Considerations on the Newton's binomial expansion

$$(x+y)^n = x^n + y^n + xy\sum_{j=1}^{n-2}\left(x^j + y^j\right)(x+y)^{n-2-j}$$

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Résumé

The binomial formula, set by Isaac Newton, is of utmost importance and has been extensively used in many different fields. This study aims at coming up with alternative expressions to the Newton's formula.

Chapitre 1

Another way to write Newton's binomial expansion.

1.1 Purpose of this chapter.

Newton's binomial expansion can be expressed differently. This new formulation allows in turn to perform other calculations which will highlight certain properties that the original formula may not be able to provide.

1.2 Another formula.

Let $n \in \mathbb{N}^*$, and $x \in \mathbb{R}^*$ and $y \in \mathbb{R}^*$. In all that follows, we assume $n \geq 3$. We can write

$$\frac{(x+y)^n - x^n}{(x+y) - x} = \sum_{j=0}^{n-1} (x+y)^{n-1-j} \, x^j = \frac{(x+y)^n - x^n}{y}$$

and likewise

$$\frac{(x+y)^n - y^n}{(x+y) - y} = \sum_{j=0}^{n-1} (x+y)^{n-1-j} \, y^j = \frac{(x+y)^n - y^n}{x}$$

Let us add these two quantities

$$\frac{(x+y)^n - x^n}{y} + \frac{(x+y)^n - y^n}{x} = \sum_{j=0}^{n-1} (x+y)^{n-1-j} \left(x^j + y^j\right)$$

and we end up with the formula

$$(x+y)^{n+1} - \left(x^{n+1} + y^{n+1}\right) = xy \sum_{j=0}^{n-1} (x+y)^{n-1-j} \left(x^j + y^j\right)$$

which, for convenience's sake, we write

$$(x+y)^n - (x^n + y^n) = xy \sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) \tag{1.1}$$

Let us now recall the Newton's binomial expansion

$$(x+y)^n = \sum_{j=0}^{n} C_n^j x^{n-j} y^j \tag{1.2}$$

wherein

$$C_n^j = \frac{n!}{(n-j)!j!} \tag{1.3}$$

allows to establish the equality

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) = \sum_{j=1}^{n-1} C_n^j x^{n-j-1} y^{j-1}$$

or lastly

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) = \sum_{j=0}^{n-2} C_n^{j+1} x^{n-2-j} y^j$$

1.3 Study of the new formula.

Let us pose

$$A_n(x,y) = \sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) \tag{1.4}$$

Let us remark first that

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) = \sum_{j=0}^{p-2} (x+y)^{n-2-j} \left(x^j + y^j\right) + \sum_{j=p-1}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right)$$

with $p \in \mathbb{N}^*$ et $p < n$, or likewise

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) = \sum_{j=0}^{p-2} (x+y)^{(n-p)+(p-2-j)} \left(x^j + y^j\right) + \sum_{j=p-1}^{n-2} (x+y)^{n-2-(j-(p-1)+p-1)} \left(x^{j-(p-1)+p-1} + y^{j-(p-1)+p-1}\right)$$

and

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) =$$

$$(x+y)^{n-p} \sum_{j=0}^{p-2} (x+y)^{p-2-j} \left(x^j + y^j\right) + \sum_{j=0}^{n-2-(p-1)} (x+y)^{n-2-(j+p-1)} \left(x^{j+p-1} + y^{j+p-1}\right)$$

and

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) =$$

$$(x+y)^{n-p} \sum_{j=0}^{p-2} (x+y)^{p-2-j} \left(x^j + y^j\right) + \sum_{j=0}^{n-2-(p-1)} (x+y)^{n-p-1-j} \left(x^{j+p-1} + y^{j+p-1}\right)$$

and finally

$$A_n(x,y) = (x+y)^{n-p} A_p(x,y) + \sum_{j=0}^{n-2-(p-1)} (x+y)^{n-p-1-j} \left(x^{j+p-1} + y^{j+p-1}\right)$$

Let us now consider the case wherein $n = p+1$, then

$$A_{p+1}(x,y) = (x+y) A_p(x,y) + \sum_{j=0}^{0} (x+y)^{-j} \left(x^{j+p-1} + y^{j+p-1}\right)$$

or likewise

$$A_{p+1}(x,y) = (x+y) A_p(x,y) + \left(x^{p-1} + y^{p-1}\right)$$

but

$$x^{p-1} + y^{p-1} = (x+y)^{p-1} - xyA_{p-1}(x,y)$$

and thus

$$A_{p+1}(x,y) = (x+y) A_p(x,y) + (x+y)^{p-1} - xyA_{p-1}(x,y) \tag{1.5}$$

Let us concentrate now more specifically on $A_n(x,y)$ and let us develop this quantity from the formula 1.4 en page 2. And so

$$\begin{aligned}
A_n(x,y) &= 3(x+y)^{n-2} + \sum_{j=0}^{n-4} (x+y)^{n-4-j}\left(x^{j+2}+y^{j+2}\right) \\
&= 3(x+y)^{n-2} + \left(x^2+y^2\right)^{n-4} + \sum_{j=0}^{n-5} (x+y)^{n-5-j}\left(x^{j+3}+y^{j+3}\right) \\
&= 3(x+y)^{n-2} + (x+y)^{n-2} - 2xy(x+y)^{n-4} + \sum_{j=0}^{n-5} (x+y)^{n-5-j}\left(x^{j+3}+y^{j+3}\right) \\
&= 4(x+y)^{n-2} - 2xy(x+y)^{n-4} + \sum_{j=0}^{n-5} (x+y)^{n-5-j}\left(x^{j+3}+y^{j+3}\right)
\end{aligned}$$

