A Navier and Stokes equations try

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Abstract

Here is proposed a solving Navier and Stokes equations three dimensions fluid model problem, described in cylindrical coordinates, composed of null radial and vertical velocities, and of a cross-radial velocity. Equations conditions verifications and calculation description leading to the expression of pressure are given. A description of the problem can be found here.

Introduction

I follow the CMI (Clay Mathematics Institute) description of the problem. The three-dimensional Navier-Stokes equations problem to solve concerns a three dimensions incompressible fluid motion. Respecting physical reasonable conditions of bounded energy and regularity, the motion could be described with the fluid's velocity u and pressure vector p fields mathematical solutions, deducted from equations linking u, p, and the fluid kinematic viscosity ν, and could be understood, or felt, with the significations and implications of such solutions. Velocity and pressure fields are functions of the spatial variable x and the time t. In cartesian coordinates, $x(t) = (x_1(t), x_2(t), x_3(t))$ represents the position of the fluid points at the time t, and $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ represents the velocity of the fluid points at the point x and the time t. In cylindrical coordinates, $x(t) = (r(t), \theta(t), z(t))$ and $u = u(x,t) = (u_r(x,t), u_\theta(x,t), u_z(x,t))$. In cartesian coordinates, forces possibly overshadowed, the Navier-Stokes equations are:

$$\text{Momentum equations, for each coordinate linked to } i: \quad \frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} - \nu \Delta u_i + \frac{\partial p}{\partial x_i} = 0$$

$$\text{Incompressibility equation}: \quad \text{div } u = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0$$

In cylindrical coordinates, the Navier-Stokes equations are:

Momentum equations, for each coordinate:

Radial coordinate:

$$\frac{\partial p}{\partial r}+\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+u_z\frac{\partial u_r}{\partial z}-\frac{u_{\theta^2}}{r}-\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}+\frac{\partial^2 u_r}{\partial z^2}-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right]=0$$

Cross-radial coordinate:

$$\frac{1}{r}\frac{\partial p}{\partial \theta}+\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+u_z\frac{\partial u_\theta}{\partial z}+\frac{u_r u_\theta}{r}-\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{\partial^2 u_\theta}{\partial z^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r^2}\right]=0$$

Vertical coordinate:

$$\frac{\partial p}{\partial z}+\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}-\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right]=0$$

Incompressibility equation: $\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}=0.$

For physically reasonable solutions, we need solutions to respect some conditions, for example, we want to make sure $u(x,t)$ does not grow large as $|x|\to\infty$. So, $|\partial_x^\alpha u^o(x)|\le C_{\alpha K}(1+|x|)^{-K}$ on $\mathbb{R}^3$, for any $\alpha\in\mathbb{N}^3$ and K, $C_{\alpha K}$ being a constant depending on α and K. We also want regularity and bounded energy conditions: $p,u\in C^\infty(\mathbb{R}^3\times[0,\infty)))$, $\int_{\mathbb{R}^3}|u(x,t)|^2dx<C$ for all $t\ge 0$ (bounded energy), initial velocity $u(x,0)=u^o(x)$ is a given, $C^\infty(\mathbb{R}^3)$ divergence-free vector field.

Try

In cylindrical coordinates, the bounded energy condition can be written as:

$$\int_0^{+\infty}\int_0^{2\pi}\int_{-\infty}^{+\infty}(u_r^2+u_\theta^2+u_z^2)\;r\,dr\,d\theta\,dz<C,\quad C\text{ being a real constant}$$

Let's suppose, for $j\in\{r,z\}$, for $i\in\{r,\theta,z\}$, when it is possible, $\lim\limits_{\substack{j\to+\infty\\ \text{or } j\to-\infty}} u_i\neq 0$.

