μ, ρ]) )), {μ, 4}, {ν, 4}, {ρ, 4}, {σ, 4}]]]];
Print[" PCMLA Eqn(15). {1,2,J8,JR},  [J8^μν,JR^ρσ] = -ⅈ(η^νρJR^μσ+η^μσJR^νρ-η^μρJR^νσ-η^νσJR^μρ)
    : ", {0} == Union[Flatten[Table[((J8[μ, ν].JR[ρ, σ] - JR[ρ, σ].J8[μ, ν]) -
       (-ⅈ (ημν[[ν, ρ]] JR[μ, σ] + ημν[[μ, σ]] JR[ν, ρ] - ημν[[μ, ρ]] JR[ν, σ] -
            ημν[[ν, σ]] JR[μ, ρ]) )), {μ, 4}, {ν, 4}, {ρ, 4}, {σ, 4}]]]];
Print[" PCMLA Eqn(16). {1,3,J8,J31} [J8^μν,J31^ρσ] =
   -ⅈ(η^νρJ31^μσ+η^μσJ31^νρ-η^μρJ31^νσ-η^νσJ31^μρ) : ",
  {0} == Union[Flatten[Table[(((J8[μ, ν].J31[ρ,  σ] - J31[ρ,  σ].J8[μ, ν])) -
       (-ⅈ (ημν[[ν, ρ]] J31[μ, σ] + ημν[[μ, σ]] J31[ν, ρ] - ημν[[μ, ρ]] J31[ν, σ] -
            ημν[[ν, σ]] J31[μ, ρ]) )), {μ, 4}, {ν, 4}, {ρ, 4}, {σ, 4}]]]]
Print[" PCMLA Eqn(17). {1,4,J8,J42} [J8^μν,J42^ρσ] =
   -ⅈ(η^νρJ42^μσ+η^μσJ42^νρ-η^μρJ42^νσ-η^νσJ42^μρ) : ",
  {0} == Union[Flatten[Table[(((J8[μ, ν].J42[ρ,  σ] - J42[ρ,  σ].J8[μ, ν])) -
       (-ⅈ (ημν[[ν, ρ]] J42[μ, σ] + ημν[[μ, σ]] J42[ν, ρ] - ημν[[μ, ρ]] J42[ν, σ] -
            ημν[[ν, σ]] J42[μ, ρ]) )), {μ, 4}, {ν, 4}, {ρ, 4}, {σ, 4}]]]]
Print[" PCMLA Eqn(18). {1,5,J8,P41},          [J8^μν,P41^ρ] = -ⅈ(η^νρP41^μ-η^μρP41^ν) : ",
  {0} == Union[Flatten[Table[(J8[μ, ν].P41[ρ, kb] - P41[ρ, kb].J8[μ, ν]) -
       (-ⅈ (ημν[[ν, ρ]] P41[μ, kb] - ημν[[μ, ρ]] P41[ν, kb]) ), {μ, 4}, {ν, 4}, {ρ, 4}]]]];
Print[" PCMLA Eqn(19). {1,6,J8,P32},          [J8^μν,P32^ρ] = -ⅈ(η^νρP32^μ-η^μρP32^ν) : ",
  {0} == Union[Flatten[Table[(J8[μ, ν].P32[ρ, kb] - P32[ρ, kb].J8[μ, ν]) -
       (-ⅈ (ημν[[ν, ρ]] P32[μ, kb] - ημν[[μ, ρ]] P32[ν, kb]) ), {μ, 4}, {ν, 4}, {ρ, 4}]]]];
Print[" PCMLA Eqn(20). {1,7,J8,P43},          [J8^μν,P43^ρ] = -ⅈ(η^νρP43^μ-η^μρP43^ν) : ",
  {0} == Union[Flatten[Table[(J8[μ, ν].P43[ρ, kb] - P43[ρ, kb].J8[μ, ν]) -
       (-ⅈ (ημν[[ν, ρ]] P43[μ, kb] - ημν[[μ, ρ]] P43[ν, kb]) ), {μ, 4}, {ν, 4}, {ρ, 4}]]]];
Print[" PCMLA Eqn(21). {1,8,J8,P34},          [J8^μν,P34^ρ] = -ⅈ(η^νρP34^μ-η^μρP34^ν) : ",
  {0} == Union[Flatten[Table[(J8[μ, ν].P34[ρ, kb] - P34[ρ, kb].J8[μ, ν]) -
       (-ⅈ (ημν[[ν, ρ]] P34[μ, kb] - ημν[[μ, ρ]] P34[ν, kb]) ), {μ, 4}, {ν, 4}, {ρ, 4}]]]];
```

```
PCMLA Eqn(14). {1,1,J8,J8} [J8^μν,J8^ρσ] = -ⅈ(η^νρJ8^μσ+η^μσJ8^νρ-η^μρJ8^νσ-η^νσJ8^μρ) : True
PCMLA Eqn(15). {1,2,J8,JR},  [J8^μν,JR^ρσ] = -ⅈ(η^νρJR^μσ+η^μσJR^νρ-η^μρJR^νσ-η^νσJR^μρ)  : True
PCMLA Eqn(16). {1,3,J8,J31} [J8^μν,J31^ρσ] = -ⅈ(η^νρJ31^μσ+η^μσJ31^νρ-η^μρJ31^νσ-η^νσJ31^μρ) : True
PCMLA Eqn(17). {1,4,J8,J42} [J8^μν,J42^ρσ] = -ⅈ(η^νρJ42^μσ+η^μσJ42^νρ-η^μρJ42^νσ-η^νσJ42^μρ) : True
PCMLA Eqn(18). {1,5,J8,P41},           [J8^μν,P41^ρ] = -ⅈ(η^νρP41^μ-η^μρP41^ν) : True
PCMLA Eqn(19). {1,6,J8,P32},           [J8^μν,P32^ρ] = -ⅈ(η^νρP32^μ-η^μρP32^ν) : True
PCMLA Eqn(20). {1,7,J8,P43},           [J8^μν,P43^ρ] = -ⅈ(η^νρP43^μ-η^μρP43^ν) : True
PCMLA Eqn(21). {1,8,J8,P34},           [J8^μν,P34^ρ] = -ⅈ(η^νρP34^μ-η^μρP34^ν) : True
```

[JR,G]:

```
In[94]:= Print[
  " PCMLA Eqn(22). {2,2,JR,JR},  [JR^μν,JR^ρσ] = -ⅈ(η^νρJR^μσ+η^μσJR^νρ-η^μρJR^νσ-η^νσJR^μρ)  : ",
  {0} == Union[Flatten[Table[((JR[μ, ν].JR[ρ, σ] - JR[ρ, σ]. JR[μ, ν]) -
       (-ⅈ (ημν[[ν, ρ]] JR[μ, σ] + ημν[[μ, σ]] JR[ν, ρ] - ημν[[μ, ρ]] JR[ν, σ] -
          ημν[[ν, σ]] JR[μ, ρ]) )), {μ, 4}, {ν, 4}, {ρ, 4}, {σ, 4}]]]];
Print[" PCMLA Eqn(23). {2,3,JR,J31},  [JR^μν,J31^ρσ] = -ⅈ/2(η^νρJ31^μσ + η^μσJ31^νρ
   - η^μρJ31^νσ - η^νσJ31^μρ) + ⅈ/2(η^μρη^νσ-η^μση^νρ)(1-η^μνη^ρσ)D31 - 1/2ϵ^μνρσD31 : ",
  {0} == Union[Flatten[Table[((( JR[μ, ν].J31[ρ,  σ] - J31[ρ,  σ]. JR[μ, ν])) -
       (-ⅈ/2 (ημν[[ν, ρ]] J31[μ, σ] + ημν[[μ, σ]] J31[ν, ρ] - ημν[[μ, ρ]] J31[ν, σ] -
          ημν[[ν, σ]] J31[μ, ρ]) + ⅈ/2 (ημν[[μ, ρ]] ημν[[ν, σ]] -
          ημν[[μ, σ]] ημν[[ν, ρ]]) (1 - ημν[[μ, ν]] ημν[[ρ, σ]]) mD31 -
        1/2 ϵλμνσ[μ, ν, ρ, σ] mD31)), {μ, 4}, {ν, 4}, {ρ, 4}, {σ, 4}]]]];
Print[" PCMLA Eqn(24). {2,4,JR,J42},  [JR^μν,J42^ρσ] = -ⅈ/2(η^νρJ42^μσ + η^μσJ42^νρ
   - η^μρJ42^νσ - η^νσJ42^μρ) + ⅈ/2(η^μρη^νσ-η^μση^νρ)(1-η^μνη^ρσ)D42 - 1/2ϵ^μνρσD42 : ",
  {0} == Union[Flatten[Table[((( JR[μ, ν].J42[ρ,  σ] - J42[ρ,  σ]. JR[μ, ν])) -
       (-ⅈ/2 (ημν[[ν, ρ]] J42[μ, σ] + ημν[[μ, σ]] J42[ν, ρ] - ημν[[μ, ρ]] J42[ν, σ] -
          ημν[[ν, σ]] J42[μ, ρ]) - ⅈ/2 (ημν[[μ, ρ]] ημν[[ν, σ]] -
          ημν[[μ, σ]] ημν[[ν, ρ]]) (1 - ημν[[μ, ν]] ημν[[ρ, σ]]) mD42 -
        1/2 ϵλμνσ[μ, ν, ρ, σ] mD42)), {μ, 4}, {ν, 4}, {ρ, 4}, {σ, 4}]]]];