As we continue our calculations in the same manner, we get

$$\begin{aligned}
A_n(x,y) = 5(x+y)^{n-2} \\
- 5xy(x+y)^{n-4} + \sum_{j=0}^{n-6} (x+y)^{n-6-j}\left(x^{j+4}+y^{j+4}\right)
\end{aligned}$$

$$\begin{aligned}
A_n(x,y) = 6(x+y)^{n-2} - 9xy(x+y)^{n-4} + 2x^2y^2(x+y)^{n-6} \\
+ \sum_{j=0}^{n-7} (x+y)^{n-7-j}\left(x^{j+5}+y^{j+5}\right)
\end{aligned}$$

$$\begin{aligned}
A_n(x,y) = 7(x+y)^{n-2} - 14xy(x+y)^{n-4} + 7x^2y^2(x+y)^{n-6} \\
+ \sum_{j=0}^{n-8} (x+y)^{n-8-j}\left(x^{j+6}+y^{j+6}\right)
\end{aligned}$$

$$\begin{aligned}
A_n(x,y) = 8(x+y)^{n-2} - 20xy(x+y)^{n-4} + 16x^2y^2(x+y)^{n-6} - 2x^3y^3(x+y)^{n-8} \\
+ \sum_{j=0}^{n-9} (x+y)^{n-9-j}\left(x^{j+7}+y^{j+7}\right)
\end{aligned}$$

$$\begin{aligned}
A_n(x,y) = 9(x+y)^{n-2} - 27xy(x+y)^{n-4} + 30x^2y^2(x+y)^{n-6} - 9x^3y^3(x+y)^{n-8} \\
+ \sum_{j=0}^{n-10} (x+y)^{n-10-j}\left(x^{j+8}+y^{j+8}\right)
\end{aligned}$$

$$\begin{aligned}
A_n(x,y) = 10(x+y)^{n-2} - 35xy(x+y)^{n-4} + 50x^2y^2(x+y)^{n-6} - 25x^3y^3(x+y)^{n-8} \\
+ 2x^4y^4(x+y)^{n-10} \\
+ \sum_{j=0}^{n-11} (x+y)^{n-11-j}\left(x^{j+9}+y^{j+9}\right)
\end{aligned}$$

$$
\begin{aligned}
A_n(x,y) = 11(x+y)^{n-2} - 44xy(x+y)^{n-4} + 77x^2y^2(x+y)^{n-6} - 55x^3y^3(x+y)^{n-8} \\
+ 11x^4y^4(x+y)^{n-10} \\
+ \sum_{j=0}^{n-12} (x+y)^{n-12-j}\left(x^{j+10} + y^{j+10}\right)
\end{aligned}
$$

It is of course possible to extend our calculations as far as we desire. As n is taking on the values 3, 4, 5, 6,..., we can deduct the respective new developments of $A_3(x,y)$, $A_4(x,y)$, $A_5(x,y)$, $A_6(x,y)$, etc.

Let us assume now that the following formulas hold for all integers less than or equal to $2k$ et $2k+1$, where $k \in \mathbb{N}^*$

$$
A_{2k}(x,y) = \sum_{j=0}^{k-1} D_{2k}^{j} (-1)^j (xy)^j (x+y)^{2(k-1-j)} \tag{1.6}
$$

$$
A_{2k+1}(x,y) = (x+y) \sum_{j=0}^{k-1} D_{2k+1}^{j} (-1)^j (xy)^j (x+y)^{2(k-1-j)} \tag{1.7}
$$

The coefficients D_{2k}^{j} et D_{2k+1}^{j} are to be made explicit if possible (and will be indeed further down this study).

Let us go back to the equation 1.5 page 3 and rewrite in the form

$$
A_{2k+2}(x,y) = (x+y)A_{2k+1}(x,y) + (x+y)^{2k} - xyA_{2k}(x,y)
$$

Let us develop now this relation

$$
\begin{aligned}
A_{2k+2}(x,y) = (x+y)^2 \sum_{j=0}^{k-1} D_{2k+1}^{j} (-1)^j (xy)^j (x+y)^{2(k-1-j)} \\
+ (x+y)^{2k} \\
- xy \sum_{j=0}^{k-1} D_{2k}^{j} (-1)^j (xy)^j (x+y)^{2(k-1-j)}
\end{aligned}
$$

$$
\Longleftrightarrow
$$

$$
\begin{aligned}
A_{2k+2}(x,y) = (x+y)^2 \sum_{j=0}^{k-1} D_{2k+1}^{j} (-1)^j (xy)^j (x+y)^{2(k-1-j)} \\
+ (x+y)^{2k} \\
+ \sum_{j=0}^{k-1} D_{2k}^{j} (-1)^{j+1} (xy)^{j+1} (x+y)^{2(k-1-j)}
\end{aligned}
$$

Let us carry on with our calculations. We obtain in an equivalent manner

$$A_{2k+2}(x,y) = \sum_{j=0}^{k-1} D_{2k+1}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-j)}$$
$$+ (x+y)^{2k}$$
$$+ \sum_{j=0}^{k-1} D_{2k}^{j} (-1)^{j+1} (xy)^{j+1} (x+y)^{2(k-1-j)}$$
$$\Longleftrightarrow$$
$$A_{2k+2}(x,y) = \sum_{j=0}^{k-1} D_{2k+1}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-j)}$$
$$+ (x+y)^{2k}$$
$$+ \sum_{j=1}^{k} D_{2k}^{j-1} (-1)^{j} (xy)^{j} (x+y)^{2(k-j)}$$
$$\Longleftrightarrow$$
$$A_{2k+2}(x,y) = \sum_{j=1}^{k-1} \left(D_{2k+1}^{j} + D_{2k}^{j-1}\right) (-1)^{j} (xy)^{j} (x+y)^{2(k-j)}$$
$$+ D_{2k+1}^{0} (x+y)^{2k} + (x+y)^{2k} + D_{2k}^{k-1} (xy)^{k}$$

and we can write

$$A_{2k+2}(x.y) = \sum_{j=0}^{k} D_{2k+2}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-j)}$$

with

$$D_{2k+2}^{0} = D_{2k+1}^{0} + 1$$
$$D_{2k+2}^{k} = D_{2k}^{k-1}$$

and

$$(\forall j \in \mathbb{N})\,(1 \leq j \leq k-1)\left(D_{2k+2}^{j} = D_{2k+1}^{j} + D_{2k}^{j-1}\right)$$

