Rule of necessitation: a non-contingent truthity, but not a tautology

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1. The axiom or rule of necessitation \( \text{N} \) states that if \( p \) is a theorem, then necessarily \( p \) is a theorem:

\[
\text{If } \vdash p \text{ then } \vdash \Box p.
\]

We show this is non-contingent (a truthity), but not tautologous (a proof). We evaluate axioms (in bold) of \( \text{N}, \text{K}, \text{T}, \text{4}, \text{B}, \text{D}, \text{5} \) to derive systems (in italics) of \( \text{K}, \text{M}, \text{T}, \text{S4}, \text{S5}, \text{D} \).

We assume the Meth8 apparatus implementing system variant \( \text{VL}4 \), where:

\[
\# \text{ necessity, universal quantifier; } \% \text{ possibility, existential quantifier; } \\
\Rightarrow \text{ Imply; } = \text{ Equivalent to; } (p=p) \text{ Tautology}
\]

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The designated proof value is T tautology. Note the meaning of \( (\%p>\#p) \): a possibility of \( p \) implies the necessity of \( p \); and some \( p \) implies all \( p \). In other words, if a possibility of \( p \) then the necessity of \( p \); and if some \( p \) then all \( p \). This shows equivalence and interchangeability of respective modal operators and quantified operators, as proved in Appendix. (That correspondence is proved by \( \text{VL}4 \) corrections to the vertices of the Square of Opposition and subsequent corrections to the syllogisms of Modus Cesare and Modus Camestros.)

Results are the 16-value truth table as row-major and horizontal; tautology is all "TTTT".

\[
\text{N: If } \vdash p \text{ then } \vdash \Box p. \\
\vdash p>\#p; \\
The necessity of \( p \) or \( \neg p \) is a theorem. \\
\vdash (p\neg \neg p)=(p=p); \\
\text{Tautology} \quad \text{TNTN TNTN TNTN TNTN} \quad \text{(N.1.1)} \\
\text{Non-contingency} \quad \text{NNNN NNNN NNNN NNNN} \quad \text{(N.2.1)}
\]

Eqs. N.1.2 and 2.2 are minimally tautologous at a level of non-contingency \( (\text{NNNN NNNN NNNN NNNN}) \) as \textit{truthity}, but not a proof at a level of tautology \( (\text{TTTT TTTT TTTT TTTT}) \).

The definitions of the other axioms are as follows (Steward, Stoupa, 2004):

\[
\text{K: } \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q); \text{ no conditions} \\
\vdash (p>q)>(\#p>\#q); \\
\text{Tautology} \quad \text{TTTT TTTT TTTT TTTT} \quad \text{(K.1.1)}
\]

\[
\text{D: } p \rightarrow (p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q); \text{ no conditions} \\
\vdash \neg p \rightarrow (\neg p \rightarrow \#q); \\
\text{Non-contingency} \quad \text{NNNN NNNN NNNN NNNN} \quad \text{(D.1.2)}
\]

\[
\text{B: } \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q); \text{ no conditions} \\
\vdash p \rightarrow (p \rightarrow \#q); \\
\text{Non-contingency} \quad \text{NNNN NNNN NNNN NNNN} \quad \text{(B.1.2)}
\]
The definitions of systems are as follows:

\[ K := \quad K \text{ (no conditions)} \]
\[
#(p \rightarrow q) \rightarrow (#p \rightarrow #q) ; \\
TTTT TTTT TTTT TTTT \\
(K.1.1)
\]
\[
#(p \rightarrow q) \rightarrow (#p \rightarrow #q) ; \\
TTTT TTTT TTTT TTTT \\
(K.1.2)
\]

alternatively, \( K \) & \( N \) is used (viz, en.wikipedia.org/wiki/Modal_logic) \]
\[
(#(p \rightarrow q) \rightarrow (#p \rightarrow #q)) \& (p \rightarrow #p) ; \\
TNTN TNTN TNTN TNTN \\
(K.2.1)
\]

Eq. \( K.2.2 \) subsequently taints all results as having some value of truth (TNTN), but not tautology (TTTT).

\[ D := \quad K \& D \text{ (serial)} \]
\[
(#(p \rightarrow q) \rightarrow (#p \rightarrow #q)) \& (#p \rightarrow %p) ; \\
TTTT TTTT TTTT TTTT \\
(D.1.1)
\]

\[ M := \quad K \& T \]
\[
(#(p \rightarrow q) \rightarrow (#p \rightarrow #q)) \& (#p \rightarrow p) ; \\
TCTT TCTT TCTT TCTT \\
(T.1.1)
\]

\[ S4 := \quad M \& 4 \text{ ; reflexive and transitive} \]
\[
((#(p \rightarrow q) \rightarrow (#p \rightarrow #q)) \& (#p \rightarrow p)) \& (#p \rightarrow ##p) ; \\
TTTT TTTT TTTT TTTT \\
(S4.1.2)
\]

\[ B := \quad M \& B \]
\[
((#(p \rightarrow q) \rightarrow (#p \rightarrow #q)) \& (#p \rightarrow p)) \& (p \rightarrow %p) ; \\
TTTT TTTT TTTT TTTT \\
(B.1.1)
\]

\[ S5 := \quad M \& 5 \text{ ; reflexive and Euclidean} \]
\[
((#(p \rightarrow q) \rightarrow (#p \rightarrow #q)) \& (#p \rightarrow p)) \& (%p \rightarrow %p) ; \\
TTTT TTTT TTTT TTTT \\
(S5.1.1)
\]

alternatively, \( M \) & \( B \) & \( 4 \)
\[
((#(p \rightarrow q) \rightarrow (#p \rightarrow #q)) \& (#p \rightarrow p)) \& (p \rightarrow %p) \& (#p \rightarrow ##p) ; \\
(S5.2.1)
\]
2. We also evaluated (Steward, Stouppa, 2004) to derive by replication some systems of interest.