Print[" PCMLA Eqn(25). {2,5,JR,P41},
```

```
[JR^μν,P41^ρ[kb]] = - i/2 (η^νρ P41^μ[kb]-η^μρ P41^ν[kb]) + 1/2 ϵ^μνρσ P41_σ[kb] : ",
  {0} == Union[Flatten[Simplify[Table[( JR[μ, ν].P41[ρ, kb] - P41[ρ, kb]. JR[μ, ν]) -
        ((- i/2 (ημν[[ν, ρ]] P41[μ, kb] - ημν[[μ, ρ]] P41[ν, kb]) ) + 1/2 Sum[ϵλμνσ[μ, ν, ρ, σ]
            ημν[[σ, τ]] P41[τ, kb], {σ, 4}, {τ, 4}]), {μ, 4}, {ν, 4}, {ρ, 4}]]]]];
Print[" PCMLA Eqn(26). {2,6,JR,P32},          [JR^μν,P32^ρ[kc]] = -
    i/2 (η^νρ P32^μ[kc]-η^μρ P32^ν[kc]) - 1/2 ϵ^μνρσ P32_σ[kc] : ",
  {0} == Union[Flatten[Simplify[Table[( JR[μ, ν].P32[ρ, kc] - P32[ρ, kc]. JR[μ, ν]) -
        ((- i/2 (ημν[[ν, ρ]] P32[μ, kc] - ημν[[μ, ρ]] P32[ν, kc]) ) - 1/2 Sum[ϵλμνσ[μ, ν, ρ,
            σ] ημν[[σ, τ]] P32[τ, kc], {σ, 4}, {τ, 4}]), {μ, 4}, {ν, 4}, {ρ, 4}]]]]];
Print["PCMLA Eqn(27). {2,7,JR,P43},          [JR^μν,P43^ρ[kd]] =
    - i(η^νρ P43^μ[kd]-η^μρ P43^ν[kd]) : ",
  {0} == Union[Flatten[Simplify[Table[( JR[μ, ν].P43[ρ, kd] - P43[ρ, kd]. JR[μ, ν]) -
        (- i (ημν[[ν, ρ]] P43[μ, kd] - ημν[[μ, ρ]] P43[ν, kd]) ) , {μ,
        4}, {ν, 4}, {ρ, 4}]]]]];
Print["PCMLA Eqn(28). {2,8,JR,P34},          [JR^μν,P34^ρ[ke]]
    = - i(η^νρ P34^μ[ke]-η^μρ P34^ν[ke]) : ",
  {0} == Union[Flatten[Simplify[Table[( JR[μ, ν].P34[ρ, ke] - P34[ρ, ke]. JR[μ, ν]) -
        (- i (ημν[[ν, ρ]] P34[μ, ke] - ημν[[μ, ρ]] P34[ν, ke]) ) , {μ,
        4}, {ν, 4}, {ρ, 4}]]]]];
Print[" PCMLA Eqn(29). {2,10,JR,D31},          [JR^μν,D31] = -i/2 J31^μν : ",
  {0} == Union[
    Flatten[Table[( JR[μ, ν].mD31 - mD31. JR[μ, ν]) - (-i/2 J31[μ, ν]), {μ, 4}, {ν, 4}]]]];
Print[" PCMLA Eqn(30). {2,11,JR,D42},          [JR^μν,D42] = +i/2 J42^μν : ", {0} ==
    Union[Flatten[Table[( JR[μ, ν].mD42 - mD42. JR[μ, ν]) - (i/2 J42[μ, ν]), {μ, 4}, {ν, 4}]]]];
```

PCMLA Eqn(22). {2,2,JR,JR}, $[JR^{\mu\nu},JR^{\rho\sigma}] = -i(\eta^{\nu\rho}JR^{\mu\sigma}+\eta^{\mu\sigma}JR^{\nu\rho}-\eta^{\mu\rho}JR^{\nu\sigma}-\eta^{\nu\sigma}JR^{\mu\rho})$: True

PCMLA Eqn(23). {2,3,JR,J31}, $[JR^{\mu\nu},J31^{\rho\sigma}] = -\frac{i}{2}(\eta^{\nu\rho}J31^{\mu\sigma} + \eta^{\mu\sigma}J31^{\nu\rho} - \eta^{\mu\rho}J31^{\nu\sigma} - \eta^{\nu\sigma}J31^{\mu\rho}) + \frac{i}{2}(\eta^{\mu\rho}\eta^{\nu\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho})(1-\eta^{\mu\nu}\eta^{\rho\sigma})D31 - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}D31$: True

PCMLA Eqn(24). {2,4,JR,J42}, $[JR^{\mu\nu},J42^{\rho\sigma}] = -\frac{i}{2}(\eta^{\nu\rho}J42^{\mu\sigma} + \eta^{\mu\sigma}J42^{\nu\rho} - \eta^{\mu\rho}J42^{\nu\sigma} - \eta^{\nu\sigma}J42^{\mu\rho}) + \frac{i}{2}(\eta^{\mu\rho}\eta^{\nu\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho})(1-\eta^{\mu\nu}\eta^{\rho\sigma})D42 - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}D42$: True

PCMLA Eqn(25). {2,5,JR,P41},
$[JR^{\mu\nu},P_{41}{}^{\rho}[kb]] = -\frac{i}{2}(\eta^{\nu\rho}P_{41}{}^{\mu}[kb]-\eta^{\mu\rho}P_{41}{}^{\nu}[kb]) + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_{41\,\sigma}[kb]$: True

PCMLA Eqn(26). {2,6,JR,P32},
$[JR^{\mu\nu},P_{32}{}^{\rho}[kc]] = -\frac{i}{2}(\eta^{\nu\rho}P_{32}{}^{\mu}[kc]-\eta^{\mu\rho}P_{32}{}^{\nu}[kc])-\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_{32\,\sigma}[kc]$: True

PCMLA Eqn(27). {2,7,JR,P43}, $[JR^{\mu\nu},P_{43}{}^{\rho}[kd]] = -i(\eta^{\nu\rho}P_{43}{}^{\mu}[kd]-\eta^{\mu\rho}P_{43}{}^{\nu}[kd])$: True

PCMLA Eqn(28). {2,8,JR,P34}, $[JR^{\mu\nu},P_{34}{}^{\rho}[ke]] = -i(\eta^{\nu\rho}P_{34}{}^{\mu}[ke]-\eta^{\mu\rho}P_{34}{}^{\nu}[ke])$: True

PCMLA Eqn(29). {2,10,JR,D31}, $[JR^{\mu\nu},D31] = -\frac{i}{2}J31^{\mu\nu}$: True

PCMLA Eqn(30). {2,11,JR,D42}, $[JR^{\mu\nu},D42] = +\frac{i}{2}J42^{\mu\nu}$: True

[J31,G] and [J42,G]:

In[103]:=
```
Print[" PCMLA Eqn(31). {3,7,J31,P43}
    [J31^μν,P43^ρ[kd]] = -1/2 i(η^νρ P41^μ[ kd]-η^μρ P41^ν[ kd]) - 1/2 ϵ^μνρσ P41σ[ kd]    : ",
  {0} == Union[Flatten[Simplify[Table[ (J31[μ, ν] .P43[ρ, kd] - P43[ρ, kd]. J31[μ, ν] ) -
       (-i 1/2 (ημν[[ν, ρ]] P41[μ, kd] - ημν[[μ, ρ]] P41[ν, kd]) - 1/2 Sum[ϵλμνσ[μ, ν, ρ, σ]
          ημν[[σ, τ]] P41[τ, kd], {σ, 4}, {τ, 4}] ), {μ, 4}, {ν, 4}, {ρ, 4}]]]]];

