Proper-Time Based Covariant Extension of Newton's Second Law

Steven Kenneth Kauffmann*

Abstract

The generic single-particle relativistic dynamical principle is usually taken to be that the time derivative of the particle's momentum is equal to the applied force. That relativistic momentum itself, however, is the particle's rest mass times the proper-time derivative of its position, so this dynamical principle is easily reverted to the original Newtonian one of the particle's mass times acceleration being equal to the force, provided that the time derivatives always refer to proper time and the force is amended by a relativistic factor of gamma, which we denote as proper force; the Lorentz covariance of the result is greatly aided by the Lorentz invariance of proper time. It is shown that the purely relativistic fourth component of this Lorentz-covariant Second Law extension pertains to the particle's power, as would be expected. Its full electromagnetic case emerges directly from a certain entirely Lorentz-invariant Lagrangian.

Proper versus perceived velocity in special relativity

An *implicit ingredient in the perceived speed* $|dx/dt|$ *of a relativistic moving object* in one spatial dimension is *its length contraction by the factor* γ^{-1} [1],

$$\gamma^{-1} \stackrel{\text{def}}{=} \left(1 - ((dx/dt)/c)^2\right)^{\frac{1}{2}};\ \ 0 < \gamma^{-1} \le 1. \tag{1a}$$

The *degree* of length contraction *is completely dependent on the inertial reference frame*, so perceived speed $|dx/dt|$ is *unsuitable* for some applications. The *effect* of length contraction on a moving object's perceived speed is *removed* by multiplying that speed by γ, *which produces the higher speed* $\gamma|dx/dt|$. So doing is *equivalent* to replacing the moving object's perceived speed $|dx/dt|$ *by its proper speed* $|dx/d\tau|$, where *Lorentz-transformation invariant proper differential time* $d\tau$ *is defined as*,

$$d\tau \stackrel{\text{def}}{=} \left((dt)^2 - (dx/c)^2\right)^{\frac{1}{2}} = \left(1 - ((dx/dt)/c)^2\right)^{\frac{1}{2}} dt = \gamma^{-1}dt \Rightarrow (dt/d\tau) = \gamma \Rightarrow |dx/d\tau| = \gamma|dx/dt|. \tag{1b}$$

The Lorentz-transformation invariant differential proper time $d\tau$ of Eq. (1b) is somewhat analogous to the Galilean-transformation invariant differential time *dt of Newtonian physics*. Also, despite *strict adherence of perceived speed to* $|dx/dt| < c$, proper speed $|dx/d\tau| = \gamma|dx/dt|$ *is unbounded* because γ is unbounded. Thus proper speed $|dx/d\tau|$ is somewhat analogous *to the unbounded speed* $|dx/dt|$ *of Newtonian physics*.

The extension to three spatial dimensions of Eq. (1b) with regard to $d\tau$ and proper velocity $(d\mathbf{r}/d\tau)$ is,

$$d\tau \stackrel{\text{def}}{=} \left((dt)^2 - |d\mathbf{r}/c|^2\right)^{\frac{1}{2}} = \left(1 - |\dot{\mathbf{r}}/c|^2\right)^{\frac{1}{2}} dt = \gamma^{-1}dt \Rightarrow (dt/d\tau) = \gamma \Rightarrow (d\mathbf{r}/d\tau) = \gamma\dot{\mathbf{r}}. \tag{2}$$

The Lorentz-transformation invariant differential proper time $d\tau$ of Eq. (2) is somewhat analogous to the Galilean-transformation invariant *dt of Newtonian physics*. Also, *despite* $|\dot{\mathbf{r}}| < c$, the Eq. (2) proper speed $|d\mathbf{r}/d\tau| = \gamma|\dot{\mathbf{r}}|$ *is unbounded*; thus it is somewhat analogous *to the unbounded speed* $|\dot{\mathbf{r}}|$ *of Newtonian physics*.

