ABOUT NON-CLOSEDNESS OF THE THREE-DIMENSIONAL NAVIER-STOKES EQUATIONS SYSTEM FOR THE VISCOUS INCOMPRESSIBLE FLUID.

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ABSTRACT. In this paper it is shown that the system of four equations formed by three-dimensional Navier-Stokes equations system for incompressible fluid and equation of continuity, is not closed, equation of continuity is excessive. This is because the three-dimensional Navier-Stokes equations system cannot have a bounded at infinity solutions to the Cauchy problem with a non-zero velocity field divergence.

The interest of the Navier-Stokes equations is so great that information about this question periodically appears in the newspaper news. The fact is that proven methods for analyzing partial differential equations in the case of the Navier-Stokes equations for the incompressible fluid do not work for an unknown reason. Equations remain elusively incomprehensible.

In year 2000, seven problems named as major mathematical problems of the third millennium were published on the website http://claymath.org/, one of those problems – Navier-Stokes equations. This problem is formulated by C. L. Fefferman by a range of questions regarding the solution of these equations, because until now it's not possible to understand what properties do they have. The question about how good the set of Navier-Stokes equations describes behavior of real viscous fluids also remains open.

In the paper written by O.A. Ladyzhenskaya [1] and published in 2003 the problem of Navier-Stokes equations was formulated in the following way: «Do Navier-Stokes equations together with initial and boundary conditions give determining description of incompressible fluid dynamics or not?»

As of 2014, the situation with the problem of the Navier-Stokes equations became almost mystical. It is described in the paper by one of the leading researchers of this problem, Terence Tao [2]. In his paper, he actually comes to the conclusion that the existing methods of analysis cannot solve the problem. To date, no important results have been achieved in solving the problem of the Navier-Stokes equations.

Really mystical situation: the Navier-Stokes equations must describe real fluids, which behavior has a certain set of properties. These properties should be visible during the analysis of the equations, but this does not happen. More precisely, it happens only for the plane case of fluid motion, but not for the three-dimensional one. Suspicion occurs that the Navier-Stokes equations have something that goes unnoticed, some unique feature …

Let's consider the system of four equations formed by the three Navier-Stokes equations system

$$\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\left(\frac{\partial^2 V_x}{\partial^2 x}+\frac{\partial^2 V_x}{\partial^2 y}+\frac{\partial^2 V_x}{\partial^2 z}\right)$$

$$\frac{\partial V_y}{\partial t}+V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_y}{\partial y}+V_z\frac{\partial V_y}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+\nu\left(\frac{\partial^2 V_y}{\partial^2 x}+\frac{\partial^2 V_y}{\partial^2 y}+\frac{\partial^2 V_y}{\partial^2 z}\right) \qquad (1)$$

$$\frac{\partial V_z}{\partial t}+V_x\frac{\partial V_z}{\partial x}+V_y\frac{\partial V_z}{\partial y}+V_z\frac{\partial V_z}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\nu\left(\frac{\partial^2 V_z}{\partial^2 x}+\frac{\partial^2 V_z}{\partial^2 y}+\frac{\partial^2 V_z}{\partial^2 z}\right)$$

and the equation of continuity (the incompressibility condition of fluid)

$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}=0 \qquad (2)$$

Four equations include four unknown functions, and, apparently, the given equations system is closed, however, this impression is deceptive. Here it will be very useful to recall systems of linear algebraic equations. As is well known, in this case the equality of the number of equations to the number of unknowns does not mean at all that the equations system is closed and uniquely solvable.

