Negation of the Riemann Hypothesis

by

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Abstract

We show that the Riemann zeta function has infinitely many non-trivial zeros off the critical line.

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§1. Proof of Negation

Definition 1.1. A real number is a cut in the real number line.

Definition 1.2. A cut in a line separates one line into two pieces.

Remark. A number is a cut in a line. A line is defined a priori. All lines can be cut so all lines are number lines. A given line is the real line by definition. A real number separates the real number line into a set of "larger" real numbers and a set of "smaller" real numbers.

Definition 1.3. For an unboundedly large finite radius about the origin, call real numbers beyond that radius "real numbers in the neighborhood of infinity' and call all other real numbers "real numbers in the neighborhood of the origin."

Theorem 1.4. *For an arbitrarily large radius R about the origin of the real number line, where R is a real number in the neighborhood of the origin, some real numbers lie outside that radius.*

Proof. Considering the positive branch of the real number line, there exist real numbers $R+n$ with $n \in \mathbb{N}$ which lie outside the radius. This is true for any R in the neighborhood of the origin. □

Definition 1.5. If x is a real number in the neighborhood of the origin then it measures "distance" from the origin. If x is a real number in the neighborhood of

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infinity then it measures "distance" from one of the two endpoints of the extended real line

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\} \quad . \tag{1.1}$$

Definition 1.6. When ∞ appears as $\widehat{\infty}$, let the hat be an instruction to delay additive absorption.

Definition 1.7. Some real numbers in the neighborhood of infinity shall be written as

$$x = \pm\big(\widehat{\infty} - b\big) \quad , \tag{1.2}$$

where b is a positive real number in the neighborhood of the origin. This definition does not describe all real numbers in the neighborhood of infinity because numbers of the form $R + n$ where $n \in \mathbb{N}$ and R is the radius of the neighborhood of the origin cannot be written in the form of equation (1.2).

Remark. All numbers on the extended real number line are real numbers except $\pm\infty$. Therefore, any number separated from an endpoint of the extended real number line by some non-vanishing "distance" is a real number. All numbers of the form of equation (1.2) are real numbers because b is a positive real number in the neighborhood of the origin which describes some "distance" from an endpoint of the extended real number line.

Theorem 1.8. *The quotient of a real number in the neighborhood of the origin divided by a real number in the neighborhood of infinity is identically zero.*

Proof. Let X be a real number in the neighborhood of the origin and let Y be a number in the neighborhood of infinity. Let Z be any non-zero real number such that

$$\frac{X}{Y} = Z \quad . \tag{1.3}$$

Since $||X|| < ||Y||$, we have $||Z|| < 1$. This means that Z must be a real number in the neighborhood of the origin. All such numbers have a multiplicative inverse. We find, therefore, that

$$\frac{X}{ZY} = 1 \qquad \Longleftrightarrow \qquad X = ZY \quad . \tag{1.4}$$

The hat on $\widehat{\infty}$ only suppresses additive absorption so

$$ZY = Z\big(\widehat{\infty} - b\big) = \widehat{\infty} - Zb \quad . \tag{1.5}$$

This delivers a contradiction because it requires that X is a real number in the neighborhood of infinity while we have already defined it to be a real number in the neighborhood of the origin. Therefore, the only possible numerical value for X/Y is 0. □

Definition 1.9. A number z is a complex number $z \in \mathbb{C}$ if and only if

$$z = x + iy \quad , \qquad \text{and} \qquad x, y \in \mathbb{R} \quad . \tag{1.6}$$

Theorem 1.10. *The Riemann hypothesis is false.*

Proof. Let

$$z_0 = -\big(\widehat{\infty} - b\big) + iy_0 \qquad \Longrightarrow \qquad z_0 \in \mathbb{C} \quad , \tag{1.7}$$

and consider the Riemann zeta function written as the Euler product

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p|prime} \frac{1}{1 - p^{-z}} \quad . \tag{1.8}$$

Evaluating ζ at z_0 yields

$$\zeta(z_0) = \prod_{p|prime} \frac{1}{1 - p^{(\widehat{\infty} - b) - iy_0}} \quad . \tag{1.9}$$

Applying the formula

$$a^{b+ic} = a^b \big[\cos(c \ln a) + i \sin(c \ln a)\big] \quad , \tag{1.10}$$

yields

$$\zeta(z_0) = \prod_{p|prime} \frac{1}{1 - \left(\dfrac{p^{\widehat{\infty}}}{p^b}\right) \big[\cos(y_0 \ln p) - i \sin(y_0 \ln p)\big]} \quad . \tag{1.11}$$

Now choose

$$y_0 \ln p' = \pm 2n\pi \quad , \tag{1.12}$$

for some prime p' with $n \in \mathbb{N}$. Under this condition we obtain

$$\zeta(z_0) = \frac{1}{1-\widehat{\infty}} \left(\prod_{\substack{p|prime \\ p \neq p'}} \frac{1}{1-p^{-z_0}} \right) \quad . \tag{1.13}$$

The coefficient of this product $1/\big(1-\widehat{\infty}\big)$ has the form of a real number in the neighborhood of the origin divided by a real number in the neighborhood of infinity. By Theorem 1.8, this coefficient is identically zero and

$$\zeta(z_0) = 0 \quad . \tag{1.14}$$

z_0 is on neither the critical line nor the real axis so this constitutes a rigorous negation of Riemann's hypothesis. □