Then,

$$\exists\epsilon_1>0\;\;\exists a\in\mathbb{R}\;\;\forall j\in]a;+\infty[\;\;|u_i|>\epsilon_1$$

$$\exists\epsilon_2>0\;\;\exists b\in\mathbb{R}\;\;\forall j\in]-\infty;b[\;\;|u_i|>\epsilon_2$$

$$\int_0^{+\infty}u_i^2\,rdr=\int_0^a u_i^2\,rdr+\int_a^{+\infty}u_i^2\,rdr$$

$$\int_{-\infty}^{+\infty}u_i^2\,dz=\int_{-\infty}^b u_i^2\,dz+\int_b^a u_i^2\,dz+\int_a^{+\infty}u_i^2\,dz$$

u_i being a continuous function, its integral on a segment is constant. So, our supposition would imply that $\int_0^{+\infty}u_i^2rdr=\int_{-\infty}^{+\infty}u_i^2dz=+\infty$, which is not coherent with the bounded energy condition. So, for $i\in\{r,\theta,z\}$, $\lim\limits_{r\to+\infty}u_i=0$

and $\lim\limits_{\substack{z\to+\infty \\ \text{or } z\to-\infty}} u_i = 0$. So, the bounded energy condition forces the functions u_i to converge to 0 at infinite extremities, and that can simply make immediately think to the function $x \to e^{-x^2}$. Searching for a three dimensions solution, a cross-radial velocity describing basically a two dimensions plan motion, we may search to construct a cross-radial velocity depending on the third dimension, adding it to the formula of the cross-radial velocity. This gives $u_\theta = u_\theta^o \; e^{-\frac{r^2}{r_a^2} - \frac{z^2}{z_a^2}}$, where r_a and z_a are time dependent added elements for the required without dimension exponential argument. After having tested this formula to find the pressure field, it appears eventual Cauchy principal value problems. Multiplying by $\sqrt{\frac{r}{r_i}}$, r_i being a non null constant, solves this, and still respects problem conditions. We define the cross-radial velocity:

$$u_\theta = u_\theta^o \sqrt{\frac{r}{r_i}} \; e^{-\frac{r^2}{r_a^2} - \frac{z^2}{z_a^2}} \qquad \text{where} \quad \begin{cases} u_\theta^o = u_\theta^o(x) = u_\theta(r,\theta,z,0) \text{ is the initial non null} \\ \text{constant velocity } u^o(x) \text{ cross-radial component,} \\ \text{which is a } C^\infty(\mathbb{R}^3)\text{divergence-free vector field } (\nabla\cdot u^o(x) = 0) \\ t \text{ represents the time } (t \geq 0) \\ r_i, r_a^o, z_a^o \text{ are non null distance constants} \\ \tau \text{ is a non null time constant} \\ r_a = \sqrt{\frac{\tau}{t+\tau} - 1 + r_a^{o^2}},\ r_a^o = r_a(0) \\ z_a = \sqrt{\frac{\tau}{t+\tau} - 1 + z_a^{o^2}},\ z_a^o = z_a(0) \end{cases}$$

This fluid moves in three dimensions because the cross-radial velocity changes according to the radial distance r and the height z. So, this velocity evolves in these two dimensions, and by definition in the azimuth theta one. The movement so happens in the three dimensions. As proved below, this fluid model fully solved the Navier-Stokes equations in three dimensions, and respect the above conditions. Here is first a mind imagination path to conceive of the fluid motion. Only the cross-radial velocity is not null, and is decreasing exponentially, vertically and radially. So, to imagine the moving fluid, you may imagine first an infinite plan with a center point, with a radially exponentially decreasing cross-radial velocity. You may add to this moving image in mind, the imagination (=image action?) that, at the zero height of the plan, and at a certain radial distance point from the center, because of the z squaring operation of the velocity formula, the cross-radial velocity is vertically maximal at this point, and also vertically symmetrically, exponentially decreasing. So the cross-radial velocity is slower and slower, radial and vertical distances increasing. You may also imagine an infinitely long cylinder, cut into an infinite number of discus, one of them holding a radially exponentially decreasing cross-radial velocity, and symmetrically referred to this one, each discus still has a radially exponentially decreasing cross-radial velocity, but also linearly both directions exponentially decreasing. We verify now the conditions of incompressibility and bounded energy. Incompressibility is verified if div $u = 0$, where $u = (u_r, u_\theta, u_z)$, which gives, div $u = 0 \Leftrightarrow \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \Leftrightarrow \frac{\partial u_\theta}{\partial \theta} = 0$, because radial and vertical velocities are null. $u^o(x)$ being constant, we have: $\dfrac{\partial u_\theta^o(x)}{\partial \theta} = 0$. Radial and vertical velocities, $\dfrac{\partial u_\theta^o}{\partial \theta}$, being null (implying $\dfrac{\partial u_\theta}{\partial \theta} = \dfrac{\partial u_\theta^o}{\partial \theta}\sqrt{\dfrac{r}{r_i}}e^{-\dfrac{r^2}{r_a^2}-\dfrac{z^2}{z_a^2}}$ being also null), incompressibility is verified.