Print[" PCMLA Eqn(32). {3,9,J31,D}           [J31^μν,D] = -i/2 J31^μν    : ",
  {0} == Union[Flatten[
    Simplify[Table[ (J31[μ, ν] .mD - mD. J31[μ, ν] ) - (-i/2 J31[μ, ν]), {μ, 4}, {ν, 4}]]]]];
```

PCMLA Eqn(31). {3,7,J31,P43}
$[J31^{\mu\nu},P_{43}{}^{\rho}[kd]] = -\frac{1}{2}i(\eta^{\nu\rho}P_{41}{}^{\mu}[\ kd]-\eta^{\mu\rho}P_{41}{}^{\nu}[\ kd]) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_{41\,\sigma}[\ kd]$: True

PCMLA Eqn(32). {3,9,J31,D} $[J31^{\mu\nu},D] = -\frac{i}{2}\ J31^{\mu\nu}$: True

In[105]:=
```
Print[" PCMLA Eqn(33). {4,8,J42,P34}
    [J42^μν,P34^ρ[ke]] = -1/2 i(η^νρ P32^μ[ ke]-η^μρ P32^ν[ ke]) + 1/2 ϵ^μνρσ P32σ[ ke]    : ",
  {0} == Union[Flatten[Simplify[Table[ (J42[μ, ν] .P34[ρ, ke] - P34[ρ, ke]. J42[μ, ν] ) -
       (-i 1/2 (ημν[[ν, ρ]] P32[μ, ke] - ημν[[μ, ρ]] P32[ν, ke]) + 1/2 Sum[ϵλμνσ[μ, ν, ρ, σ]
          ημν[[σ, τ]] P32[τ, ke], {σ, 4}, {τ, 4}] ), {μ, 4}, {ν, 4}, {ρ, 4}]]]]];

Print[" PCMLA Eqn(34). {4,9,J42,D}           [J42^μν,D] = +i/2 J42^μν    : ",
  {0} == Union[Flatten[
    Simplify[Table[ (J42[μ, ν] .mD - mD. J42[μ, ν] ) - (i/2 J42[μ, ν]), {μ, 4}, {ν, 4}]]]]];
```

```
PCMLA Eqn(33). {4,8,J42,P34}
  [J42^μν,P34^ρ[ke]] = -1/2 i (η^νρ P32^μ[ ke]-η^μρ P32^ν[ ke]) + 1/2 ε^μνρσ P32σ[ ke]    : True

PCMLA Eqn(34). {4,9,J42,D}            [J42^μν,D] = +i/2 J42^μν    : True
```

[P41,G] and [P32,G]:

```
In[107]:= Print[" PCMLA Eqn(35). {5,8,P41,P34},
      [P41^μ[kb],P34^ν[ke]]  =  -2ikb ke ( η^μν D31 - J31^μν)    : ",
    {0} == Union[Flatten[Simplify[Table[(P41[μ, kb].P34[ν, ke] - P34[ν, ke].P41[μ, kb]) -
          (- 2 i kb ke ( mD31 ημν[[μ, ν]] -  J31[μ, ν])) , {μ, 4}, {ν, 4}]]]]];
  Print[" PCMLA Eqn(36). {5,9,P41,D},            [P41^μ[kb],D]  =  +i 1/2 P41^μ[kb]    : ",
    {0} ==  Union[Flatten[Simplify[
        Table[(P41[μ, kb].mD - mD.P41[μ, kb]) - (+i 1/2 P41[μ, kb]) , {μ, 4}, {ν, 4}]]]]];

PCMLA Eqn(35). {5,8,P41,P34},
  [P41^μ[kb],P34^ν[ke]]  =  -2ikb ke ( η^μν D31 - J31^μν)    : True

PCMLA Eqn(36). {5,9,P41,D},            [P41^μ[kb],D]  =  +i 1/2 P41^μ[kb]    : True

In[109]:= Print[" PCMLA Eqn(37). {6,7,P32,P43},
      [P32^μ[kc],P43^ν[kd]]  =  +2ikc kd ( η^μν D42 + J42^μν)    : ",
    {0} == Union[Flatten[Simplify[Table[(P32[μ, kc].P43[ν, kd] - P43[ν, kd].P32[μ, kc]) -
          (+2 i kc kd ( mD42 ημν[[μ, ν]] +  J42[μ, ν])) , {μ, 4}, {ν, 4}]]]]];
  Print[" PCMLA Eqn(38). {6,9,P32,D},            [P32^μ[kc],D]  =  - 1/2 i P32^μ[kc]    : ",
    {0} ==  Union[Flatten[Simplify[
        Table[(P32[μ, kc].mD - mD.P32[μ, kc]) - (- 1/2 i P32[μ, kc]) , {μ, 4}, {ν, 4}]]]]];

PCMLA Eqn(37). {6,7,P32,P43},
  [P32^μ[kc],P43^ν[kd]]  =  +2ikc kd ( η^μν D42 + J42^μν)    : True

PCMLA Eqn(38). {6,9,P32,D},            [P32^μ[kc],D]  =  - 1/2 i P32^μ[kc]    : True
```

[P43,G] and [P34,G]:

In[111]:=
```
Print[" PCMLA Eqn(39). {7,8,P43,P34},
    [P43^μ[kd],P34^ν[ke]]  =  +2ⅈkd ke ( η^μν D + JR^μν)   : ",
  {0} == Union[Flatten[Simplify[Table[(P43[μ, kd].P34[ν, ke] - P34[ν, ke].P43[μ, kd]) -
      (+ 2 ⅈ kd ke ( mD ημν[[μ, ν]] + JR[μ, ν])) , {μ, 4}, {ν, 4}]]]]];
Print[" PCMLA Eqn(40). {7,9,P43,D},            [P43^μ[kd],D]  =  +ⅈ P43^μ[kd]   : ",
  {0} == Union[Flatten[Simplify[
     Table[(P43[μ, kd].mD - mD.P43[μ, kd]) - (+ⅈ P43[μ, kd] ) , {μ, 4}, {ν, 4}]]]]];
Print[" PCMLA Eqn(41). {7,10,P43,D31},            [P43^μ[kd],D31]
    =  - 1/2 ⅈ P41^μ[kd]   : ", {0} == Union[Flatten[Simplify[
     Table[(P43[μ, kd].mD31 - mD31.P43[μ, kd]) - (- 1/2 ⅈ P41[μ, kd] ) , {μ, 4}, {ν, 4}]]]]];
```

PCMLA Eqn(39). {7,8,P43,P34}, [P43^μ[kd],P34^ν[ke]] = +2ⅈkd ke (η^μν D + JR^μν) : True

PCMLA Eqn(40). {7,9,P43,D}, [P43^μ[kd],D] = +ⅈ P43^μ[kd] : True

PCMLA Eqn(41). {7,10,P43,D31}, [P43^μ[kd],D31] = − 1/2 ⅈ P41^μ[kd] : True

In[114]:=
```
Print[" PCMLA Eqn(42). {8,9,P34,D},            [P34^μ[ke],D]  =  - ⅈ P34^μ[ke]   : ",
  {0} == Union[Flatten[Simplify[
     Table[(P34[μ, ke].mD - mD.P34[μ, ke]) - (-ⅈ P34[μ, ke] ) , {μ, 4}, {ν, 4}]]]]];
Print[" PCMLA Eqn(43). {8,11,P34,D42},            [P34^μ[ke],D42]
    =  + 1/2 ⅈP32^μ[ke]   : ", {0} == Union[Flatten[Simplify[
     Table[(P34[μ, ke].mD42 - mD42.P34[μ, ke]) - (+ 1/2 ⅈ P32[μ, ke] ) , {μ, 4}, {ν, 4}]]]]];
```

PCMLA Eqn(42). {8,9,P34,D}, [P34^μ[ke],D] = − ⅈ P34^μ[ke] : True

PCMLA Eqn(43). {8,11,P34,D42}, [P34^μ[ke],D42] = + 1/2 ⅈP32^μ[ke] : True

[D,G]:

In[116]:=
```
Print[" PCMLA Eqn(44) {9,10,D,D31},            [D,D31]  =  + 1/2 ⅈD31  : ",
  {0} == Union[Flatten[Simplify[(mD.mD31 - mD31.mD) - (+ 1/2 ⅈ mD31) ]]]];
Print[" PCMLA Eqn(45). {9,11,D,D42},            [D,D42]  =  - 1/2 ⅈD42  : ",
  {0} == Union[Flatten[Simplify[(mD.mD42 - mD42.mD) - (- 1/2 ⅈ mD42) ]]]];
```

PCMLA Eqn(44) {9,10,D,D31}, [D,D31] = + 1/2 ⅈD31 : True

PCMLA Eqn(45). {9,11,D,D42}, [D,D42] = − 1/2 ⅈD42 : True

That completes the list of non-commuting generators forming a closed Lie algebra. The other commutators vanish.

Sub-Algebras

Lorentz Sub-Algebras

In[118]:=
```
Print[" The J8^μν matrices represent the Lorentz algebra
   because [J8^μν,J8^ρσ] = -ⅈ(η^νρJ8^μσ+η^μσJ8^νρ-η^μρJ8^νσ-η^νσJ8^μρ) : ",
  {0} == Union[Flatten[Table[((J8[μ, ν].J8[ρ, σ] - J8[ρ, σ].J8[μ, ν]) -
       (-ⅈ (ημν[[ν, ρ]] J8[μ, σ] + ημν[[μ, σ]] J8[ν, ρ] - ημν[[μ, ρ]] J8[ν, σ] -
           ημν[[ν, σ]] J8[μ, ρ]))), {μ, 4}, {ν, 4}, {ρ, 4}, {σ, 4}]]]];
Print[" The JR^μν matrices represent the Lorentz algebra because
   [JR^μν,JR^ρσ] = -ⅈ(η^νρJR^μσ+η^μσJR^νρ-η^μρJR^νσ-η^νσJR^μρ)  : ",
  {0} == Union[Flatten[Table[((JR[μ, ν].JR[ρ, σ] - JR[ρ, σ]. JR[μ, ν]) -
       (-ⅈ (ημν[[ν, ρ]] JR[μ, σ] + ημν[[μ, σ]] JR[ν, ρ] - ημν[[μ, ρ]] JR[ν, σ] -
           ημν[[ν, σ]] JR[μ, ρ]))), {μ, 4}, {ν, 4}, {ρ, 4}, {σ, 4}]]]];
```