Similarly, we have

$$A_{2k+3}(x,y) = (x+y)\,A_{2k+2}(x,y) + (x+y)^{2k} - xyA_{2k+1}(x,y)$$

Let us make it more explicit

$$A_{2k+3}(x,y) = (x+y) \sum_{j=0}^{k} D_{2k+2}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-j)}$$
$$+ (x+y)^{2k+1}$$
$$- xy\,(x+y) \sum_{j=0}^{k-1} D_{2k+1}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-1-j)}$$

hence

$$A_{2k+3}(x,y) = \sum_{j=0}^{k} D_{2k+2}^{j} (-1)^j (xy)^j (x+y)^{2(k-j)+1} \\ + (x+y)^{2k+1} \\ + \sum_{j=0}^{k-1} D_{2k+1}^{j} (-1)^{j+1} (xy)^{j+1} (x+y)^{2(k-1-j)+1}$$

which is equivalent to

$$A_{2k+3}(x,y) = (x+y)^{2k+1} \\ + \sum_{j=0}^{k} D_{2k+2}^{j} (-1)^j (xy)^j (x+y)^{2(k-j)+1} \\ + \sum_{j=1}^{k-1} D_{2k+1}^{j-1} (-1)^j (xy)^j (x+y)^{2(k-j)+1}$$

and also

$$A_{2k+3}(x,y) = \left(D_{2k+2}^{0} + 1\right)(x+y)^{2k+1} \\ + \sum_{j=1}^{k} \left(D_{2k+2}^{j} + D_{2k+1}^{j-1}\right) (-1)^j (xy)^j (x+y)^{2(k-j)+1}$$

and we can finally write

$$A_{2k+3}(x,y) = (x+y) \sum_{j=0}^{k} D_{2k+2}^{j} (-1)^j (xy)^j (x+y)^{2(k-j)}$$

with

$$D_{2k+2}^{0} = D_{2k+1}^{0} + 1$$

and

$$(\forall j \in \mathbb{N})\,(1 \leq j \leq k)\left(D_{2k+3}^{j} = D_{2k+2}^{j} + D_{2k+1}^{j-1}\right)$$

This concludes our mathematical induction and we can write at last

$$(\forall k \in \mathbb{N}^*)\left(A_{2k} = \sum_{j=0}^{k-1} D_{2k}^{j} (-1)^j (xy)^j (x+y)^{2(k-1-j)}\right) \qquad (1.8)$$

with

$$D_{2k}^{0} = D_{2k-1}^{0} + 1 \Longleftrightarrow D_{2k}^{0} = 2k \qquad (1.9)$$

and

$$D_{2k}^{k-1} = D_{2k-2}^{k-2} = \cdots = D_{4}^{1} = 2 \qquad (1.10)$$

and

$$(\forall j \in \mathbb{N})\,(1 \leq j \leq k-1)\left(D_{2k}^{j} = D_{2k-1}^{j} + D_{2k-2}^{j-1}\right) \tag{1.11}$$

and as well

$$(\forall k \in \mathbb{N}^{*})\left(A_{2k+1} = (x+y)\sum_{j=0}^{k-1} D_{2k+1}^{j}\,(-1)^{j}\,(xy)^{j}\,(x+y)^{2(k-1-j)}\right) \tag{1.12}$$

with

$$D_{2k+1}^{0} = D_{2k}^{0} + 1 \Longleftrightarrow D_{2k}^{0} = 2k+1 \tag{1.13}$$

and

$$(\forall j \in \mathbb{N})\,(1 \leq j \leq k)\left(D_{2k+1}^{j} = D_{2k}^{j} + D_{2k-1}^{j-1}\right) \tag{1.14}$$

1.4 Values taken by the coefficients D_h^j where $(h \in \mathbb{N})$ and $(h \geq 3)$

We have, as we just established it

$$(\forall h \in \mathbb{N})\,(h \geq 3)\left(D_h^0 = h\right)$$

Let us take $j = 1$. We have

$$D_h^1 = D_{h-1}^1 + D_{h-2}^0$$

We can then write

$$\left.\begin{aligned} D_h^1 &= D_{h-1}^1 + D_{h-2}^0 \\ D_{h-1}^1 &= D_{h-2}^1 + D_{h-3}^0 \\ &\cdots \\ &\cdots \\ &\cdots \\ D_5^1 &= D_4^1 + D_3^0 \end{aligned}\right\} \Longrightarrow D_h^1 = \sum_{j=0}^{h-5} D_{h-2-j}^0 + D_4^1$$

but

$$D_{h-2-j}^0 = h - 2 - j$$

and, according to the relation 1.10 established in page 7

$$D_4^1 = 2$$

hence we get

$$D_h^1 = \sum_{j=0}^{h-5} (h-2-j) + 2 = ((h-2) + (h-3) + (h-4) + \cdots + 3) + 2$$

and therefore

$$2D_h^1 = h\,(h+3)$$

and finally

$$(\forall h \in \mathbb{N}^{*})\,(h \geq 3)\left(D_h^1 = \frac{h\,(h-3)}{2}\right) \tag{1.15}$$