Proper time produces the Lorentz-covariant extension of Newton's Second Law

The usual presentation of single-particle relativistic dynamics is,

$$(d\mathbf{p}/dt) = \mathbf{f}, \tag{3a}$$

where $\mathbf{f}$ is the force and the relativistic single-particle momentum $\mathbf{p}$ is given by,

$$\mathbf{p} = m\gamma\dot{\mathbf{r}}. \tag{3b}$$

From Eq. (2) we see that Eq. (3b) can be rewritten,

$$\mathbf{p} = m(d\mathbf{r}/d\tau), \tag{3c}$$

so Eq. (3a) becomes,

$$m(d(d\mathbf{r}/d\tau)/dt) = \mathbf{f}. \tag{3d}$$

*Retired, American Physical Society Senior Life Member, E-mail: SKKauffmann@gmail.com

From Eq. (2), $(dt/d\tau) = \gamma$, so we multiply the left side of Eq. (3d) by $(dt/d\tau)$ and its right side by γ, yielding,

$$m(d(d\mathbf{r}/d\tau)/dt)(dt/d\tau) = \gamma\mathbf{f}. \tag{3e}$$

We simplify the expression on the left side of Eq. (3e) and we denote $\gamma\mathbf{f}$ *as the proper force* $\mathbf{F}$ to obtain,

$$m(d^2\mathbf{r}/d\tau^2) = \mathbf{F}, \tag{3f}$$

the three *ordinary force components* of the relativistic extension of Newton's Second Law. An *example* of Eq. (3f) pertains to the proper force exerted by an electromagnetic field on a particle of charge e, namely,

$$\mathbf{F} = e\gamma\left(\mathbf{E} + ((\dot{\mathbf{r}}/c)\times\mathbf{B})\right). \tag{3g}$$

The still missing *fourth component* of the *Lorentz-covariant four-vector completion of* Eq. (3f) *must read*,

$$m(d^2x^0/d\tau^2) = F^0, \text{ where, of course, } x^0 = ct. \tag{3h}$$

Relativistically, $\left(F^0c\right)$ *should relate to particle power* $(\mathbf{F}\cdot\dot{\mathbf{r}})$. To *test* this, we *first* boil down Eq. (3h) to,

$$\begin{gathered}\left(F^0c\right) = mc(d^2x^0/d\tau^2) = mc^2(d(dt/d\tau)/d\tau) = mc^2(d\gamma/d\tau) = \\ mc^2(d\gamma/dt)(dt/d\tau) = mc^2\gamma\left(d\left(1-(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}/c^2)\right)^{-\frac{1}{2}}/dt\right) = \gamma^4 m(\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}}),\end{gathered} \tag{3i}$$

then we *next* similarly boil down Eq. (3f) to,

$$\mathbf{F} = m(d^2\mathbf{r}/d\tau^2) = m(d(\gamma\dot{\mathbf{r}})/d\tau) = m\gamma(d(\gamma\dot{\mathbf{r}})/dt) = m\gamma((d\gamma/dt)\dot{\mathbf{r}} + \gamma\ddot{\mathbf{r}}) = m\gamma\left(\gamma^3(\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}})\left(\dot{\mathbf{r}}/c^2\right) + \gamma\ddot{\mathbf{r}}\right), \tag{3j}$$

from which we obtain that,

$$(\mathbf{F}\cdot\dot{\mathbf{r}}) = \gamma^2 m(\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}})\left(\gamma^2|\dot{\mathbf{r}}/c|^2 + 1\right) = \gamma^2 m(\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}})\left(\gamma^2\left(1-\gamma^{-2}\right)+1\right) = \gamma^4 m(\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}}) = \left(F^0c\right), \tag{3k}$$

where the last equality follows from Eq. (3i). Inserting the Eq. (3k) value for F^0 into Eq. (3h) then yields,

$$m(d^2x^0/d\tau^2) = (\mathbf{F}\cdot(\dot{\mathbf{r}}/c)), \tag{3l}$$

which *combined with* Eq. (3f) *is the full Lorentz-covariant four-vector extension of Newton's Second Law*,

$$m(d^2x^\mu/d\tau^2) = F^\mu = ((\mathbf{F}\cdot(\dot{\mathbf{r}}/c)),\,\mathbf{F}). \tag{3m}$$

Since $(dt/d\tau) = \gamma = \left(1-|\dot{\mathbf{r}}/c|^2\right)^{-\frac{1}{2}} \to 1$ in the nonrelativistic limit that $|\dot{\mathbf{r}}/c| \to 0$, it is *transparent* that Eq. (3m) *reduces to the Newtonian Second Law three-vector equation system*,

$$m\ddot{\mathbf{r}} = \mathbf{f}, \tag{3n}$$

in the nonrelativistic limit that $|\dot{\mathbf{r}}/c| \to 0$.

