Mellin Transforms of Some Functions

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ABSTRACT

This paper deals only with the Mellin transforms of some functions and their relationship with the gamma functions.

Mellin Transform

The Mellin transform of a function is an integral transform defined by

(1) $$M\{f\} = \int_0^\infty f(x)x^{s-1}dx = M(s)$$

1) The Mellin transform of $f(x) = e^{-x}$:

$$M(s) = \int_0^\infty e^{-x}x^{s-1}dx \qquad \Re(s)>0$$

$$M(s) = \int_0^\infty x^{s-1}\left\{\sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!}\right\}dx$$

$$M(s) = \int_0^\infty x^{s-1}\left\{1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right\}dx$$

$$M(s) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{s+k}}{(s+k)k!}\Bigg|_0^\infty$$

(2) $$M(s) = \lim_{R\to\infty} \sum_{k=0}^{\infty} \frac{(-1)^k R^{s+k}}{(s+k)k!}$$

$$M(s) = \Gamma(s)$$

2) The Mellin transform of $f(x) = 2e^{-x^2}$

$$M(s) = \int_0^\infty 2e^{-x^2} x^{s-1} dx \qquad \Re(s) > 0$$

$$M(s) = 2\int_0^\infty x^{s-1} \left\{1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right\} dx$$

$$M(s) = 2\int_0^\infty x^{s-1} \left\{\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!}\right\} dx$$

(3) $$M(s) = 2\lim_{R\to\infty} \sum_{k=0}^{\infty} \frac{(-1)^k R^{s+2k}}{(s+2k)k!}$$

Let $y = x^2$ such that $dy = 2x\,dx$:

$$M(s) = \int_0^\infty 2e^{-y} (y^{1/2})^{s-1} \frac{dy}{2y^{1/2}}$$

$$M(s) = \int_0^\infty e^{-y} y^{\frac{s}{2}-1} dy$$

$$M(s) = \Gamma\left(\frac{s}{2}\right)$$

3) The Mellin transform of $f(x) = \frac{1}{2} e^{-\sqrt{x}}$

$$M(s) = \int_0^\infty \frac{1}{2} e^{-\sqrt{x}} x^{s-1} dx \qquad \Re(s) > 0$$

$$M(s) = \frac{1}{2}\int_0^\infty x^{s-1} \left\{1 - x^{1/2} + \frac{x}{2!} - \frac{x^{3/2}}{3!} + \cdots\right\} dx$$

$$M(s) = \frac{1}{2}\int_0^\infty x^{s-1} \left\{\sum_{k=0}^{\infty} \frac{(-1)^k x^{\frac{k}{2}}}{k!}\right\} dx$$

(4) $$M(s)=\frac{1}{2}\lim_{R\to\infty}\sum_{k=0}^{\infty}\frac{(-1)^k R^{s+\frac{k}{2}}}{(s+k/2)k!}$$

Let $y=\sqrt{x}$ such that $2y\,dy=dx$:

$$M(s)=\frac{1}{2}\int_0^{\infty}e^{-y}(y^2)^{s-1}2y\,dy$$

$$M(s)=\int_0^{\infty}e^{-y}y^{2s-1}dy$$

$$M(s)=\Gamma(2s)$$

4) The Mellin transform of $\sum_{n=1}^{\infty}e^{-nx}=\frac{1}{e^x-1}$ $\quad x>0$

$$M(s)=\int_0^{\infty}\left\{\sum_{n=1}^{\infty}e^{-nx}\right\}x^{s-1}dx$$

Let $y=nx$ such that $dy=n\,dx$:

$$M(s)=\int_0^{\infty}\left\{\sum_{n=1}^{\infty}e^{-y}\right\}\left(\frac{y}{n}\right)^{s-1}\frac{dy}{n}$$

$$M(s)=\left\{\sum_{n=1}^{\infty}n^{-s}\right\}\int_0^{\infty}e^{-y}y^{s-1}dy$$

(5) $$M(s)=\left\{\sum_{n=1}^{\infty}n^{-s}\right\}\left\{\lim_{R\to\infty}\sum_{k=0}^{\infty}\frac{(-1)^k R^{s+k}}{(s+k)k!}\right\}$$