Clearly

$$(\forall h \in \mathbb{N}^*)\,(h \geq 3)\,\left(D_h^1 \in \mathbb{N}\right)$$

Making similar calculations, we find for every integer $h \geq 3$

$$D_h^2 = \frac{h\,(h-4)\,(h-5)}{6} \tag{1.16}$$

$$D_h^3 = \frac{h\,(h-5)\,(h-6)\,(h-7)}{24} \tag{1.17}$$

There as well

$$(\forall h \in \mathbb{N}^*)\,(h \geq 3)\,\left(D_h^2 \in \mathbb{N}\right)$$

$$(\forall h \in \mathbb{N}^*)\,(h \geq 3)\,\left(D_h^3 \in \mathbb{N}\right)$$

We then remark that the relations 1.11 and 1.14 established in page 8, as well as those ((see relations 1.15, 1.16 et 1.17) established in pages 8 and 9 allow us to affirm

$$(\forall h \in \mathbb{N}^*)\,(h \geq 3)\left(\forall j \in \left\{0, 1, \cdots, \frac{h-4}{2}\right\}\right)\left(D_h^j \in \mathbb{N}\right)$$

let us assume now, h being chosen as even, and for all $j \in \left\{0, 1, \cdots, \frac{h-2}{2}\right\}$ the formula

$$D_h^j = \frac{h\,(h-(j+2))!}{(j+1)!\,(h-2\,(j+1))!} \tag{1.18}$$

true until rank h, for all even integer lower than or equal to h

Let us assume as well that, for all $j \in \left\{0, 1, \cdots, \frac{h-4}{2}\right\}$, until rank $h-1$, the formula

$$D_{h-1}^j = \frac{(h-1)\,((h-1)-(j+2))!}{(j+1)!\,((h-1)-2\,(j+1))!} \tag{1.19}$$

is true. Then

$$D_{h-1}^{j-1} = \frac{(h-1)\,(h-1-(j+1)!)}{j!\,(h-1-2j)!} = \frac{(h-1)\,(h-(j+2)!)}{j!\,(h-1-2j)!}$$

The relation 1.11 established in page 8 allow us to write

$$D_{h+1}^{j} = \frac{h\,(h-(j+2))!}{(j+1)!\,(h-2\,(j+1))!} + \frac{(h-1)\,(h-(j+2))!}{j!\,(h-1-2j)!}$$

$$\Longleftrightarrow$$

$$D_{h+1}^{j} = \frac{(h-(j+2))!}{j!}\left(\frac{h}{(j+1)\,(h-2\,(j+1))!} + \frac{(h-1)}{(h-1-2j)!}\right)$$

$$\Longleftrightarrow$$

$$D_{h+1}^{j} = \frac{(h-(j+2))!}{j!}\left(\frac{h\,(h-1-2j)+(h-1)\,(j+1)}{(h-1-2j)!\,(j+1)}\right)$$

$$\Longleftrightarrow$$

$$D_{h+1}^{j} = \frac{(h-(j+2))!}{(j+1)!\,(h-1-2j)!}\,(h\,(h-1)-2jh+(h-1)\,j+(h-1))$$

$$\Longleftrightarrow$$

$$D_{h+1}^{j} = \frac{(h-(j+2))!}{(j+1)!\,(h-1-2j)!}\left(h^2-1-(h+1)\,j\right)$$

and finally

$$D_{h+1}^{j} = \frac{(h-(j+2))!}{(j+1)!\,(h-1-2j)!}\,(h+1)\,(h-1-j)$$

We can therefore write

$$D_{h+1}^{j} = \frac{(h+1)\,(h-(j+1))!}{(j+1)!\,(h+1-2\,(j+1))!} \tag{1.20}$$

We could make similar calculations if we take h as odd

We verify that

$$(\forall h \in \mathbb{N}^*)\,(h \geq 3)\left(D_h^0 = h\right)$$

and, as we denote the ensemble of even integers as $2\mathbb{N}$

$$(\forall h = 2k \in 2\mathbb{N}^*)\,(h \geq 4)\left(D_{2k}^{k-1} = 2\right)$$

At the end of this mathematical induction, we have then established

$$\begin{aligned}&(\forall k \in \mathbb{N}^*)\,(\forall j \in \{0,1,2,\cdots,k-1\})\\ &\qquad\left(D_{2k+1}^{j} = \frac{(2k+1)\,(2k-(j+1))!}{(j+1)!\,(2k+1-2\,(j+1))!}\right)\\ &\qquad\quad\left(D_{2(k+1)}^{j} = \frac{2\,(k+1)\,(2\,(k+1)-1-(j+1))!}{(j+1)!\,(2\,(k+1)-2\,(j+1))!}\right)\end{aligned} \tag{1.21}$$

Let us remark that for all integer h

$$h-2\,(j+1)+(j+1) = h-(j+1)$$

We can write

$$D_h^j = \frac{h\,(h-(j+1))!}{(h-(j+1))\,(j+1)!\,(h-2\,(j+1))!}$$

and also

$$D_h^j = \frac{h}{h-(j+1)}C_{h-(j+1)}^{j+1}$$

1.5 Study on the coefficients D_h^j

For the following odd integers h, we verify the relations

$$k=1 \Longleftrightarrow h=2k+1=3$$
$$D_3^0 = 3C_0^0$$

$$k=2 \Longleftrightarrow h=2k+1=5$$
$$D_5^0 = 5C_1^0$$
$$D_5^1 = 5C_1^1$$

$$k=3 \Longleftrightarrow h=2k+1=7$$
$$D_7^0 = 7C_2^0$$
$$D_7^1 = 7C_2^1$$
$$D_7^2 = 7C_2^2$$

$$k=4 \Longleftrightarrow h=2k+1=9$$
$$D_9^0 = 9C_3^0$$
$$D_9^1 = 9C_3^1$$
$$D_9^2 = 9C_3^2 + 3C_0^0$$
$$D_9^1 = 9C_3^3$$

$$k=5 \Longleftrightarrow h=2k+1=11$$
$$D_{11}^0 = 11C_4^0$$
$$D_{11}^1 = 11C_4^1$$
$$D_{11}^2 = 11\left(C_4^2 + C_1^0\right)$$
$$D_{11}^3 = 11\left(C_4^3 + C_1^1\right)$$
$$D_{11}^4 = 11C_4^4$$

$$k=6 \Longleftrightarrow h=2k+1=13$$
$$D_{13}^0 = 13C_5^0$$
$$D_{13}^1 = 13C_5^1$$
$$D_{13}^2 = 13\left(C_5^2 + 2C_2^0\right)$$
$$D_{13}^3 = 13\left(C_5^3 + 2C_2^1\right)$$
$$D_{13}^4 = 13\left(C_5^4 + 2C_2^2\right)$$
$$D_{13}^5 = 13C_5^5$$