Eq. (3m) shows that the concept of inertial mass, which is *the same* as relativistic rest mass, *is just as relevant to special-relativistic physics as it is to Newtonian physics*. Indeed, the development of Higgs field physics [2] has put considerable flesh on the bones of the inertial mass concept. An interesting *relativistic issue* is the *existence* of particles, such as *photons*, which have *zero inertial mass* (these are asserted *to not couple at all to the Higgs field*). A zero-inertial-mass particle which has nonzero momentum $|\mathbf{p}| > 0$ *has infinite proper speed* $|d\mathbf{r}/d\tau| = \lim_{m\to 0}(|\mathbf{p}|/m) = \infty$ that corresponds *to perceived speed* c. To *demonstrate the last fact*, we *invert* the Eq. (2) relation of proper velocity $(d\mathbf{r}/d\tau)$ to perceived velocity $\dot{\mathbf{r}}$, which is,

$$(d\mathbf{r}/d\tau) = \gamma\dot{\mathbf{r}} = \left(1-|\dot{\mathbf{r}}/c|^2\right)^{-\frac{1}{2}}\dot{\mathbf{r}}.$$

This relation's *inverse* comes out to be,

$$\dot{\mathbf{r}} = (d\mathbf{r}/d\tau)\left(1+|(d\mathbf{r}/d\tau)/c|^2\right)^{-\frac{1}{2}}, \tag{3o}$$

which has the asymptotic form,

$$\dot{\mathbf{r}} \sim c\left((d\mathbf{r}/d\tau)/|d\mathbf{r}/d\tau|\right) \text{ as } |(d\mathbf{r}/d\tau)/c| \to \infty. \tag{3p}$$

This result shows that zero-inertial-mass particles of nonzero momentum $|\mathbf{p}| > 0$, *which thus have infinite proper speed* $|d\mathbf{r}/d\tau| = \lim_{m\to 0}(|\mathbf{p}|/m) = \infty$, must therefore *have perceived speed* $|\dot{\mathbf{r}}|$ *equal to* c.

Electromagnetic case of the Lorentz-covariant Newton's Second Law extension

A case of the *full* Eq. (3m) Newton's Second Law extension emerges *directly* from the following *entirely Lorentz-invariant Lagrangian* for the interaction of a charged particle with the electromagnetic field,

$$L((dx_\nu/d\tau),\, x_\nu) = -\tfrac{1}{2}m(dx_\nu/d\tau)(dx^\nu/d\tau) - (e/c)(dx_\nu/d\tau)A^\nu, \tag{4a}$$

whose equation-of-motion prescription is the natural,

$$\left(d\left(\partial L/\partial\left(dx_\mu/d\tau\right)\right)/d\tau\right) = \left(\partial L/\partial x_\mu\right). \tag{4b}$$

Eqs. (4a) and (4b) yield the equation of motion,

$$-m\left(d^2x^\mu/d\tau^2\right) - (e/c)\left(dA^\mu/d\tau\right) = -(e/c)\left(dx_\nu/d\tau\right)\left(\partial A^\nu/\partial x_\mu\right), \tag{4c}$$

which we rearrange as,

$$m\left(d^2x^\mu/d\tau^2\right) = (e/c)\left[\left(dx_\nu/d\tau\right)\left(\partial A^\nu/\partial x_\mu\right) - \left(dA^\mu/d\tau\right)\right]. \tag{4d}$$