$$M(s)=\zeta(s)\Gamma(s)\qquad \Re(s)>1$$

5) The Mellin transform of $2\sum_{n=1}^{\infty}e^{-\pi n^2x^2}$ $\quad x\neq 0$

$$M(s)=\int_0^{\infty}\left\{2\sum_{n=1}^{\infty}e^{-\pi n^2x^2}\right\}x^{s-1}dx$$

$$M(s)=2\sum_{n=1}^{\infty}\left\{\int_0^{\infty}x^{s-1}e^{-\pi n^2x^2}dx\right\}$$

$$M(s) = 2\sum_{n=1}^{\infty}\left(\int_0^{\infty} x^{s-1}\left\{1 - \pi n^2 x^2 + \frac{(\pi n^2 x^2)^2}{2!} - \frac{(\pi n^2 x^2)^3}{3!} + \cdots\right\} dx\right)$$

$$M(s) = 2\sum_{n=1}^{\infty}\left(\int_0^{\infty} x^{s-1}\left\{\sum_{k=0}^{\infty}\frac{(-1)^k(\pi n^2 x^2)^k}{k!}\right\} dx\right)$$

(6)
$$M(s) = 2\sum_{n=1}^{\infty}\left\{\lim_{R\to\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(\pi n^2)^k R^{s+2k}}{(s+2k)k!}\right\}$$

Let $y = \pi n^2 x^2$ such that $dy = 2\pi n^2 x\, dx$:

$$M(s) = \int_0^{\infty}\left\{2\sum_{n=1}^{\infty} e^{-y}\right\}\left(\frac{y^{1/2}}{\sqrt{\pi}n}\right)^{s-1}\frac{\sqrt{\pi}n\, dy}{2\pi n^2 y^{1/2}}$$

$$M(s) = \pi^{-\frac{s}{2}}\left\{\sum_{n=1}^{\infty} n^{-s}\right\}\int_0^{\infty} y^{\frac{s}{2}-1} e^{-y}\, dy$$

$$M(s) = \pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{s}{2}\right) \qquad \Re(s) > 1$$

The Inverse Mellin Transform

$$f(x) = M^{-I}\{M(s)\}$$

$$f(x) = \frac{1}{2\pi i}\oint M(s) x^{-s} ds$$

1) The inverse Mellin transform of $\Gamma(s)$:

$$f(x) = \frac{1}{2\pi i}\oint \Gamma(s) x^{-s} ds$$

$$f(x) = \frac{1}{2\pi i}\oint\left\{\lim_{R\to\infty}\sum_{k=0}^{\infty}\frac{(-1)^k R^{s+k}}{(s+k)k!}\right\} x^{-s} ds$$

$$f(x) = \frac{1}{2\pi i}\oint\left\{\frac{R^s}{s} - \frac{R^{s+1}}{s+1} + \frac{R^{s+2}}{(s+2)2!} - \frac{R^{s+3}}{(s+3)3!} + \cdots\right\} x^{-s} ds$$

The poles are at s = 0, -1, -2, -3, ...

$$f(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$f(x) = e^{-x}$$

2) The inverse Mellin transform of $\Gamma\left(\frac{s}{2}\right)$:

$$f(x) = \frac{1}{2\pi i} \oint \Gamma\left(\frac{s}{2}\right) x^{-s} ds$$

$$f(x) = \frac{1}{2\pi i} \oint \left\{ 2 \lim_{R \to \infty} \sum_{k=0}^{\infty} \frac{(-1)^k R^{s+2k}}{(s+2k)k!} \right\} x^{-s} ds$$

$$f(x) = \frac{1}{2\pi i} \oint 2 \left\{ \frac{R^s}{s} - \frac{R^{s+2}}{s+2} + \frac{R^{s+4}}{(s+4)2!} - \frac{R^{s+6}}{(s+6)3!} + \cdots \right\} x^{-s} ds$$