$k=7 \Longleftrightarrow h=2k+1=15$

$$
\begin{aligned}
D_{15}^{0} &= 15C_{6}^{0}\\
D_{15}^{1} &= 15C_{6}^{1}\\
D_{15}^{2} &= 15\left(C_{6}^{2}+3C_{3}^{0}\right)\\
D_{15}^{3} &= 15\left(C_{6}^{3}+3C_{3}^{1}\right)\\
D_{15}^{4} &= 15\left(C_{6}^{4}+3C_{3}^{2}+3C_{0}^{0}\right)\\
D_{15}^{5} &= 15\left(C_{6}^{5}+3C_{3}^{3}\right)\\
D_{15}^{6} &= 15C_{6}^{6}
\end{aligned}
$$

$k=8 \Longleftrightarrow h=2k+1=17$

$$
\begin{aligned}
D_{17}^{0} &= 17C_{7}^{0}\\
D_{17}^{1} &= 17C_{7}^{1}\\
D_{17}^{2} &= 17\left(C_{7}^{2}+5C_{4}^{0}\right)\\
D_{17}^{3} &= 17\left(C_{7}^{3}+5C_{4}^{1}\right)\\
D_{17}^{4} &= 17\left(C_{7}^{4}+5C_{4}^{2}+C_{0}^{0}\right)\\
D_{17}^{5} &= 17\left(C_{7}^{5}+5C_{4}^{3}+C_{1}^{1}\right)\\
D_{17}^{6} &= 17\left(C_{7}^{6}+5C_{4}^{4}\right)\\
D_{17}^{7} &= 17C_{7}^{7}
\end{aligned}
$$

$k=9 \Longleftrightarrow h=2k+1=19$

$$
\begin{aligned}
D_{19}^{0} &= 19C_{8}^{0}\\
D_{19}^{1} &= 19C_{8}^{1}\\
D_{19}^{2} &= 19\left(C_{8}^{2}+7C_{5}^{0}\right)\\
D_{19}^{3} &= 19\left(C_{8}^{3}+7C_{5}^{1}\right)\\
D_{19}^{4} &= 19\left(C_{8}^{4}+7C_{5}^{2}+3C_{2}^{0}\right)\\
D_{19}^{5} &= 19\left(C_{8}^{5}+7C_{5}^{3}+3C_{2}^{1}\right)\\
D_{19}^{5} &= 19\left(C_{8}^{6}+7C_{5}^{4}+2C_{2}^{2}\right)\\
D_{19}^{6} &= 19\left(C_{8}^{7}+7C_{5}^{5}\right)\\
D_{19}^{8} &= 19C_{8}^{8}
\end{aligned}
$$

$k = 10 \Longleftrightarrow h = 2k+1 = 21$

$$D_{21}^0 = 21C_9^0$$
$$D_{21}^1 = 21C_9^1$$
$$D_{21}^2 = 21\left(C_9^2 + 19C_6^0\right)$$
$$D_{21}^3 = 21\left(C_9^3 + 19C_6^1\right)$$
$$D_{21}^4 = 21\left(C_9^4 + 19C_6^2 + 14C_3^0\right)$$
$$D_{21}^5 = 21\left(C_9^5 + 19C_6^3 + 14C_3^1\right)$$
$$D_{21}^6 = 21\left(C_9^6 + 19C_6^4 + 14C_3^2 + 3C_0^0\right)$$
$$D_{21}^7 = 21\left(C_9^7 + 19C_6^5 + 14C_3^3\right)$$
$$D_{21}^8 = 21\left(C_9^7 + 19C_6^6\right)$$
$$D_{21}^9 = 21C_9^9$$

We are brought to assume that for all odd integer $2k+1$, greater than or equal to 3, each coefficient D_{2k+1}^j can be expressed as follows

$$D_{2k+1}^j = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^l C_{k-1-2l}^{j-2l}$$

In order to demonstrate the validity of this formula for all integer k, we are going to develop, to the extent posible, the coefficient F_{2k+1}^l against k and l

for any integer k, we verify the relations

$$D_{2k+1}^0 = (2k+1)\,C_{k-1}^0$$
$$D_{2k+1}^1 = (2k+1)\,C_{k-1}^1$$

We can always write, with $k \geq 4$

$$D_{2k+1}^2 = (2k+1)\,C_{k-1}^2 + \left(D_{2k+1}^2 - (2k+1)\,C_{k-1}^2\right) C_{k-4}^0$$

But, in accordance with the relations 1.3 et 1.21 established in pages 2 and 10

$$D_{2k+1}^2 - (2k+1)\,C_{k-1}^2 = (2k+1)\left(\frac{(2k-3)!}{3!\,(2k+1-6)!} - \frac{(k-1)!}{2!\,(k-3)!}\right)$$

and similarly

$$D_{2k+1}^2 - (2k+1)\,C_{k-1}^2 = (2k+1)\left(\frac{(2k-3)!}{3!\,(2k-5)!} - \frac{(k-1)!}{2!\,(k-3)!}\right)$$

and

$$D_{2k+1}^2 - (2k+1)\,C_{k-1}^2 = (2k+1)\left(\frac{(2k-3)\,(2k-4)}{3!} - \frac{(k-1)\,(k-2)}{2!}\right)$$

and

$$D_{2k+1}^2 - (2k+1)\,C_{k-1}^2 = (2k+1)\left(\frac{(2k-3)\,(k-2)}{3} - \frac{(k-1)\,(k-2)}{2}\right)$$

and also

$$D^2_{2k+1} - (2k+1)\,C^2_{k-1} = (2k+1)\left(\frac{2\,(2k-3)\,(k-2) - 3\,(k-1)\,(k-2)}{6}\right)$$

and finally

$$D^2_{2k+1} - (2k+1)\,C^2_{k-1} = \frac{(2k+1)\,(k-2)\,(k-3)}{6}$$

Let us pose

$$F^1_{2k+1} = D^2_{2k+1} - (2k+1)\,C^2_{k-1} = \frac{(2k+1)\,(k-2)\,(k-3)}{3!}$$