Except if $\alpha = (0,0,0)$, $|\partial_x^\alpha u_\theta^o(x)| = 0$ ($u_\theta^o(x)$ is constant), in which case $C_{\alpha K}$ is easily findable so that $|\partial_x^\alpha u^o(x)| \leq C_{\alpha K}(1+|x|)^{-K}$ for any K because $(1+|x|)^{-K} > 0$, but if $\alpha = (0,0,0)$, then, $u_\theta^o(x)$ being a non null constant, $|\partial_x^\alpha u_\theta^o(x)| = |u_\theta^o(x)|$ and so, a strictly positive constant $C_{\alpha K}$ can be also chosen so that $|u_\theta^o(x)| \leq C_{\alpha K}(1+|x|)^{-K}$, for any K. So, $|\partial_x^\alpha u^o(x)| \leq C_{\alpha K}(1+|x|)^{-K}$ on $\mathbb{R}^3$, for any $\alpha \in \mathbb{N}^3$ and K.

The energy of the fluid remains finite if $\displaystyle\int_{\mathbb{R}^3} |u|^2\, r\, dr\, d\theta\, dz$ is upper-bounded.

$$\int_{\mathbb{R}^3} |u|^2\, r\, dr\, d\theta\, dz = \int_{\mathbb{R}^3} u_\theta^2\, r\, dr\, d\theta\, dz = \int_{\mathbb{R}^3} u_\theta^{o^2} \frac{r^2}{r_i} e^{-\dfrac{2r^2}{r_a^2}}\, e^{-\dfrac{2z^2}{z_a^2}}\, dr\, d\theta\, dz =$$

$u_\theta^{o^2}\displaystyle\int_{\mathbb{R}^3} \frac{r^2}{r_i} e^{-\dfrac{2r^2}{r_a^2}}\, e^{-\dfrac{2z^2}{z_a^2}}\, dr\, d\theta\, dz$, which is a converging integral, consequently upper-bounded. Bounded energy condition is also verified. Given the precedent informations, we can simplify the momentum equations to calculate the pressure field:

$$\begin{aligned}
&\frac{\partial p}{\partial r} = \frac{u_\theta^2}{r} \\
&\frac{\partial p}{\partial \theta} = -r\frac{\partial u_\theta}{\partial t} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2}\right] \\
&\frac{\partial p}{\partial z} = 0
\end{aligned}$$

Which gives, integrating $\dfrac{\partial p}{\partial r}$: $P(r,\theta,z,t) = \sqrt{\dfrac{\pi}{8}}\, u_\theta^{o^2}\, \dfrac{r_a}{r_i}\, e^{-\dfrac{2z^2}{z_a^2}}\, \mathrm{erf}(\sqrt{2}\dfrac{r}{r_a}) + a(\theta, z)$

"erf" designating the error function, defined on $\mathbb{R}$: $\mathrm{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-t^2}\, dt$

$$\frac{\partial p}{\partial \theta} = \frac{\partial a(\theta,z)}{\partial \theta} = -r\frac{\partial u_\theta}{\partial t} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2}\right]$$

$$\frac{\partial u_\theta}{\partial t} = -u_\theta \ \frac{\tau}{(t+\tau)^2} \ \left(\frac{z^2}{z_a^2} + \frac{r^2}{r_a^2}\right)$$

$$\frac{\partial u_\theta}{\partial r} = u_\theta \ \sqrt{\frac{r}{r_i}}\left(\frac{1}{2r} - \frac{2r}{r_a^2}\right)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) = \frac{u_\theta}{r}\left(\sqrt{\frac{r}{r_i}}\left(\frac{1}{2r} - \frac{2r}{r_a^2}\right) + r\left(\sqrt{\frac{r}{r_i}}\left(\frac{1}{2r} - \frac{2r}{r_a^2}\right)^2 - \frac{\sqrt{\frac{r}{r_i}}(r_a^2 + 12r^2)}{4r_a^2 r^2}\right)\right)$$

$$\frac{\partial^2 u_\theta}{\partial z^2} = u_\theta(4\frac{z^2}{z_a^4} - \frac{2}{z_a^2})$$