We now apply the chain rule to calculate,

$$\left(dA^\mu/d\tau\right) = \left(\partial A^\mu/\partial x_\nu\right)\left(dx_\nu/d\tau\right), \tag{4e}$$

a result we substitute into Eq. (4d), followed by taking the common factor of $(dx_\nu/d\tau)$ outside of the square brackets to obtain,

$$m\left(d^2x^\mu/d\tau^2\right) = (e/c)\left(dx_\nu/d\tau\right)\left[\left(\partial A^\nu/\partial x_\mu\right) - \left(\partial A^\mu/\partial x_\nu\right)\right]. \tag{4f}$$

This equation *clearly has the relativistic format of* Eq. (3m). We now boil down the right side of Eq. (4f) in the cases where $\mu = i$, where i takes on the values 1, 2 or 3. We will thus obtain *precisely the proper force described by* Eqs. (3f) *and* (3g). Our first order of business with the right side of Eq. (4f) is to note that,

$$\left(dx_\nu/d\tau\right) = \left(dt/d\tau\right)\left(dx_\nu/dt\right) = \gamma\left(dx_\nu/dt\right) = \begin{cases} \gamma c & \text{if } \nu = 0, \\ \gamma\left(-\dot{x}^j\right) & \text{if } \nu = j, \text{ where } j = 1,2,3. \end{cases} \tag{4g}$$

For $\mu = i$, where $i = 1$, 2 or 3, Eqs. (4f) and (4g) yield,

$$\begin{aligned} m\left(d^2x^i/d\tau^2\right) &= e\gamma\left[-\left(\partial A^0/\partial x^i\right) - (1/c)\dot{A}^i\right] + (e/c)\gamma\sum_{j=1}^{3}\left(\dot{x}^j\right)\left[\left(\partial A^j/\partial x^i\right) - \left(\partial A^i/\partial x^j\right)\right] = \\ & e\gamma\left(-\left(\nabla_{\mathbf{r}}A^0\right) - (1/c)\dot{\mathbf{A}}\right)^i + (e/c)\gamma\left(\left(\nabla_{\mathbf{r}}\left(\dot{\mathbf{r}}\cdot\mathbf{A}\right)\right) - \left(\left(\dot{\mathbf{r}}\cdot\nabla_{\mathbf{r}}\right)\mathbf{A}\right)\right)^i = \\ & e\gamma\left(\mathbf{E} + \left((\dot{\mathbf{r}}/c)\times\left(\nabla_{\mathbf{r}}\times\mathbf{A}\right)\right)\right)^i = e\gamma\left(\mathbf{E} + \left((\dot{\mathbf{r}}/c)\times\mathbf{B}\right)\right)^i, \end{aligned} \tag{4h}$$

which accords with Eqs. (3f) *and* (3g). We *likewise* use Eqs. (4f) and (4g) to boil down the $\mu = 0$ case,

$$\begin{aligned} m\left(d^2x^0/d\tau^2\right) &= e\gamma\left[(1/c)\dot{A}^0 - (1/c)\dot{A}^0\right] + (e/c)\gamma\sum_{j=1}^{3}\left(-\dot{x}^j\right)\left[(1/c)\dot{A}^j + \left(\partial A^0/\partial x^j\right)\right] = \\ & \left(e\gamma(\dot{\mathbf{r}}/c)\cdot\left(-(1/c)\dot{\mathbf{A}} - \left(\nabla_{\mathbf{r}}A^0\right)\right)\right) = \left(e\gamma\left(\mathbf{E}\right)\cdot\left(\dot{\mathbf{r}}/c\right)\right) = \left(e\gamma\left(\mathbf{E} + \left((\dot{\mathbf{r}}/c)\times\mathbf{B}\right)\right)\cdot\left(\dot{\mathbf{r}}/c\right)\right), \end{aligned} \tag{4i}$$

since $\left(\left((\dot{\mathbf{r}}/c)\times\mathbf{B}\right)\cdot\left(\dot{\mathbf{r}}/c\right)\right) = 0$. Eqs. (4i) *and* (4h) *together accord with the fundamental* Eqs. (3l) *and* (3m).

References

[1] Length contraction–Wikipedia, https://en.wikipedia.org/wiki/Length_contraction.

[2] Higgs field–Simple Wikipedia, https://simple.wikipedia.org/wiki/Higgs_field.