Let the three arbitrarily chosen functions $V_x = V_x(x,y,z,t)$, $V_y = V_y(x,y,z,t)$, $V_z = V_z(x,y,z,t)$ describe the velocity field. Further, it is assumed that the functions V_x, V_y, V_z are continuous together with their partial derivatives in coordinates to the third order inclusive. The second partial derivatives of these functions with respect to time and one of the coordinates are also continuous. Substituting these functions into the first equation (or any other equation, as the first equation is chosen for definiteness) of the system (1), it is possible by appropriate choice of the pressure function $P = P(x,y,z,t)$ to achieve the fulfillment of this equation. Thus, it can be said that one of the equations of the system (1) will always be satisfied for an arbitrarily specified velocity field V_x, V_y, V_z. Let's rotate the velocity field V_x, V_y, V_z and the pressure field P around an arbitrarily selected Z' axis, orthogonal to the XY coordinate plane at an arbitrary angle α (hereinafter, speaking of the rotation of the velocity field, it is always assumed that this also causes the rotation of the pressure field P, the rotation of the velocity field is considered for a fixed point in time). In this case, for an arbitrarily chosen velocity field, the first equation of the system (1) will no longer be satisfied. What conditions must the velocity field satisfy so that when it is rotated the first equation continues to be satisfied? The answer is very simple: in the initial state, the velocity field must satisfy the system of two equations, namely, the first and second equations of system (1). In this case, the two equations mentioned will be fulfilled after the rotation of the velocity field relative to the axis Z' by an arbitrary angle α. This is possible to prove by mathematical analysis of the field turning process (it is not complicated, but very lengthy for this paper). Turning the field at a small angle $d\alpha$, using the field continuity property and requiring the realization of the first equation of the system (1), we obtain the condition: in the initial state, the velocity field must satisfy the first and second equations of system (1). Physically, this result is absolutely clear. The equations system (1) is the record of the impulse conservation law for the three components of impulse. If the velocity field satisfies only the first equation of the system (1), this means that in this field the impulse conservation law is realized only for X component of impulse. When the velocity field rotates around the Z' axis, the contribution of the Y component of impulse will be made in the X component. Hence it is clear that the conservation law of the X component of the impulse can be fulfilled after the field is rotated only if the conservation law of the X and Y components of impulse were fulfilled before the field was rotated.

Similarly, if the first equation of the system (1) is satisfied when the field is rotated around the Y' axis, orthogonal to the XZ coordinate plane, then this field will satisfy the first and third equations of system (1). And finally, if the first equation of the system (1) is satisfied when the field is rotated around to any arbitrary directional axis, then such a field will satisfy all three equations of system (1).

So far nothing has been said about the velocity field divergence. Consider velocity field with nonzero divergence, i.e.

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \mathrm{Div}\,\boldsymbol{V} \neq 0$$

Non-zero divergence of the velocity field in the incompressible fluid means that the sources (sinks) of the fluid are continuously distributed throughout the fluid volume (this will violate the mass conservation law), which from a physical point of view looks absolutely ridiculous. However, abstracting from the physical sense, mathematically, in the most general form it can be written as

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = D(x, y, z, t) \tag{3}$$

where $D(x, y, z, t)$- is a continuos function of coordinates and time.

Let's suppose that there is a velocity field that satisfies the three equations of the system (1) and equation (3). This means that the first (or any other) equation of the system (1) and equation (3) will both be fulfilled when the field is rotated relative to an arbitrarily located and arbitrarily directed axis. Obviously, the equations of the system (1) will be satisfied (because of the assumptions made), but equation (3) in this case can only be satisfied if

$$D(x, y, z, t) = const$$

(this constant may be a function only of time, but in this case, it doesn't matter). All mentioned above doesn't mean that there are solutions for the system (1), for which Div $\boldsymbol{V} = const$. All mentioned above means that they might exist, as the method of analysis used here doesn't allow these solutions to be cut off. However, there is another way to do that. As mentioned above, the constant divergence in the incompressible fluid is a continuous distribution of the sources (sinks) of the fluid itself by volume of the fluid. Moreover, the distribution density of sources is constant throughout the volume of the fluid. Applying the Cauchy problem, the presence of a velocity field with a constant divergence will lead to an unlimited increase in velocities (or any one velocity) with increasing distance from the origin. This means that all solutions of the system (1) that are bounded at infinity will have zero velocity field divergence. Thus, the solutions of the three-dimensional Cauchy problem for an incompressible fluid will be all solutions of system (1) bounded at infinity. All such solutions will automatically satisfy continuity equation (2), i.e. the continuity equation turns out to be unnecessary.