The poles are at s = 0, -2, -4, -6, ...

$$f(x) = 2\left\{ 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots \right\}$$

$$f(x) = 2e^{-x^2}$$

3) The inverse Mellin transform of $\Gamma(2s)$:

$$f(x) = \frac{1}{2\pi i} \oint \Gamma(2s) x^{-s} ds$$

$$f(x) = \frac{1}{2\pi i} \oint \frac{1}{2} \left\{ \lim_{R \to \infty} \sum_{k=0}^{\infty} \frac{(-1)^k R^{s+k/2}}{(s+k/2)k!} \right\} x^{-s} ds$$

$$f(x) = \frac{1}{2\pi i} \oint \frac{1}{2} \left\{ \frac{R^s}{s} - \frac{R^{s+1/2}}{s+1/2} + \frac{R^{s+1}}{(s+2/2)2!} - \frac{R^{s+3/2}}{(s+3/2)3!} + \cdots \right\} x^{-s} ds$$

The poles are at s = 0, -1/2, -1, -3/2, ...

$$f(x) = \frac{1}{2}\left\{ 1 - x^{1/2} + \frac{x}{2!} - \frac{x^{3/2}}{3!} + \cdots \right\} = \frac{1}{2} e^{-\sqrt{x}}$$

4) The inverse Mellin transform of $\zeta(s)\Gamma(s)$:

$$f(x)=\frac{1}{2\pi i}\oint \zeta(s)\Gamma(s)x^{-s}ds$$

$$f(x)=\frac{1}{2\pi i}\oint\left(\sum_{n=1}^{\infty}n^{-s}\right)\left(\lim_{R\to\infty}\sum_{k=0}^{\infty}\frac{(-1)^k R^{s+k}}{(s+k)k!}\right)x^{-s}ds$$

$$f(x)=\sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\oint\left\{\frac{R^s}{s}-\frac{R^{s+1}}{s+1}+\frac{R^{s+2}}{(s+2)2!}-\frac{R^{s+3}}{(s+3)3!}+\cdots\right\}n^{-s}x^{-s}ds\right)$$

The poles are at s = 0, -1, -2, -3, ...

$$f(x)=\sum_{n=1}^{\infty}\left(1-nx+\frac{n^2x^2}{2!}-\frac{n^3x^3}{3!}+\cdots\right)$$

$$f(x)=\sum_{n=1}^{\infty}e^{-nx}\qquad x>0.$$

5) The inverse Mellin transform of $M(s)=\pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{s}{2}\right)$:

$$f(x)=\frac{1}{2\pi i}\oint \pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{s}{2}\right)x^{-s}ds$$

$$f(x)=\frac{1}{2\pi i}\oint\left(2\sum_{n=1}^{\infty}\left\{\lim_{R\to\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(\pi n^2)^k R^{s+2k}}{(s+2k)k!}\right\}\right)x^{-s}ds$$

$$f(x)=2\sum_{n=1}^{\infty}\left\{\frac{1}{2\pi i}\oint\left(1-\frac{(\pi n^2)R^{s+2}}{(s+2)}+\frac{(\pi n^2)^2R^{s+4}}{(s+4)2!}-\frac{(\pi n^2)^3R^{s+6}}{(s+6)3!}+\cdots\right)x^{-s}ds\right\}$$

The poles are at s = 0, -2, -4, -6, ...

$$f(x)=2\sum_{n=1}^{\infty}\left\{1-(\pi n^2)x^2+\frac{(\pi n^2)^2x^4}{2!}-\frac{(\pi n^2)^3x^6}{3!}+\cdots\right\}$$

$$f(x)=2\sum_{n=1}^{\infty}\left\{1-\pi n^2x^2+\frac{(\pi n^2x^2)^2}{2!}-\frac{(\pi n^2x^2)^3}{3!}+\cdots\right\}$$

$$f(x)=2\sum_{n=1}^{\infty}e^{-\pi n^2x^2}\qquad x\neq 0.$$