\[ \mathbf{K}: \square(p \supset q) \supset (\square p \supset \square q) \]  
\[ (p \supset q) > (#p > #q) ; \]  
\[ \text{(3.1.1)} \]  
\[ \text{TTTT TTTT TTTT TTTT} \]  
\[ \text{(3.1.2)} \]  

Axion \( T \): \[ p \supset p \]  
\[ #p > q ; \]  
\[ \text{(3.2.1)} \]  
\[ \text{TTTT TTTT TTTT TTTT} \]  
\[ \text{(3.2.2)} \]  

\( \mathbf{M} \), obtained by extending system \( \mathbf{K} \) with rule \( T \) [not Gödel's system \( T \)]  
\[ (#p > q) > (#p > #q) > (#p > q) ; \]  
\[ \text{(3.3.1)} \]  
\[ \text{TCTT TCTT TCTT TCTT} \]  
\[ \text{(3.3.2)} \]  

"The strongest system from these modal logics that is perfectly straightforward to formulate in a sequent system and to prove cut-free is system \( \mathbf{G-M} \) (for Gentzen system \( \mathbf{M} \))."

We remark that the subsequent derivations of \( S4 \), \( B \), and \( S5 \) are tautologous, as are \( \mathbf{K} \) and \( T \) as demonstrated in section 1.

2. We found other mistakes in (Steward, Stouppa, 2004).

2.1. "The following lemma is a straightforward exercise in theoremhood over \( \mathbf{K} \):

**LEMMA 6**  If \( A \supset B \) is a theorem of \( \mathbf{M} \), then so are:

\[ \text{(L.6.0.1)} \]  
\[ \begin{align*} 
1. & \, A \land C \supset \land C; \\
2. & \, A \lor C \supset \lor C; \\
3. & \, \square A \supset \square B; \\
4. & \, \langle\rangle A \supset \langle\rangle B. 
\end{align*} \]

\[ \text{(L.6.1)} \]  
\[ \text{TNTT TNTT TTTT TTTT} \]  
\[ \text{(L.6.2)} \]  
\[ \text{TNTT TTTT TTTT TTTT} \]  
\[ \text{(L.6.3)} \]  
\[ \text{TNTT TTTT TTTT TTTT} \]  
\[ \text{(L.6.4)} \]  

To map Eq. L.6.0.1 we use Eq. 3.3.2.

\[ ((#(p > q) > (#p > #q)) > (#p > q)) > (p > q) ; \]  
\[ \text{TNTT TTTT TTTT TTTT} \]  
\[ \text{(L.6.0.2)} \]  

We then reuse Eq. L.6.0.2 to map L.6.1.2 - 6.4.2.

\[ (((#(p > q) > (#p > #q)) > (#p > q)) > (p > q)) > ((p & r) > (q & r)) ; \]  
\[ \text{TTTT TTTT TTTT TTTT} \]  
\[ \text{TCTT TCTT TCTT TCTT} \]  
\[ \text{(L.6.1)} \]  
\[ \text{TTTT TTTT TTTT TTTT} \]  
\[ \text{TCTT TCTT TCTT TCTT} \]  
\[ \text{(L.6.2)} \]  
\[ \text{TCTT TCTT TCTT TCTT} \]  
\[ \text{(L.6.3)} \]  
\[ \text{TCTT TCTT TCTT TCTT} \]  
\[ \text{(L.6.4)} \]  

2.2. These inference rules were flagged by Meth8, with page number for equation.

**LET:**  \( p \, \text{uc_Gamma} \); \( q \, \text{uc_Delta} \); \( r \, A \); \( s \, B \)

\( (p & r) > (%p & #r) ; \)  
\[ \text{TNTT TTTT TTTT TTTT} \]  
\[ \text{(315, []1)} \]  
\[ \text{TTTT NTNN TTTT NTNN} \]  
\[ \text{(323, []2)} \]  
\[ \text{TNTT TTTT TTTT TTTT} \]  
\[ \text{(324, []5)} \]
"we recommend the reader works ... example (A ⊃ B ⊃ C) ⊃ (A ⊃ C) ⊃ B ⊃ C"  (321.1)

((p>q)>r)>p)>q)>r ; TFFF TTTT TFFF TTTT  (321.2)

We conclude that N the axiom or rule of necessitation is not tautologous. Because system M as derived and rendered is not tautologous, system G-M also not tautologous.

What follows is that systems derived from using M are tainted, regardless of the tautological status of the result so masking the defect, such as systems S4, B, and S5.

We also find that Gentzen-sequent proof is suspicious, perhaps due to its non bi-valent lattice basis in a vector space.

References

Steward, Charles; Stouppa, Phiniki. (2004). A systematic proof theory for several modal logics; also at textproof.com/supervision/phiniki04sbm.pdf