In a similar way, we would find

$$D^3_{2k+1} = (2k+1)\left(C^3_{k-1} + F^1_{2k+1}C^1_{k-4}\right)$$

and

$$D^4_{2k+1} = (2k+1)\left(C^4_{k-1} + F^1_{2k+1}C^2_{k-4} + F^2_{2k+1}C^0_{k-7}\right)$$

which gives us

$$F^2_{2k+1} = \left(\left(D^4_{2k+1} - (2k+1)\,C^4_{k-1}\right) - \left(D^2_{2k+1} - (2k+1)\,C^2_{k-1}\right)C^2_{k-4}\right)$$

Making similar calculations as the previous ones, and with coefficients D^j_{2k+1} and C^l_{k-j} being made explicit, we find

$$F^2_{2k+1} = \frac{(2k+1)\,(k-3)\,(k-4)\,(k-5)\,(k-6)}{5!}$$

We are therefore brought to assume that, for all integer $k \geq 1$, the equality

$$F^l_{2k+1} = \frac{(2k+1)\,(k-1-l)!}{(2l+1)!\,(k-1-3l)!} \tag{1.22}$$

is true, with the integer l such as

$$0 \leq l \leq \lfloor \frac{k}{3} \rfloor$$

Let us calculate the difference

$$F^l_{2k+1} - F^l_{2k-1} = \frac{(2k+1)\,(k-1-l)!}{(2l+1)!\,(k-1-3l)!} - \frac{(2k-1)\,(k-2-l)!}{(2l+1)!\,(k-2-3l)!}$$

Then

$$F^l_{2k+1} - F^l_{2k-1} = \frac{(k-2-l)!}{(2l+1)!\,(k-2-3l)!}\left(\frac{(2k+1)\,(k-1-l) - (2k-1)\,(k-1-3l)}{(k-1-3l)}\right)$$

and

$$F^l_{2k+1} - F^l_{2k-1} = \frac{(k-2-l)!}{(2l+1)!\,(k-1-3l)!}\left((2k+1)\,(k-1-l) - (2k-1)\,(k-1-3l)\right)$$

and also

$$F_{2k+1}^{l}-F_{2k-1}^{l}=\frac{(k-2-l)!}{(2l+1)!\,(k-1-3l)!}\left(2\left(2l+1\right)\left(k-1\right)\right)$$

and finally

$$F_{2k+1}^{l}-F_{2k-1}^{l}=\frac{2\left(k-1\right)\left(k-2-l\right)!}{(2l)!\,(k-1-3l)!} \tag{1.23}$$

Let us use a mathematical induction to show the existence of the relation

$$\left(\forall k\in\mathbb{N}^{*}\right)\left(k\geq 1\right)\left(\forall j\in\mathbb{N}\right)\left(0\leq j\leq k-1\right)\left(D_{2k+1}^{j}=\sum_{l=0}^{\lfloor\frac{k}{3}\rfloor}F_{2k+1}^{l}C_{k-1-3l}^{j-2l}\right)$$

wherein each coefficient F_{2k+1}^{l} is expressed by the formula 1.22 established in page 14.

Let us assume that, for all integer $j\leq k-2$, the relation

$$D_{2k-1}^{j}=\sum_{l=0}^{\lfloor\frac{k-1}{3}\rfloor}F_{2k-1}^{l}C_{k-2-3l}^{j-2l}$$

is true until the rank $2k-1$, with

$$F_{2k-1}^{l}=\frac{(2k-1)\,(k-2-l)!}{(2l+1)!\,(k-2-3l)!}$$

Let us calculate now the difference

$$D_{2k-1}^{j}-D_{2k-3}^{j-1}=D_{2k-2}^{j}$$

and also

$$D_{2k-2}^{j}=\sum_{l=0}^{\lfloor\frac{k-1}{3}\rfloor}F_{2k-1}^{l}C_{k-2-3l}^{j-2l}-\sum_{l=0}^{\lfloor\frac{k-2}{3}\rfloor}F_{2k-3}^{l}C_{k-3-2l}^{j-2l-1}$$

Then, we are faced with two cases.

Case 1 : $\lfloor\frac{k-1}{3}\rfloor=\lfloor\frac{k-2}{3}\rfloor=m$

This case is equivalent to $k\equiv 0\ (3)$ ou $k\equiv 2\ (3)$.

Let us recall that

$$\left(C_{k-2-3l}^{j-2l}=C_{k-3-3l}^{j-2l-1}+C_{k-3-3l}^{j-2l}\right)\Longleftrightarrow\left(C_{k-3-3l}^{j-2l-1}=C_{k-2-3l}^{j-2l}-C_{k-3-3l}^{j-2l}\right)$$

We then have

$$D_{2k-2}^{j}=\sum_{l=0}^{m}\left(F_{2k-1}^{l}C_{k-2-3l}^{j-2l}-F_{2k-3}^{l}C_{k-3-3l}^{j-2l-1}\right)$$

which is equivalent to

$$D_{2k-2}^{j}=\sum_{l=0}^{m}\left(F_{2k-1}^{l}C_{k-2-3l}^{j-2l}-F_{2k-3}^{l}\left(C_{k-2-3l}^{j-2l}-C_{k-3-3l}^{j-2l}\right)\right)$$

and

$$D_{2k-2}^{j} = \sum_{l=0}^{m} \left(\left(F_{2k-1}^{l} - F_{2k-3}^{l}\right) C_{k-2-3l}^{j-2l} + F_{2k-3}^{l} C_{k-3-3l}^{j-2l} \right) \tag{1.24}$$

with $m = \lfloor \frac{k-1}{3} \rfloor = \lfloor \frac{k-2}{3} \rfloor$

Case 2 : $\lfloor \frac{k-1}{3} \rfloor = \lfloor \frac{k-2}{3} \rfloor + 1 = m + 1$

This case is equivalent to $k \equiv 1 \ (3)$.