```
The J8^μν matrices represent the Lorentz algebra
  because [J8^μν,J8^ρσ] = -ⅈ(η^νρJ8^μσ+η^μσJ8^νρ-η^μρJ8^νσ-η^νσJ8^μρ) : True

The JR^μν matrices represent the Lorentz algebra
  because  [JR^μν,JR^ρσ] = -ⅈ(η^νρJR^μσ+η^μσJR^νρ-η^μρJR^νσ-η^νσJR^μρ)  : True
```

Poincare Sub-Algebras

In[120]:=
```
Print[" The J8^μν matrices represent the Lorentz algebra because
   [J8^μν,J8^ρσ] = -ⅈ(η^νρJ8^μσ+η^μσJ8^νρ-η^μρJ8^νσ-η^νσJ8^μρ), PCMLA Eqn(47) : ",
  {0} == Union[Flatten[Table[((J8[μ, ν].J8[ρ, σ] - J8[ρ, σ].J8[μ, ν]) -
       (-ⅈ (ημν[[ν, ρ]] J8[μ, σ] + ημν[[μ, σ]] J8[ν, ρ] - ημν[[μ, ρ]] J8[ν, σ] -
           ημν[[ν, σ]] J8[μ, ρ]))), {μ, 4}, {ν, 4}, {ρ, 4}, {σ, 4}]]]];
P8[μ_, kb_, kc_] := P41[μ, kb] + P32[μ, kc]
Print["The J8^μν and P8^ρ, for P8^μ = P41^μ + P32^μ, PCMLA Eqn(46),  "]
Print[" are a rep of the Poincare algebra because of the commutation
   relations   [J8^μν,P_8^ρ] = -ⅈ(η^νρP_8^μ-η^μρP_8^ν), PCMLA Eqn(48) : ",
  {0} == Union[Flatten[Table[(J8[μ, ν].P8[ρ, kb, kc] - P8[ρ, kb, kc].J8[μ, ν]) -
      (-ⅈ (ημν[[ν, ρ]] P8[μ, kb, kc] - ημν[[μ, ρ]] P8[ν, kb, kc])),
     {μ, 4}, {ν, 4}, {ρ, 4}]]], ", and"];
Print["All the commutators [P8^μ,P8^ν] vanish, PCMLA Eqn(49): ", {0} == Union[Flatten[
   Table[(P8[μ, kb, kc].P8[ν, kb, kc] - P8[ν, kb, kc].P8[μ, kb, kc]), {μ, 4}, {ν, 4}]]]]
Print["Thus J8^μν and P8^μ obey the commutation rules of the Poincare algebra. "]
Print["Note that P8^μ = P41^μ + P32^μ is an arbitrary linear
  combination of P41 and P32, P8^μ = α P41'^μ + β P32'^μ  if you give kb =
  α kb' and kc = β kc' new values. You can do that because kb and kc are
  arbitrary: the commutation relations are homogeneous in kb and in kc. "]
```

The $J8^{\mu\nu}$ matrices represent the Lorentz algebra because
$[J8^{\mu\nu},J8^{\rho\sigma}] = -i(\eta^{\nu\rho}J8^{\mu\sigma}+\eta^{\mu\sigma}J8^{\nu\rho}-\eta^{\mu\rho}J8^{\nu\sigma}-\eta^{\nu\sigma}J8^{\mu\rho})$, PCMLA Eqn(47) : True

The $J8^{\mu\nu}$ and $P8^{\rho}$, for $P8^{\mu} = P41^{\mu} + P32^{\mu}$, PCMLA Eqn(46),

are a rep of the Poincare algebra because of the commutation relations $[J8^{\mu\nu},P_8{}^{\rho}] = -i(\eta^{\nu\rho}P_8{}^{\mu}-\eta^{\mu\rho}P_8{}^{\nu})$, PCMLA Eqn(48) : True, and

All the commutators $[P8^{\mu},P8^{\nu}]$ vanish, PCMLA Eqn(49): True

Thus $J8^{\mu\nu}$ and $P8^{\mu}$ obey the commutation rules of the Poincare algebra.

Note that $P8^{\mu} = P41^{\mu} + P32^{\mu}$ is an arbitrary linear combination of P41 and P32, $P8^{\mu} = \alpha\, P41'^{\mu} + \beta\, P32'^{\mu}$ if you give kb = α kb' and kc = β kc' new values. You can do that because kb and kc are arbitrary: the commutation relations are homogeneous in kb and in kc.

The $J8^{\mu\nu}$ and $P8^{\rho}$, for $P8^{\mu} = P41^{\mu} + P32^{\mu}$, PCMLA Eqn(46),

are a rep of the Poincare algebra because of the commutation relations $[J8^{\mu\nu},P_8{}^{\rho}] = -i(\eta^{\nu\rho}P_8{}^{\mu}-\eta^{\mu\rho}P_8{}^{\nu})$, PCAB88M Eqn(48) : True, and

All the commutators $[P8^{\mu},P8^{\nu}]$ vanish, PCAB88M Eqn(49): True

Thus $J8^{\mu\nu}$ and $P8^{\mu}$ obey the commutation rules of the Poincare algebra.

Note that $P8^{\mu} = P41^{\mu} + P32^{\mu}$ is an arbitrary linear combination of P41 and P32, $P8^{\mu} = \alpha\, P41'^{\mu} + \beta\, P32'^{\mu}$ if you give kb = α kb' and kc = β kc' new values. You can do that because kb and kc are arbitrary: the commutation relations are homogeneous in kb and in kc.

In[127]:=
```
Print[" The JR^μν matrices represent the Lorentz algebra because
    [JR^μν,JR^ρσ] = -i(η^νρ JR^μσ+η^μσ JR^νρ-η^μρ JR^νσ-η^νσ JR^μρ), PCMLA Eqn(50) : ",
  {0} == Union[Flatten[Table[((JR[μ, ν].JR[ρ, σ] - JR[ρ, σ]. JR[μ, ν]) -
        (-i (ημν[[ν, ρ]] JR[μ, σ] + ημν[[μ, σ]] JR[ν, ρ] - ημν[[μ, ρ]] JR[ν, σ] -
             ημν[[ν, σ]] JR[μ, ρ]) )), {μ, 4}, {ν, 4}, {ρ, 4}, {σ, 4}]]]];
Print["The JR^μν and P43^ρ  form a rep of the Poincare algebra because of the
    commutation relations   [JR^μν,P43^ρ] = -i(η^νρ P43^μ-η^μρ P43^ν), PCMLA Eqn(51): ",
  {0} == Union[Flatten[Simplify[Table[( JR[μ, ν].P43[ρ, kd] - P43[ρ, kd]. JR[μ, ν]) -
        (- i (ημν[[ν, ρ]] P43[μ, kd] - ημν[[μ, ρ]] P43[ν, kd]) ) ,
      {μ, 4}, {ν, 4}, {ρ, 4}]]]], ", and"];
Print["all the commutators [P43^μ,P43^ν] vanish, PCMLA Eqn(52): ", {0} ==
  Union[Flatten[Table[(P43[μ, kd].P43[ν, kd] - P43[ν, kd].P43[μ, kd]), {μ, 4}, {ν, 4}]]]]
Print[" Thus JR^μν and P43^μ obey the commutation rules of the Poincare algebra. "]
```

The $JR^{\mu\nu}$ matrices represent the Lorentz algebra because
$[JR^{\mu\nu},JR^{\rho\sigma}] = -i(\eta^{\nu\rho}JR^{\mu\sigma}+\eta^{\mu\sigma}JR^{\nu\rho}-\eta^{\mu\rho}JR^{\nu\sigma}-\eta^{\nu\sigma}JR^{\mu\rho})$, PCMLA Eqn(50) : True

The $JR^{\mu\nu}$ and $P43^{\rho}$ form a rep of the Poincare algebra because of the commutation relations $[JR^{\mu\nu},P_{43}{}^{\rho}] = -i(\eta^{\nu\rho}P_{43}{}^{\mu}-\eta^{\mu\rho}P_{43}{}^{\nu})$, PCMLA Eqn(51): True, and

all the commutators $[P43^{\mu},P43^{\nu}]$ vanish, PCMLA Eqn(52): True

Thus $JR^{\mu\nu}$ and $P43^{\mu}$ obey the commutation rules of the Poincare algebra.

