Remark on the paper of Zheng Jie Sun and Ling Zhu

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Abstract : In this short review note we show that the new proof of theorem 1.1 given by Zheng Jie Sun and Ling Zhu in the paper *Simple proofs of the Cusa-Huygens-type and Becker-Stark-type inequalities* is logically incorrect and present another simple proof of the same.

Keywords : Cusa-Huygens inequality; circular inequality; logically incorrect; mathematical mistake

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1 Remarks

The sharp circular inequality[1, 4]

$$\frac{sinx}{x} < \frac{2 + cosx}{3}; \ x \in (0, \pi/2) \tag{1.1}$$

is known as Cusa-Huygens inequality. C.-P. Chen, W.-S. Cheung[2] and József Sándor[5] extended and sharpened inequality (1.1) independently. Their common result is as stated below:

$$\left(\frac{2 + cosx}{3}\right)^{\theta} < \frac{sinx}{x} < \left(\frac{2 + cosx}{3}\right)^{\vartheta}; \ x \in (0, \pi/2) \tag{1.2}$$

with the best positive constants $\theta \approx 1.1137399$ and $\vartheta = 1$.

In 2013, Zheng Jie Sun and Ling Zhu[6, Theorem 1.1] presented new proof of inequalities in (1.2). The authors of this paper[6] obtained that

$$g(x) > \left(\frac{x\,cosx - sinx}{x\,sinx} + \frac{sinx}{2+cosx}\right) ln\left(\frac{sinx}{x}\right) \tag{1.3}$$

where $g(x) = \frac{x\,cosx - sinx}{x\,sinx}\, ln\left(\frac{2+cosx}{3}\right) + \frac{sinx}{2+cosx}\, ln\left(\frac{sinx}{x}\right)$.

Using (1.3) they proved (1.2). We explain how intermediate result (1.3) is logically incorrect as follows:
By virtue of (1.1) we have

$$ln\left(\frac{sinx}{x}\right) < ln\left(\frac{2+cosx}{3}\right)$$

which gives

$$\frac{x\,cosx-sinx}{x\,sinx}\, ln\left(\frac{2+cosx}{3}\right) < \frac{x\,cosx-sinx}{x\,sinx}\, ln\left(\frac{sinx}{x}\right)$$

since $x\,cosx - sinx < 0$ as $cosx < \frac{sinx}{x}$[3].
This in turn results

$$g(x) < \left(\frac{x\,cosx-sinx}{x\,sinx} + \frac{sinx}{2+cosx}\right) ln\left(\frac{sinx}{x}\right).$$

In what follows, result in (1.3) is logically incorrect. The authors of [6] still proved their main result (1.2)[6, Theorem 1.1,] using this incorrect result (1.3), which is a mathematical mistake. So their proof as they claimed cannot be considered as new proof of inequalities in (1.2). However, the they gave new and simple proof of another theorem [6, Theorem 1.2].

2 Main Result

We give simple proof of (1.2) by using following lemma.

Lemma 1. *(l'Hôpital's Rule [7] of monotonicity): Let f, g be two real valued functions which are continuous on $[a, b]$ and derivable on (a, b) and $g' \neq 0$. Then the functions $\frac{f(x)-f(a)}{g(x)-g(a)}$ and $\frac{f(x)-f(b)}{g(x)-g(b)}$ are increasing(or decreasing) on (a, b) if f'/g' is increasing(or decreasing) on (a, b). The monotonicity in the conclusion is strict if f'/g' is strictly monotone.*

Simple Proof of Double Inequality (1.2):

Consider, $f(x) = \frac{ln(sinx/x)}{ln\left(\frac{2+cosx}{3}\right)} = \frac{f_1(x)}{f_2(x)}$

where $f_1(x) = ln(sinx/x)$ and $f_2(x) = ln\left(\frac{2+cosx}{3}\right)$ with $f_1(0+) = 0 = f_2(0)$. By differentiation

$$\frac{f_1'(x)}{f_2'(x)} = \frac{(sinx - x\,cosx)(2+cosx)}{x\,sin^2x} = \frac{f_3(x)}{f_4(x)}$$

where $f_3(x) = (sinx - x\,cosx)(2 + cosx)$ and $f_4(x) = x\,sin^2x$ with $f_3(0) = 0 = f_4(0)$. Again differentiating we get

$$\begin{aligned}\frac{f_3'(x)}{f_4'(x)} &= \frac{2\,x\,cosx + 2\,x - sinx}{2\,x\,cosx + sinx} \\ &= 1 + \frac{2\,x - 2\,sinx}{2\,x\,cosx + sinx} \\ &= 1 + \frac{2 - 2\,sinx/x}{2\,cosx + sinx/x} \\ &= 1 + g(x)\,h(x)\end{aligned}$$

where $g(x) = 2 - 2\,\frac{sinx}{x}$ and $h(x) = \frac{1}{2\,cosx + \frac{sinx}{x}}$.

Now $cosx$ and $sinx/x$ are clearly positive decreasing functions and $sin/x < 1$, we have that $g(x)$ and $h(x)$ are both positive increasing functions which are differentiable on $(0, \pi/2)$. Therefore $h(x), h'(x) > 0$ and $g(x), g'(x) > 0$. Hence, $[g(x)\,h(x)]' > 0$, which shows that $f_3'(x)/f_4'(x)$ is strictly increasing in $(0, \pi/2)$. By Lemma 1, $f(x)$ is also strictly increasing in $(0, \pi/2)$. Consequently, $\theta = f(\pi/2) = \frac{ln(2/\pi)}{ln(2/3)} \approx 1.1137399$ and $\vartheta = f(0+) = 1$ by l'Hôpital's rule. □

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