We then have

$$D_{2k-2}^{j} = F_{2k-1}^{m+1} C_{k-2-3(m+1)}^{j-2(m+1)} + \sum_{l=0}^{m} \left(((F_{2k-1}^{l} - F_{2k-3}^{l}) C_{k-2-3l}^{j-2l} + F_{2k-3}^{l} C_{k-3-2l}^{j-2l} \right)$$

but

$$(k - 1 = 3m + 1) \Longleftrightarrow (k - 2 < 3m + 1 < 3\,(m + 1))$$

and therefore

$$D_{2k-2}^{j} = \sum_{l=0}^{m} \left(((F_{2k-1}^{l} - F_{2k-3}^{l}) C_{k-2-3l}^{j-2l} + F_{2k-3}^{l} C_{k-3-2l}^{j-2l} \right) \tag{1.25}$$

with $m = \lfloor \frac{k-1}{3} \rfloor - 1 = \lfloor \frac{k-2}{3} \rfloor$. avec $m = \lfloor \frac{k-1}{3} \rfloor - 1 = \lfloor \frac{k-2}{3} \rfloor$.

Further to the review of these two cases, we then remark that for each case

$$\sum_{l=0}^{m} F_{2k-3}^{l} C_{k-3-3l}^{j-2l} = D_{2k-3}^{j}$$

hence we get

$$D_{2k-2}^{j} - D_{2k-3}^{j} = D_{2k-4}^{j-1}$$

and finally we obtain the equality

$$D_{2k-4}^{j-1} = \sum_{l=0}^{\lfloor \frac{k-2}{3} \rfloor} F_{2k-4}^{l} C_{k-2-3l}^{j-2l} \tag{1.26}$$

with, in accordance with the relation 1.23 established page 15

$$F_{2k-4}^{l} = F_{2k-1}^{l} - F_{2k-3}^{l}$$

We are now have to prove the validity of the equality 1.26, for all integer $k \in \mathbb{N}$ with $k \geq 4$. We make sure first, by a simple calculation, that this equality is true when k takes successively the values 4, 5 et $6 \cdots$, whereas j takes its values in its domain.

We then assume that this equality is true until the rank $2k$, for all $j \leq (k - 1)$, that is

$$D_{2k}^{j} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k}^{l} C_{k-3l}^{j+1-2l}$$

We can now remark that the calculations made to get the formula of $D^j_{h=2k}$ depending on the coefficients F^l_{2k} and the binomial coefficients C^{j+1-2l}_{k-3l} are generalizable to any value of h in $\mathbb{N}$. We just have to verify by mathematical induction the correctness of the formulation of the odd index coefficients $D^j_{h=2k+1}$ to obtain a result that is valid, irrespective of the parity of this index h.

Let us go back to the initial hypothesis on the odd index coefficients, and let us utilize what we just established. We verify

$$D^j_{2k+1} = D^j_{2k} + D^{j-1}_{2k-1}$$

with

$$D^j_{2k} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F^l_{2k} C^{j+1-2l}_{k-3l}$$

and

$$D^{j-1}_{2k-1} = \sum_{l=0}^{\lfloor \frac{k-1}{3} \rfloor} F^l_{2k-1} C^{j-2l}_{k-2-3l}$$

Further to the calculations we just made in pages 16 and 16, we have

$$D^j_{2k} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor = m} F^l_{2k-2} C^{j-2l}_{k-1-3l} + F^l_{2k-1} C^{j-2l}_{k-2-3l}$$

$$\Longleftrightarrow D^j_{2k} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor = m} \left(F^l_{2k+1} - F^l_{2k-1}\right) C^{j-2l}_{k-1-3l} + F^l_{2k-1} C^{j-2l}_{k-2-3l}$$

$$\Longleftrightarrow D^j_{2k} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor = m} F^l_{2k+1} C^{j-2l}_{k-1-3l} - \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor = m} F^l_{2k-1} C^{j-2l-1}_{k-1-3l}$$

$$\Longleftrightarrow D^j_{2k} = D^j_{2k+1} - D^{j-1}_{2k-1}$$

This result is in agreement with the equality 1.14 established in page 8.

As we know how to express the coefficients F^l_{2k-2} and F^l_{2k-1} against l et de k, we can now calculate F^l_{2k+1}. We thus find

$$D^j_{2k+1} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F^l_{2k+1} C^{j-2l}_{k-1-3l}$$

with

$$F^l_{2k+1} = \frac{(2k+1)\,(k-1-l)!}{(2l+1)!\,(k-1-3l)!}$$

Our mathematical induction is complete for every coefficient D^j_h

Let us now summarize all the results we have obtained over the previous pages (see equations 1.8 et 1.12 in page 7)

$$(\forall n \in \mathbb{N})\,(n \geq 3) \left((x^n + y^n) = x^n + y^n + xy \sum_{j=1}^{n-2} A_n\,(x, y) \right)$$

with, for $n = 2k$ (see equation 1.8 in page 7)

$$A_{2k}(x,y) = \sum_{j=0}^{k-1} D_{2k}^{j} (-1)^j (xy)^j (x+y)^{2(k-1-j)}$$

and

$$(\forall k \in \mathbb{N}^*) (k > 1) (\forall j \in \mathbb{N}) (j \leq k-1) \left(D_{2k}^{j} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k}^{l} C_{k-3l}^{j+1-2l} \right)$$

and

$$F_{2k}^{l} = \frac{2k(k-1-l)!}{(2l)!(k-3l)!}$$

and for $n = 2k+1$ (see equation 1.12 in page 8)

$$A_{2k+1}(x,y) = (x+y) \sum_{j=0}^{k-1} D_{2k+1}^{j} (-1)^j (xy)^j (x+y)^{2(k-1-j)}$$

and

$$(\forall k \in \mathbb{N}^*) (\forall j \in \mathbb{N}) (j \leq k-1) \left(D_{2k+1}^{j} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l} C_{k-1-3l}^{j-2l} \right)$$

and

$$F_{2k+1}^{l} = \frac{(2k+1)(k-1-l)!}{(2l+1)!(k-1-3l)!}$$

1.6 Study of $A_{2k+1}(x,y)$ where $k \in \mathbb{N}^*$

We will show in this paragraph how we can further factorize the quantity $A_{2k+1}(x,y)$ Using the previous results, we can write

$$A_{2k+1}(x,y) = \sum_{j=0}^{k-1} \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l} C_{k-1-3l}^{j-2l} (-1)^j (xy)^j (x+y)^{2(k-1-j)}$$