```
In[131]:= Print[" PCMLA Eqn(53). The JR^μν matrices represent the Lorentz
     algebra because [JR^μν,JR^ρσ] = -ⅈ(η^νρJR^μσ+η^μσJR^νρ-η^μρJR^νσ-η^νσJR^μρ) : ",
   {0} == Union[Flatten[Table[((JR[μ, ν].JR[ρ, σ] - JR[ρ, σ]. JR[μ, ν]) -
        (-ⅈ (ημν[[ν, ρ]] JR[μ, σ] + ημν[[μ, σ]] JR[ν, ρ] - ημν[[μ, ρ]] JR[ν, σ] -
            ημν[[ν, σ]] JR[μ, ρ]))), {μ, 4}, {ν, 4}, {ρ, 4}, {σ, 4}]]]];
Print[" PCMLA Eqn(54). The JR^μν and P34^ρ  form a rep of the Poincare algebra
     because of the commutation relations   [JR^μν,P34^ρ] = -ⅈ(η^νρP34^μ-η^μρP34^ν) : ",
   {0} == Union[Flatten[Simplify[Table[( JR[μ, ν].P34[ρ, kd] - P34[ρ, kd]. JR[μ, ν]) -
        (- ⅈ (ημν[[ν, ρ]] P34[μ, kd] - ημν[[μ, ρ]] P34[ν, kd]) ) ,
      {μ, 4}, {ν, 4}, {ρ, 4}]]]], ", and"];
Print["all the commutators [P34^μ,P34^ν] vanish, PCMLA Eqn(55): ", {0} ==
   Union[Flatten[Table[(P34[μ, ke].P34[ν, ke] - P34[ν, ke].P34[μ, ke]), {μ, 4}, {ν, 4}]]]]
Print[" Thus JR^μν and P34^μ obey the commutation rules of the Poincare algebra. "]
```

PCMLA Eqn(53). The JR$^{\mu\nu}$ matrices represent the Lorentz algebra because [JR$^{\mu\nu}$,JR$^{\rho\sigma}$] = $-i(\eta^{\nu\rho}$JR$^{\mu\sigma}+\eta^{\mu\sigma}JR^{\nu\rho}-\eta^{\mu\rho}JR^{\nu\sigma}-\eta^{\nu\sigma}JR^{\mu\rho}$) : True

PCMLA Eqn(54). The JR$^{\mu\nu}$ and P34$^{\rho}$ form a rep of the Poincare algebra because of the commutation relations [JR$^{\mu\nu}$,$P_{34}{}^{\rho}$] = $-i(\eta^{\nu\rho}P_{34}{}^{\mu}-\eta^{\mu\rho}P_{34}{}^{\nu}$) : True, and

all the commutators [P34$^{\mu}$,P34$^{\nu}$] vanish, PCMLA Eqn(55): True

Thus JR$^{\mu\nu}$ and P34$^{\mu}$ obey the commutation rules of the Poincare algebra.

Conformal Sub-Algebra

```
In[135]:= Print[" PCMLA Eqn(56). The JR^μν matrices represent the Lorentz
      algebra because [JR^μν,JR^ρσ] = -i(η^νρJR^μσ+η^μσJR^νρ-η^μρJR^νσ-η^νσJR^μρ) : ",
    {0} == Union[Flatten[Table[((JR[μ, ν].JR[ρ, σ] - JR[ρ, σ]. JR[μ, ν]) -
          (-i (ημν[[ν, ρ]] JR[μ, σ] + ημν[[μ, σ]] JR[ν, ρ] - ημν[[μ, ρ]] JR[ν, σ] -
              ημν[[ν, σ]] JR[μ, ρ]) )), {μ, 4}, {ν, 4}, {ρ, 4}, {σ, 4}]]]];
  Print[" PCMLA Eqn(57). The JR^μν and P43^ρ obey [JR^μν,P43^ρ] = - i(η^νρP43^μ-η^μρP43^ν) : ",
    {0} == Union[Flatten[Simplify[Table[( JR[μ, ν].P43[ρ, kd] - P43[ρ, kd]. JR[μ, ν]) -
          (- i (ημν[[ν, ρ]] P43[μ, kd] - ημν[[μ, ρ]] P43[ν, kd]) ) , {μ,
          4}, {ν, 4}, {ρ, 4}]]]]];
  Print["  PCMLA Eqn(57).  The JR^μν and P34^ρ obey  [JR^μν,P34^ρ] = - i(η^νρP34^μ-η^μρP34^ν) : ",
    {0} == Union[Flatten[Simplify[Table[( JR[μ, ν].P34[ρ, ke] - P34[ρ, ke]. JR[μ, ν]) -
          (- i (ημν[[ν, ρ]] P34[μ, ke] - ημν[[μ, ρ]] P34[ν, ke]) ) , {μ,
          4}, {ν, 4}, {ρ, 4}]]]]];
  Print["  PCMLA Eqn(58). The conformal algebra comm. relations for P34^μ and
      P43^ν when kd ke = 1,       [P43^μ,P34^ν]  =  +2ikd ke ( η^μνD + JR^μν)   : ",
    {0} == Union[Flatten[Simplify[Table[(P43[μ, kd].P34[ν, ke] - P34[ν, ke].P43[μ, kd]) -
          (+ 2 i kd ke ( mD ημν[[μ, ν]] +  JR[μ,  ν])) , {μ, 4}, {ν, 4}]]]]];
  Print["  PCMLA Eqn(59).  The conformal algebra comm. relations for P43^μ and
      D,           [P43^μ,D]  =  +iP43^μ   : ", {0} == Union[Flatten[Simplify[
        Table[(P43[μ, kd].mD - mD.P43[μ, kd]) - (+i P43[μ, kd] ) , {μ, 4}, {ν, 4}]]]]];
  Print[" PCMLA Eqn(59). The conformal algebra comm. relations for P34^ν  and
      D,         [P34^μ,D]  =  -iP34^μ   : ", {0} == Union[Flatten[Simplify[
        Table[(P34[μ, ke].mD - mD.P34[μ, ke]) - (-i P34[μ, ke] ) , {μ, 4}, {ν, 4}]]]]];
  Print["  PCMLA Eqn(60).  The JR^μν, P43^ρ, P34^ρ, D obey  [JR^μν,D]
      = 0, [P43^μ,P43^ν] = 0, [P34^μ,P34^ν] = 0 : ",
    {0} == Union[Flatten[Simplify[Table[( JR[μ, ν].mD - mD. JR[μ, ν]) , {μ, 4}, {ν, 4}]]]],
    " , ", {0} == Union[Flatten[
       Simplify[Table[( P43[μ, ke].P43[ν, ke] - P43[ν, ke]. P43[μ, ke]) , {μ, 4}, {ν, 4}]]]],
    " , ", {0} == Union[Flatten[Simplify[
        Table[( P34[μ, ke].P34[ν, ke] - P34[ν, ke]. P34[μ, ke]) , {μ, 4}, {ν, 4}, {ρ, 4}]]]]];
  Print["Thus JR^μν, P43^μ, P34^μ, and D obey the commutation rules of the conformal
     algebra when kd ke = 1, which can be arranged since kd and ke are
     arbitrary. The commutation relations are homogeneous in kd and ke."]
```

PCMLA Eqn(56). The $JR^{\mu\nu}$ matrices represent the Lorentz
algebra because $[JR^{\mu\nu}, JR^{\rho\sigma}] = -i(\eta^{\nu\rho}JR^{\mu\sigma} + \eta^{\mu\sigma}JR^{\nu\rho} - \eta^{\mu\rho}JR^{\nu\sigma} - \eta^{\nu\sigma}JR^{\mu\rho})$: True

PCMLA Eqn(57). The $JR^{\mu\nu}$ and $P43^{\rho}$ obey $[JR^{\mu\nu}, P_{43}{}^{\rho}] = -i(\eta^{\nu\rho}P_{43}{}^{\mu} - \eta^{\mu\rho}P_{43}{}^{\nu})$: True

PCMLA Eqn(57). The $JR^{\mu\nu}$ and $P34^{\rho}$ obey $[JR^{\mu\nu}, P_{34}{}^{\rho}] = -i(\eta^{\nu\rho}P_{34}{}^{\mu} - \eta^{\mu\rho}P_{34}{}^{\nu})$: True

PCMLA Eqn(58). The conformal algebra comm. relations for $P34^{\mu}$ and
$P43^{\nu}$ when kd ke = 1, $[P43^{\mu}, P34^{\nu}] = +2i\,kd\,ke\,(\eta^{\mu\nu}D + JR^{\mu\nu})$: True

PCMLA Eqn(59). The conformal algebra comm.
relations for $P43^{\mu}$ and D, $[P43^{\mu}, D] = +iP43^{\mu}$: True

PCMLA Eqn(59). The conformal algebra comm.
relations for $P34^{\nu}$ and D, $[P34^{\mu}, D] = -iP34^{\mu}$: True

PCMLA Eqn(60). The $JR^{\mu\nu}$, $P43^{\rho}$, $P34^{\rho}$, D obey
$[JR^{\mu\nu}, D] = 0$, $[P43^{\mu}, P43^{\nu}] = 0$, $[P34^{\mu}, P34^{\nu}] = 0$: True , True , True

Thus $JR^{\mu\nu}$, $P43^{\mu}$, $P34^{\mu}$, and D obey the commutation rules of
the conformal algebra when kd ke = 1, which can be arranged since kd and
ke are arbitrary. The commutation relations are homogeneous in kd and ke.

Commuting generators:

In[143]:=
```
Print["Family ID Numbers and Names of Pairs of Commuting Generator Families: "]
Table[
  If[commutingGENbi[[n]][[1]] ≤ commutingGENbi[[n]][[2]], {commutingGENbi[[n]][[1]],
    commutingGENbi[[n]][[2]], genNAME[[commutingGENbi[[n]][[1]]]]
    , genNAME[[commutingGENbi[[n]][[2]]]]} ]
  , {n, Length[commutingGENbi]}];
Complement[%, {Null}]
Length[%];
```

Family ID Numbers and Names of Pairs of Commuting Generator Families:

Out[145]=
```
{{1, 9, J8, D}, {1, 10, J8, D31}, {1, 11, J8, D42}, {2, 9, JR, D}, {3, 3, J31, J31},
 {3, 4, J31, J42}, {3, 5, J31, P41}, {3, 6, J31, P32}, {3, 8, J31, P34}, {3, 10, J31, D31},
 {3, 11, J31, D42}, {4, 4, J42, J42}, {4, 5, J42, P41}, {4, 6, J42, P32}, {4, 7, J42, P43},
 {4, 10, J42, D31}, {4, 11, J42, D42}, {5, 5, P41, P41}, {5, 6, P41, P32}, {5, 7, P41, P43},
 {5, 10, P41, D31}, {5, 11, P41, D42}, {6, 6, P32, P32}, {6, 8, P32, P34}, {6, 10, P32, D31},
 {6, 11, P32, D42}, {7, 7, P43, P43}, {7, 11, P43, D42}, {8, 8, P34, P34}, {8, 10, P34, D31},
 {9, 9, D, D}, {10, 10, D31, D31}, {10, 11, D31, D42}, {11, 11, D42, D42}}
```

Abelian Subalgebras

In[147]:=
```
Print["Check the list of Families of Commuting Generators: ",
 commutingGEN, " . Check: ", Union[Flatten[Table[
    Union[Flatten[Table[gen[[commutingGEN[[i, 1]], m]].gen[[commutingGEN[[i, 2]], n]] -
        gen[[commutingGEN[[i, 2]], n]].gen[[commutingGEN[[i, 1]], m]],
       {m, Length[gen[[commutingGEN[[i, 1]]]] ]},
       {n, Length[gen[[commutingGEN[[i, 2]]]] ]}]]], {i, Length[commutingGEN]}]]] == {0}]
```

Check the list of Families of Commuting Generators:
```
{{1, 9}, {1, 10}, {1, 11}, {2, 9}, {3, 3}, {3, 4}, {3, 5}, {3, 6}, {3, 8},
 {3, 10}, {3, 11}, {4, 4}, {4, 5}, {4, 6}, {4, 7}, {4, 10}, {4, 11}, {5, 5},
 {5, 6}, {5, 7}, {5, 10}, {5, 11}, {6, 6}, {6, 8}, {6, 10}, {6, 11}, {7, 7},
 {7, 11}, {8, 8}, {8, 10}, {9, 9}, {10, 10}, {10, 11}, {11, 11}} . Check: True
```

If a family of generators is part of an Abelian subalgebra, the members of the family must commute with each other. Thus, by inspection of the list “commutingGEN”, one sees that families (3,4,5,6,7,8,9,10,11) can be in an Abelian subalgebra. Define the list “abPOSSIBLE” = {3,4,5,6,7,8,9,10,11}.

More than one family can be an Abelian subalgebra if all the matrices in all the families commute. Thus any subset of “abPOSSIBLE” = {3,4,5,6,7,8,9,10,11} is a potential Abelian subalgebra. Call the list of subsets “subabPOSSIBLE “.

In[148]:=
```
abPOSSIBLE = {3, 4, 5, 6, 7, 8, 9, 10, 11};
subabPOSSIBLE = Subsets[abPOSSIBLE];
Print["There are ", Length[subabPOSSIBLE],
 " possible sets of generator families that can form a Abelian subalgebra."]
```