An interesting result can be obtained when trying to solve some simplest boundary problem for the Navier-Stokes equations using a velocity field with constant divergence. So, for example, one-dimensional problem of fluid motion between flat walls in the case of a divergence-free velocity field becomes two-dimensional in case of constant divergence. In the set of equations obtained in this case, internal contradictions arise, with the result that the solution of this problem simply does not exist. A similar result is obtained in other problems. It can be assumed that the system of equations (1) has no solutions at all with a constant velocity field divergence. It would mean that all solutions of equations system (1) satisfy the equation (2). We may probably prove this in the course of further research in the light of newly discovered circumstances.

The system of equations (1) consists of three equations, containing four unknown functions, hence it is not closed. Solution for this equations system can be performed according to the following scheme, for example. You can arbitrarily choose one of the velocities, for example, $V_x(x, y, z, t)$, choosing a function that tends to zero at infinity and fades with the time and has integrals that are bounded for any points of time t

$$\iiint_{-\infty}^{+\infty} |V_x| dxdydz$$

$$\iiint_{-\infty}^{+\infty} V_x^2 dxdydz$$

The limitation of the first integral means the limitation and localization of the fluid impulse associated with the velocity V_x. The limitation of the second integral means the limitation of the kinetic energy. The value of the integral

$$\iiint_{-\infty}^{+\infty} V_x dxdydz$$

doesn't have to depend from time, this is a requirement of the impulse conservation law.

Substituting V_x to equations system (1), we will get a closed system of three equations for three unknown functions V_y, V_z and P. Since V_x was chosen arbitrarily, it is clear that there are infinitely many solutions. Basically, it cannot be argued that absolutely all solutions obtained in this way will be limited at infinity. But it is also obvious that there will be solutions, and there will be infinitely many of them. A physically adequate system of equations (1) describing a dissipative process cannot respond to a localized and energetically limited effect by an unlimited increase in velocities at infinity. All said does not exclude a local unlimited growth of velocities (blowup), but with a limitation on the impulse and kinetic energy of the entire mass of the fluid.

So, if the equations system (1) has infinitely many solutions, is there the only solution for three-dimensional Cauchy problem for a given initial velocity field? Probably not, loss of determinism is possible. This question remains open in this paper, it requires additional research.

All mentioned above about non-closedness of equations system is relative also for two-dimensional case of fluid flow. Two-dimensional system of equations is non-closed as well, equation of continuity is excessive. However there is no such arbitrariness as in three-dimensional case. Let's look at the two-dimensional system of equations:

$$\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 V_x}{\partial^2 x} + \frac{\partial^2 V_x}{\partial^2 y}\right)$$

$$\frac{\partial V_y}{\partial t} + V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 V_y}{\partial^2 x} + \frac{\partial^2 V_y}{\partial^2 y}\right) \quad (4)$$

This system of equations contain three unknown functions and it seems that it's possible to choose one of velocities while second velocity and pressure can be found from system of equations (4). In this case, this can not be done, the obstacle is the equation of continuity

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0 \quad (5)$$

All solutions of equations system (4) bounded in infinity satisfy the equation of continuity (5), but not all solutions of continuity equation (5) will satisfy the equations system (4). If randomly choose one of velocities we can find second velocity from equation of continuity (with accuracy of function of only one coordinate and time). Then we obtain the solution of the continuity equation (5), i.e. we find the velocity field. And it is not necessary that this velocity field will satisfy the system of equations (4). This equations system, for given initial conditions, has only one solution, as shown in [3].

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