In an ordered field, there is an important mapping \( | - | \) called absolute value function. We shall define \( | - | \) for \( \mathbb{R} \).

The absolute value function is defined by \( | - | : \mathbb{R} \rightarrow [0, \infty) \).

**Remarks:** The image \( | a | \) of \( a \in \mathbb{R} \) is called the absolute of \( a \). We have the following simple but important properties of the absolute value function.

1. \( | a | \geq 0, \quad \forall \ a \in \mathbb{R} \)
2. \( | a | = 0, \quad iff \ a = 0 \)
3. \( -a \leq | a | \quad and \quad a \leq | a |, \quad \forall \ a \in \mathbb{R} \)
4. \( -| a | \leq a \leq | a |, \quad \forall \ a \in \mathbb{R} \)
5. \( | ab | = | a | | b |, \quad \forall \ a, b \in \mathbb{R} \)
6. \( \frac{a}{b} = \frac{|a|}{|b|}, \quad \forall \ a, b \in \mathbb{R} \)
7. \( a^2 = | a |^2, \quad \forall \ a \in \mathbb{R} \)

**Triangle Inequality**

Let \( a, b \in \mathbb{R} \). Show that \( | a + b | \leq | a | + | b | \).

**Proof:**
\[
(| a + b |)^2 = (a + b)^2 = a^2 + 2ab + b^2 \leq | a |^2 + 2 | ab | + | b |^2 = | a |^2 + 2 | a | | b | + | b |^2 = (| a | + | b |)^2
\]

Take positive square root
\[
| a + b | \leq | a | + | b |
\]

**Remarks:**

1. Extension of the above theorem to complex number.
   Show that \( | a + b | \leq | a | + | b |, \quad \text{where} \ a, b \in \mathbb{C} \).
   **Proof:**
   \[
   (| a + b |)^2 \leq | a |^2 + 2 | ab | + | b |^2
   = | a |^2 + 2 | a | | b | + | b |^2 = (| a | + | b |)^2
   \]
   Take positive square root
   \[
   | a + b | \leq | a | + | b |
   \]
   **Note:** Some steps are missing because \( \mathbb{C} \) is not an ordered field.

2. Extension of the above theorem to vector.
   Show that \( | a + b | \leq | a | + | b |, \quad \text{where} \ a, b \) are vectors.
   **Proof:**
   \[
   (| a + b |)^2 \leq | a |^2 + 2 | a | | b | + | b |^2
   = (| a | + | b |)^2
   \]
   Take positive square root
\[ |a + b| \leq |a| + |b| \]

**Note:** Some steps are missing because vector is defined for dot and cross (undergraduate syllabus). In addition, vector is not ordered.

### Further Examples

1. Show that \(|a| - |b| \leq |a-b|\)

   **Proof:**
   \[
   (|a| - |b|)^2 = |a|^2 - 2|a||b| + |b|^2 \\
   = |a|^2 - 2ab + |b|^2 \\
   \leq a^2 - 2ab + b^2 = (a-b)^2 = (|a-b|)^2
   \]

   Take positive square root

   \[ |a| - |b| \leq |a-b| \]

2. Show that \(|\,|a| - |b|\,| \leq |a-b|\)

   **Proof:**
   \[
   (||a| - |b||)^2 = (|a| - |b|)^2 \\
   = |a|^2 - 2|a||b| + |b|^2 \\
   = |a|^2 - 2ab + |b|^2 \\
   \leq a^2 - 2ab + b^2 = (a-b)^2 = (|a-b|)^2
   \]

   Take positive square root

   \[ ||a| - |b|| \leq |a-b| \]

**Reference:**