```
There are 512 possible sets of generator families that can form a Abelian subalgebra.
```

In[151]:=
```
(*Find the Abelian subalgebras.*)
abSUBalgebras = {};
For[  n = 1, n ≤ Length[subabPOSSIBLE], n++, If[
  Union[Flatten[Table[MemberQ[ commutingGENbi, {i, j}],
      {i, subabPOSSIBLE[[n]]}, {j, subabPOSSIBLE[[n]]}]]] == {True},
  AppendTo[ abSUBalgebras, subabPOSSIBLE[[n]] ]  ]  ]
Clear[n]
abSUBalgebras;
```

In[155]:=
```
Print["There are ", Length[abSUBalgebras],
 " Abelian subalgebras if you count families with just one matrix
   like D, D31, D42. If you don't count the singles, then there are ",
 Length[abSUBalgebras] - 3, "  Abelian subalgebras."]
```

```
There are 80
  Abelian subalgebras if you count families with just one matrix like D, D31, D42. If you
   don't count the singles, then there are 77  Abelian subalgebras.
```

In[156]:=
```
subALGEBRAtable = Table[{n, abSUBalgebras[[n]], Table[genNAME[[ abSUBalgebras[[n, m ]] ]],
     {m, Length[abSUBalgebras[[n]] ]}]}, {n, Length[abSUBalgebras]}  ];
Print["The Abelian subalgebras are listed with (A) index number for
   reference, (B) list of families in the subalgebra by numerical
   ID, (C) list of families in the subalgebra by name:"]
subALGEBRAtable
```