We then have for each k, and for all j and all l

$$\begin{aligned} &F_{2k+1}^{l} C_{k-1-3l}^{j-2l} (-1)^j (xy)^j (x+y)^{2(k-1-l)} \\ &\quad = F_{2k+1}^{l} C_{k-1-3l}^{j-2l} (-1)^{j-2l+2l} (xy)^{j-2l+2l} (x+y)^{2(k-1-3l+3l-(j-2l)-2l)} \\ &= \left(F_{2k+1}^{l} C_{k-1-3l}^{j-2l} (-1)^{j-2l} (xy)^{j-2l} (x+y)^{2(k-1-3l-(j-2l))} \right) (-1)^{2l} (xy)^{2l} (x+y)^{2l} \end{aligned}$$

We can therefore write $A_{2k+1}(x.y)$ in the following manner

$$\begin{aligned} &A_{2k+1}(x.y) \\ &= (x+y) \\ &\sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l} (-1)^{2l} (x+y)^{2j} \\ &\sum_{j=0}^{k-1} C_{k-1-3l}^{j-2l} (-1)^{j-2l} (xy)^{j-2l} (x+y)^{2(k-1-3l-(j-2l))} \end{aligned}$$

If j varies from 0 to $k-1$, then $j-2l$ varies from 0 to $k-1-2l$, and as we have necessarily

$$j-2l \leq k-1-3l$$

we get

$$A_{2k+1}(x.y)$$
$$= (x+y)\sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l}(-1)^{2l}(x+y)^{2j}\sum_{j=0}^{k-1-3l} C_{k-1-3l}^{j}(-1)^{j}(xy)^{j}(x+y)^{2(k-1-3l-j)}$$

but

$$\sum_{j=0}^{k-1-3l} C_{k-1-3l}^{j}(-1)^{j}(xy)^{j}(x+y)^{2(k-1-3l-j)}$$
$$= \left((x+y)^{2}-xy\right)^{k-1-3l}$$
$$= \left(x^{2}+xy+y^{2}\right)^{k-1-3l}$$

and lastly

$$A_{2k+1}(x.y) = (x+y)\sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l}(-1)^{2l}(x+y)^{2j}\left(x^{2}+xy+y^{2}\right)^{k-1-3l}$$

If, in addition, we assume that $2k+1$ is a prime integer strictly greater than 3, then

$$(2k+1 \not\equiv 0 \quad (3)) \Longleftrightarrow (k \not\equiv 1 \quad (3))$$

Therefore, $k-1-3l$ does not vanish for any value taken by l and $A_{2k+1}(x,y)$ is divisble by $\left(x^{2}+xy+y^{2}\right)$.

Lastly, by denoting $\mathbb{P}$ as the ensemble of the prime integers, for all $n \in \mathbb{P}-\{2,3\}$

$$A_{n}(x,y) =$$
$$(x+y)\left(x^{2}+xy+y^{2}\right)\sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l}(-1)^{2l}(x+y)^{2j}\left(x^{2}+xy+y^{2}\right)^{k-2-3l} \quad (1.27)$$

1.7 Expressing Newton's binomial expansion differently.

We are getting now close to the end of this study, the purpose of which was to express the Newton binomial expansion in other manners.

As enounced (see relation 1.2 in page 2) and later established (see relation 1.1

in page 1), we have

$$\begin{aligned}(x+y)^n &= \sum_{j=0}^{n} C_n^j x^{n-j} y^j \\ &= x^n + y^n + \sum_{j=1}^{n-1} C_n^j x^{n-j} y^j \\ &= x^n + y^n + xy \sum_{j=0}^{n-2} C_n^{j+1} x^{n-2-j} y^j \\ &= x^n + y^n + xy \sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right)\end{aligned}$$

Moreover, depending on whether the integer n is even, odd, or odd prime, the binomial expansion can be equally expressed as follows

$n = 2k$ **even**

$$(x+y)^{2k} = x^{2k} + y^{2k} + xy \sum_{j=0}^{k-1} D_{2k}^j (-1)^j (xy)^j (x+y)^{2(k-1-j)}$$

with

$$D_{2k}^j = \frac{2k\,(2k-1-(j+1))!}{(j+1)!\,(2k-2\,(j+1))!}$$

as established in page 7 (see equation 1.8).

$n = 2k+1$ **odd**

$$(x+y)^{2k+1} = x^{2k+1} + y^{2k+1} + xy\,(x+y) \sum_{j=0}^{k-1} D_{2k+1}^j (-1)^j (xy)^j (x+y)^{2(k-1-j)}$$

with

$$D_{2k+1}^j = \frac{(2k+1)\,(2k-(j+1))!}{(j+1)!\,(2k+1-2\,(j+1))!}$$

as established in page 8 (see equation 1.12).

$n > 3$ **odd prime**

$$(x+y)^n = x^n + y^n$$

$$+$$

$$xy\,(x+y)\left(x^2+xy+y^2\right) \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^l (-1)^{2l} (x+y)^{2j} \left(x^2+xy+y^2\right)^{k-2-3l}$$

with

$$F_{2k+1}^l = \frac{(2k+1)\,(k-1-l)!}{(2l+1)!\,(k-1-3l)!}$$

as established in page 19 (see equation 1.27).

Outlining these results concludes this study.