$\frac{\partial p}{\partial \theta}$ being independent of θ, we have: $a(\theta,z) = \frac{\partial p}{\partial \theta}\theta + b(z)$.

$$a(\theta,z) = ru_\theta\theta \ \frac{\tau}{(t+\tau)^2} \ \left(\frac{z^2}{z_a^2} + \frac{r^2}{r_a^2}\right)$$
$$+ \frac{\nu u_\theta \theta}{r}\left(\sqrt{\frac{r}{r_i}}\left(\frac{1}{2r} - \frac{2r}{r_a^2}\right) + r\left(\sqrt{\frac{r}{r_i}}\left(\frac{1}{2r} - \frac{2r}{r_a^2}\right)^2 - \frac{\sqrt{\frac{r}{r_i}}(r_a^2 + 12r^2)}{4r_a^2 r^2}\right)\right)$$
$$+ \nu u_\theta \theta(4\frac{z^2}{z_a^4} - \frac{2}{z_a^2} - \frac{1}{r^2}) + b(z)$$

$$\frac{\partial p}{\partial z} = -2z\sqrt{\frac{\pi}{2}} \ \frac{r_a u_\theta^{o^2}}{z_a^2 r_i} \ e^{-\frac{2z^2}{z_a^2}} \ \mathrm{erf}(\sqrt{2}\frac{r}{r_a}) + \frac{\partial a(\theta,z)}{\partial z}$$
$$= -2z\sqrt{\frac{\pi}{2}} \ \frac{r_a u_\theta^{o^2}}{z_a^2 r_i} \ e^{-\frac{2z^2}{z_a^2}} \ \mathrm{erf}(\sqrt{2}\frac{r}{r_a}) - \frac{2z}{z_a^2} r\theta u_\theta \frac{\tau}{(t+\tau)^2} \ \left(\frac{z^2}{z_a^2} + \frac{r^2}{r_a^2}\right) + r\theta u_\theta \frac{\tau}{(t+\tau)^2} \ \left(\frac{2z}{z_a^2}\right)$$
$$- \frac{2z}{z_a^2} \ \frac{\nu u_\theta \theta}{r} \ \left(\sqrt{\frac{r}{r_i}}\left(\frac{1}{2r} - \frac{2r}{r_a^2}\right) + r\left(\sqrt{\frac{r}{r_i}}\left(\frac{1}{2r} - \frac{2r}{r_a^2}\right)^2 - \frac{\sqrt{\frac{r}{r_i}}(r_a^2 + 12r^2)}{4r_a^2 r^2}\right)\right)$$
$$+ \nu\theta u_\theta\left(-\frac{8z^3}{z_a^6} + \frac{12z}{z_a^4} + \frac{2z}{z_a^2 r^2}\right) + \frac{\partial b(z)}{\partial z}$$
$$= 0$$

$$b(z) = \nu u_\theta \theta \left(-4 \left(\frac{z_a^2 + z^2}{z_a^4} \right) + 6 \frac{1}{z_a^2} + \frac{1}{r^2} \right) - \frac{1}{r} \left(\sqrt{\frac{r}{r_i}} \left(\frac{1}{2r} - \frac{2r}{r_a^2} \right) \right)$$

$$+ \left(\sqrt{\frac{r}{r_i}} \left(\frac{1}{2r} - \frac{2r}{r_a^2} \right)^2 - \frac{\sqrt{\frac{r}{r_i}}(r_a^2 + 12r^2)}{4r_a^2 r^2} \right)$$

$$+ r\theta u_\theta \frac{\tau}{(t+\tau)^2} - r\theta u_\theta \frac{\tau}{(t+\tau)^2} \left(\left(\frac{r^2}{r_a^2} + 1 \right) + \frac{z^2}{z_a^2} \right) - \sqrt{\frac{\pi}{8}} \frac{r_a}{r_i} u_\theta^{o^2} e^{\frac{-2z^2}{z_a^2}} \operatorname{erf}(\sqrt{2}\frac{r}{r_a}) + A.$$

Finally, $P(r, \theta, z, t) = \mathrm{A}$. We find a constant pressure in all $\mathbb{R}^3$. Velocity and pressure expressions are smooth $C^\infty(\mathbb{R}^3 \times [0, \infty])$ functions. It has been verified that the presented fluid model satisfies all the conditions to be a three dimensions Navier-Stokes equations solution proposition.