```
The Abelian subalgebras are listed with (A) index number for reference, (B) list of families
  in the subalgebra by numerical ID, (C) list of families in the subalgebra by name:
```

Out[158]=
```
{{1, {3}, {J31}}, {2, {4}, {J42}}, {3, {5}, {P41}}, {4, {6}, {P32}},
 {5, {7}, {P43}}, {6, {8}, {P34}}, {7, {9}, {D}}, {8, {10}, {D31}}, {9, {11}, {D42}},
 {10, {3, 4}, {J31, J42}}, {11, {3, 5}, {J31, P41}}, {12, {3, 6}, {J31, P32}},
 {13, {3, 8}, {J31, P34}}, {14, {3, 10}, {J31, D31}}, {15, {3, 11}, {J31, D42}},
 {16, {4, 5}, {J42, P41}}, {17, {4, 6}, {J42, P32}}, {18, {4, 7}, {J42, P43}},
 {19, {4, 10}, {J42, D31}}, {20, {4, 11}, {J42, D42}}, {21, {5, 6}, {P41, P32}},
 {22, {5, 7}, {P41, P43}}, {23, {5, 10}, {P41, D31}}, {24, {5, 11}, {P41, D42}},
 {25, {6, 8}, {P32, P34}}, {26, {6, 10}, {P32, D31}}, {27, {6, 11}, {P32, D42}},
 {28, {7, 11}, {P43, D42}}, {29, {8, 10}, {P34, D31}}, {30, {10, 11}, {D31, D42}},
 {31, {3, 4, 5}, {J31, J42, P41}}, {32, {3, 4, 6}, {J31, J42, P32}},
 {33, {3, 4, 10}, {J31, J42, D31}}, {34, {3, 4, 11}, {J31, J42, D42}},
 {35, {3, 5, 6}, {J31, P41, P32}}, {36, {3, 5, 10}, {J31, P41, D31}},
 {37, {3, 5, 11}, {J31, P41, D42}}, {38, {3, 6, 8}, {J31, P32, P34}},
 {39, {3, 6, 10}, {J31, P32, D31}}, {40, {3, 6, 11}, {J31, P32, D42}},
 {41, {3, 8, 10}, {J31, P34, D31}}, {42, {3, 10, 11}, {J31, D31, D42}},
 {43, {4, 5, 6}, {J42, P41, P32}}, {44, {4, 5, 7}, {J42, P41, P43}},
 {45, {4, 5, 10}, {J42, P41, D31}}, {46, {4, 5, 11}, {J42, P41, D42}},
 {47, {4, 6, 10}, {J42, P32, D31}}, {48, {4, 6, 11}, {J42, P32, D42}},
 {49, {4, 7, 11}, {J42, P43, D42}}, {50, {4, 10, 11}, {J42, D31, D42}},
 {51, {5, 6, 10}, {P41, P32, D31}}, {52, {5, 6, 11}, {P41, P32, D42}},
 {53, {5, 7, 11}, {P41, P43, D42}}, {54, {5, 10, 11}, {P41, D31, D42}},
 {55, {6, 8, 10}, {P32, P34, D31}}, {56, {6, 10, 11}, {P32, D31, D42}},
 {57, {3, 4, 5, 6}, {J31, J42, P41, P32}}, {58, {3, 4, 5, 10}, {J31, J42, P41, D31}},
 {59, {3, 4, 5, 11}, {J31, J42, P41, D42}}, {60, {3, 4, 6, 10}, {J31, J42, P32, D31}},
 {61, {3, 4, 6, 11}, {J31, J42, P32, D42}}, {62, {3, 4, 10, 11}, {J31, J42, D31, D42}},
 {63, {3, 5, 6, 10}, {J31, P41, P32, D31}}, {64, {3, 5, 6, 11}, {J31, P41, P32, D42}},
 {65, {3, 5, 10, 11}, {J31, P41, D31, D42}}, {66, {3, 6, 8, 10}, {J31, P32, P34, D31}},
 {67, {3, 6, 10, 11}, {J31, P32, D31, D42}}, {68, {4, 5, 6, 10}, {J42, P41, P32, D31}},
 {69, {4, 5, 6, 11}, {J42, P41, P32, D42}}, {70, {4, 5, 7, 11}, {J42, P41, P43, D42}},
 {71, {4, 5, 10, 11}, {J42, P41, D31, D42}}, {72, {4, 6, 10, 11}, {J42, P32, D31, D42}},
 {73, {5, 6, 10, 11}, {P41, P32, D31, D42}}, {74, {3, 4, 5, 6, 10}, {J31, J42, P41, P32, D31}},
 {75, {3, 4, 5, 6, 11}, {J31, J42, P41, P32, D42}},
 {76, {3, 4, 5, 10, 11}, {J31, J42, P41, D31, D42}},
 {77, {3, 4, 6, 10, 11}, {J31, J42, P32, D31, D42}},
 {78, {3, 5, 6, 10, 11}, {J31, P41, P32, D31, D42}},
 {79, {4, 5, 6, 10, 11}, {J42, P41, P32, D31, D42}},
 {80, {3, 4, 5, 6, 10, 11}, {J31, J42, P41, P32, D31, D42}}}
```

Some Abelian subalgebras that may be of interest:

In[159]:=
```
(*  "P41", "P32", "D31", "D42" . Includes P8,
part of Poincare algebra above. This is number 73 in subALGEBRAtable.*)
{{5, 5, "P41", "P41"}, {5, 6, "P41", "P32"}, {5, 10, "P41", "D31"}, {5, 11, "P41", "D42"}};
 {{6, 6, "P32", "P32"}, {6, 10, "P32", "D31"}, {6, 11, "P32", "D42"}};
{{10, 10, "D31", "D31"}, {10, 11, "D31", "D42"}};
{11, 11, "D42", "D42"};
```

In[162]:=
```
(*  P41, P43, D42 . This is number 53 in subALGEBRAtable.*)
{{5, 5, "P41", "P41"}, {5, 7, "P41", "P43"}, {5, 11, "P41", "D42"}};
{{7, 7, "P43", "P43"}, {7, 11, "P43", "D42"}};
{11, 11, "D42", "D42"};
```

In[165]:=
```
(*  P32, P34, D31  . This is number 55 in subALGEBRAtable.*)
{{6, 6, "P32", "P32"}, {6, 8, "P32", "P34"}, {6, 10, "P32", "D31"}};
{{8, 8, "P34", "P34"}, {8, 10, "P34", "D31"}};
{10, 10, "D31", "D31"};
```

Center

Find all 8x8 matrices, the center c, that commute with all elements G of the group, [c,G] = 0.

In[168]:=
```
centerM = Table[am[i, j], {i, 8}, {j, 8}];
centerM // MatrixForm;
centerMvar = Variables[centerM];
LHS = Union[Flatten[Table[Simplify[ centerM.gen[[i, m]] - gen[[i, m]].centerM],
     {i, Length[gen]}, {m, Length[gen[[i]] ]}  ]]];
Table[LHS[[i]], {i, 10}];
{Length[LHS], Length[LHS[[1]]], 37 * 64};
CENTEReqns = Map[zip, LHS]; (*zip[f_]:= f==0*)
liSOLN = Solve[CENTEReqns, centerMvar];
Print["There is(are) ", Length[liSOLN],
 " solution(s) for matrices c that commute with all matrices of the algebra."]
Print["The matrix(ces) is(are) c = ",
 Table[(centerM /. liSOLN[[n]]) // MatrixForm, {n, Length[liSOLN]}]]
```

... Solve: Equations may not give solutions for all "solve" variables.

```
There is(are) 1 solution(s) for matrices c that commute with all matrices of the algebra.

The matrix(ces) is(are) c =
```

$$\left\{\begin{pmatrix} am[7,7] & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & am[7,7] & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & am[7,7] & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & am[7,7] & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & am[7,7] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & am[7,7] & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & am[7,7] & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & am[7,7] \end{pmatrix}\right\}$$

Therefore, only multiples of the unit matrix commute with all 37 linearly independent generators.

Transformations

A generator G generates a transformation T = exp[±i θ G], with minus(-) for momenta generating translations and plus(+) otherwise. θ is a parameter. Use "θ" with J, use "x" with P, and use "s" with D.

In[178]:= `{J8[1, 2] // MatrixForm, MatrixExp[ⅈ J8[1, 2] θ] // MatrixForm} (*Samples*)`

Out[178]= $$\left\{\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} e^{\frac{i\theta}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-\frac{i\theta}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{\frac{i\theta}{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-\frac{i\theta}{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\frac{i\theta}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-\frac{i\theta}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{\frac{i\theta}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{i\theta}{2}} \end{pmatrix}\right\}$$

In[179]:= `{JR[1, 2] // MatrixForm, MatrixExp[ⅈ JR[1, 2] θ] // MatrixForm}`

Out[179]= $$\left\{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\frac{i\theta}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-\frac{i\theta}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{\frac{i\theta}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{i\theta}{2}} \end{pmatrix}\right\}$$

In[180]:= `{J31[1, 2] // MatrixForm, MatrixExp[ⅈ J31[1, 2] θ] // MatrixForm}`

Out[180]= $$\left\{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{i\theta}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{i\theta}{2} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right\}$$

In[181]:= `{J42[1, 2] // MatrixForm, MatrixExp[ⅈ J42[1, 2] θ] // MatrixForm}`

Out[181]= $$\left\{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{i\theta}{2} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{i\theta}{2} & 0 & 0 & 0 & 1 \end{pmatrix}\right\}$$

In[182]:=
```
{P41[1, kb] // MatrixForm, MatrixExp[-i P41[1, kb] x] // MatrixForm}
```

Out[182]=

$$\left\{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i\,kb & 0 & 0 & 0 & 0 & 0 & 0 \\ i\,kb & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & kb\,x & 0 & 0 & 0 & 0 & 1 & 0 \\ kb\,x & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right\}$$

In[183]:=
```
{P32[1, kc] // MatrixForm, MatrixExp[-i P32[1, kc] x] // MatrixForm}
```

Out[183]=

$$\left\{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\,kc & 0 & 0 & 0 & 0 \\ 0 & 0 & -i\,kc & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -kc\,x & 1 & 0 & 0 & 0 \\ 0 & 0 & -kc\,x & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right\}$$

In[184]:=
```
{P43[1, kd] // MatrixForm, MatrixExp[-i P43[1, kd] x] // MatrixForm}
```

Out[184]=

$$\left\{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i\,kd & 0 & 0 \\ 0 & 0 & 0 & 0 & -i\,kd & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -kd\,x & 1 & 0 \\ 0 & 0 & 0 & 0 & -kd\,x & 0 & 0 & 1 \end{pmatrix}\right\}$$

In[185]:=
```
{P34[1, ke] // MatrixForm, MatrixExp[-i P34[1, ke] x] // MatrixForm}
```

Out[185]=

$$\left\{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i\,ke \\ 0 & 0 & 0 & 0 & 0 & 0 & i\,ke & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & ke\,x \\ 0 & 0 & 0 & 0 & 0 & 1 & ke\,x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right\}$$

In[186]:=
```
{mD // MatrixForm, MatrixExp[i mD s] // MatrixForm}
```

Out[186]=

$$\left\{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{i}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-s/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-s/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{s/2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{s/2} \end{pmatrix}\right\}$$

In[187]:=
```
{mD31 // MatrixForm, MatrixExp[ⅈ mD31 s] // MatrixForm}
```

Out[187]=

$$\left\{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{i}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{i}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{s}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{s}{2} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right\}$$

In[188]:=
```
{mD42 // MatrixForm, MatrixExp[ⅈ mD42 s] // MatrixForm}
```

Out[188]=

$$\left\{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{i}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{2} & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{s}{2} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{s}{2} & 0 & 0 & 0 & 1 \end{pmatrix}\right\}$$

Structure Constants, adjoint rep, Cartan-Killing metric:

tGEN37 is a simple list of the 37 generator matrices, unlike gen37[[i,m]] which is split into 11 families and their members.

In[189]:=
```
tGEN37 = Flatten[Table[gen37[[i, m]], {i, Length[gen37]}, {m, Length[gen37[[i]] ]}], 1];
tGEN37names =
  Flatten[Table[{genNAME[[i]], m}, {i, Length[gen37]}, {m, Length[gen37[[i]] ]}], 1];
{tGEN37names[[3]], tGEN37[[3]] // MatrixForm,
  tGEN37names[[28]], tGEN37[[28]] // MatrixForm};
Print["There are ", Length[tGEN37], " matrices in tGEN37."]
```

```
There are 37 matrices in tGEN37.
```

Examples follow:

In[193]:=
```
Print["Each J family has six matrices J^μν with indices in the order {μ,ν} = ",
 Flatten[ Table[{μ, ν}, {μ, 3}, {ν, μ + 1, 4}], 1], " . "]
Print["Thus the matrix tGEN37[[9]] = ", tGEN37[[9]] // MatrixForm,
 " with tGEN37names[[9]] = ", tGEN37names[[9]], " is the matrix JR^14 = JR^xt."]
```

```
Each J family has six matrices J^μν with indices in the order {μ,ν} =
 {{1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}} .
```

Thus the matrix tGEN37[[9]] =
$$\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&\frac{i}{2}&0&0\\ 0&0&0&0&\frac{i}{2}&0&0&0\\ 0&0&0&0&0&0&0&-\frac{i}{2}\\ 0&0&0&0&0&0&-\frac{i}{2}&0 \end{pmatrix}$$

with tGEN37names[[9]] = {JR, 3} is the matrix JR^{14} = JR^{xt}.

In[195]:=
```
Print["Each P family has four matrices P^μ with indices in the order μ = ",
 Table[μ, {μ, 4}]]
Print["Thus the matrix tGEN37[[28]] = ", tGEN37[[28]] // MatrixForm,
 " with tGEN37names[[28]] = ", tGEN37names[[28]], " is the matrix P43^2 = P43^y."]
```

Each P family has four matrices P^μ with indices in the order μ = {1, 2, 3, 4}

Thus the matrix tGEN37[[28]] =
$$\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&-kd&0&0\\ 0&0&0&0&kd&0&0&0 \end{pmatrix}$$

with tGEN37names[[28]] = {P43, 2} is the matrix $P43^2$ = $P43^y$.

In[197]:=
```
Print["Each D family has one matrix D with no indices."]
Print["Thus the matrix tGEN37[[35]] = ", tGEN37[[35]] // MatrixForm,
 " with tGEN37names[[35]] = ", tGEN37names[[35]], " is the matrix D."]
```

Each D family has one matrix D with no indices.

Thus the matrix tGEN37[[35]] =
$$\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&\frac{i}{2}&0&0&0\\ 0&0&0&0&0&\frac{i}{2}&0&0\\ 0&0&0&0&0&0&-\frac{i}{2}&0\\ 0&0&0&0&0&0&0&-\frac{i}{2} \end{pmatrix}$$

with tGEN37names[[35]] = {D, 1} is the matrix D.

structC[i,j,k] is the structure constant s_k^{ij} in the sum: $[g^i, g^j] = i\, s_k^{ij}\, g^k$, where g^i is the i^{th} generator in tGEN37 and i,j,k = 1,2,... 37.

In[199]:=
```
(*Find the structure constants structC[i,j,k].*)
For[i = 1, i ≤ 37, i++, For[j = 1, j ≤ 37, j++,
  strucConstLHS = Complement[Union[Flatten[tGEN37[[i]].tGEN37[[j]] -
       tGEN37[[j]].tGEN37[[i]] - Sum[ⅈ s[k] tGEN37[[k]], {k, 37}]]]], {0}];
  strucConstEQN = Map[zip, strucConstLHS]; (*zip[f_]:= f==0*)
  strucConstSOL = Solve[strucConstEQN, Table[s[k], {k, 37}]];
  Table[structC[i, j, k] = s[k] /. strucConstSOL[[1]], {k, 37}]]]
```

In[200]:=
```
Print["For example, [JR^14,JR^24] = +ⅈημν[[4,4]]JR[1,2], which is ",
 Union[Flatten[(tGEN37[[9]]. tGEN37[[11]] - tGEN37[[11]]. tGEN37[[9]]) -
     (+ⅈ ημν[[4, 4]] tGEN37[[7]])]] == {0},
 " , involves the three generators tGEN37[[9]], tGEN37[[11]], tGEN37[[7]],
   with names tGEN37names[[9]], tGEN37names[[11]], tGEN37names[[7]] = ",
 tGEN37names[[9]], " , ", tGEN37names[[11]], " , ", tGEN37names[[7]], " ."]
Print["Note that each J family has six matrices J^μν with indices in the order {μ,ν} = ",
 Flatten[ Table[{μ, ν}, {μ, 3}, {ν, μ + 1, 4}], 1],
 " and the JR matrices are the 7,8,9,10,11,12^th matrices in tGEN37. "]
Print["Thus the structure constant structC[9,11,7]
   should be structC[9,11,7] = + ημν[[4,4]] = ",
 + ημν[[4, 4]], ": ",  structC[9, 11, 7] == + ημν[[4, 4]]  ]
```

```
For example, [JR^14,JR^24] = +ⅈημν[[4,4]]JR[1,2], which is True
  , involves the three generators tGEN37[[9]], tGEN37[[11]], tGEN37[[7]], with names
   tGEN37names[[9]], tGEN37names[[11]], tGEN37names[[7]] = {JR, 3} , {JR, 5} , {JR, 1} .

Note that each J family has six matrices J^μν with indices in the order {μ,ν} =
 {{1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}}
  and the JR matrices are the 7,8,9,10,11,12^th matrices in tGEN37.

Thus the structure constant structC[9,11,7] should be structC[9,11,7] = + ημν[[4,4]] = -1: True
```

Adjoint rep.

mT[n] is the 37x37 matrix for the *n*th generator in the adjoint rep. According to ***Group Theory in a Nutshell***, A. Zee, page 365 , the adjoint rep is defined to be the set of 37 matrices with components

$$\left(mT^i\right)^j_k = -i s^{ij}_k \ .$$

In[203]:=
```
mT[i_] := mT[i] = -ⅈ Table[structC[i, j, k], {j, 37}, {k, 37}]
```

In[204]:=
```
mT[1] // MatrixForm; (* sample *)
Table[{i, Union[Flatten[mT[i]]]}, {i, 37}] ;
Print["None of the mT[a], a=1,...37, matrices is null: ",
 Union[Table[Length[Union[Flatten[mT[i]]]] > 1, {i, 37}] ]]
```

```
None of the mT[a], a=1,...37, matrices is null: {True}
```

In[207]:=
```
Print["The mT^i matrices of the adjoint rep obey the Lie algebra: ",
 Union[Flatten[Table[mT[i].mT[j] - mT[j].mT[i] - (
      ⅈ Sum[structC[i, j, k] mT[k], {k, 37}]), {i, 37}, {j, 37}]]] == {0}]
```

```
The mT^i matrices of the adjoint rep obey the Lie algebra: True
```

Cartan-Killing metric gCK is not invertible.

$g_{CK}^{ab} = \mathrm{Tr}(T^a T^b) = -s_d^{ac} s_c^{bd}$ = gCK[[a,b]]

In[208]:=
```
gCK = Table[  -Sum[structC[a, c, d] structC[b, d, c], {c, 37}, {d, 37}] , {a, 37}, {b, 37}];
Print["The determinant of the Cartan-Killing metric vanishes, Det(gCK^ab) = 0: ",
 Det[gCK] == 0 ]
Print["Therefore the metric gCK^ab has no inverse."]
```

```
The determinant of the Cartan-Killing metric vanishes, Det(gCK^ab) = 0: True

Therefore the metric gCK^ab has no inverse.
```

END PROGRAM...