THE ARBELOS IN WASAN GEOMETRY, PROBLEMS OF IZUMIYA AND NAITŌ

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ABSTRACT. We generalize two sangaku problems involving an arbelos proposed by Izumiya and Naitō, and show the existence of six non-Archimedean congruent circles.

1. Introduction

In this article we generalize two sangaku problems involving an arbelos proposed by Izumiya (泉屋德太郎静政) and Naitō (内藤豐次郎). Let $\alpha$, $\beta$ and $\gamma$ be the three semicircles with diameters $AO$, $BO$ and $AB$, respectively for a point $O$ on the segment $AB$ constructed on the same side of $AB$. The area surrounded by the three semicircles is called arbelos (see Figure 1). The radical axis of $\alpha$ and $\beta$ is called the axis. Let $r_A$ and $r_B$ be the radii of $A$ and $B$, respectively, and let $\delta_\alpha$ (resp. $\delta_\beta$) be the incircle of the curvilinear triangle made by $\alpha$ (resp. $\beta$), $\gamma$ and the axis. The two circles $\delta_\alpha$ and $\delta_\beta$ have common radius $r_A = ab/(a + b)$ and are called the twin circles of Archimedes.

Figure 1.

Izumiya’s problems appeared in a sangaku in Saitama hung in 1866, which is as follows [6] (see Figure 2).

**Problem 1.** If $\alpha$ and $\beta$ are congruent and the tangent of $\alpha$ from $B$ meets $\gamma$ in a point $C$, show that the inradius of the curvilinear triangle formed by $\alpha$, $\gamma$ and the perpendicular from $C$ to $AB$ equals $a/9$. 
Naitō’s problem appeared in a sangaku in Fukushima hung in 1983 (the sangaku seems to be made in modern day times), which is as follows [3] (see Figure 3).

**Problem 2.** If \( \alpha \) and \( \beta \) are congruent, show that the radius of the circle touching the remaining external common tangent of \( \alpha \) and \( \delta_\alpha \) and the arc of \( \gamma \) cut by the tangent at the midpoint equals \( a/9 \).

2. **Generalization**

We now consider the case in which the semicircles \( \alpha \) and \( \beta \) are not always congruent. We use the next proposition (see Figure 4).

**Proposition 2.1.** For a point \( P \) on the segment \( AB \), let \( h \) be the perpendicular to \( AB \) at \( P \). If \( \delta_1 \) is the circle touching \( h \) at \( P \) from the side opposite to \( B \) and the tangent of \( \beta \) from \( A \) and \( \delta_2 \) is the circle touching \( \alpha \) externally \( \gamma \) internally and \( h \) from the same side as \( \delta_1 \), then \( \delta_1 \) and \( \delta_2 \) are congruent.

**Proof.** The radius of \( \delta_2 \) is proportional to the distance between its center and the radical axis of \( \alpha \) and \( \gamma \) [1, p. 108], while \( \delta_2 \) coincides with \( \beta \) if \( P = B \). Also the radius of \( \delta_1 \) is proportional to the distance between its center and the point \( A \), and \( \delta_1 \) coincides with \( \beta \) if \( P = B \). \( \square \)
Theorem 2.2. Let $C$ be the point of intersection of $\gamma$ and the tangent of $\alpha$ from $B$ and let $D$ be the foot of perpendicular from $C$ to $AB$. The incircle of the curvilinear triangle made by $\alpha$, $\gamma$ and $CD$ is denoted by $\varepsilon_1$. Let $u$ be the remaining external common tangent of $\alpha$ and $BC$. The circle touching $u$ and the arc of $\gamma$ cut by $u$ at the midpoint is denoted by $\varepsilon_2$. The incircle of the curvilinear triangle made by $\gamma$, $\delta_\beta$ and the axis is denoted by $\varepsilon_3$. The circle touching the tangent of $\beta$ from $A$ and $CD$ at $D$ from the side opposite to $B$ is denoted by $\varepsilon_4$. The smallest circle passing through the point of intersection of $\beta$ and $BC$ and touching the axis is denoted by $\varepsilon_5$. The smallest circle passing through the point of intersection of $BC$ and $u$ and touching the line $CD$ is denoted by $\varepsilon_6$. Then the following statements hold.

(i) The six circles $\varepsilon_1$, $\varepsilon_2$, $\cdots$, $\varepsilon_6$ are congruent and have common radius

$$\frac{a^2b}{(a + 2b)^2}.$$ 

(ii) The circle $\varepsilon_1$ touches the line $t$, and the circle $\varepsilon_2$ touches $\gamma$ at $C$.

Proof. We assume that $r_i$ is the radius of $\varepsilon_i$, $d = a + 2b$, $E$ is the point of intersection of $BC$ and $\beta$, $F$ is the foot of perpendicular from $E$ to the axis, $G$ is the point of tangency of $\alpha$ and $BC$, $H$ is the center of $\alpha$, and $BC$ meets the axis and $u$ in points $J$ and $K$, respectively (see Figure 6).

Since the three segments $CA$, $GH$ and $EO$ are parallel and $H$ is the midpoint of $AO$, $G$ is the midpoint of $CE$. While the line $BC$ is the internal common tangent of $\alpha$ and $\delta_\alpha$ [2, p. 212]. Therefore $G$ is also the midpoint of $JK$. Hence $|EJ| = |CK|$, i.e., the circles $\varepsilon_5$ and $\varepsilon_6$ are congruent. Since the triangles $BGH$, $BEO$ and $OFE$ are similar, $a/d = |OE|/(2b) = |EF|/|OE|$. Therefore $|OE| = 2ab/d$ and $|EF| = 2a^2b/d^2$. Hence $r_5 = a^2b/d^2 = r_6$, and $|OF| = 4ab\sqrt{(a + b)b}/d^2$ from the right triangle $OFE$.

The last equation implies $|EF| = a|OF|/(2\sqrt{(a + b)b})$. Let $x = |BD|$. Then $|CD| = ax/(2\sqrt{(a + b)b}$ from the similar triangles $OFE$ and $BDC$. Therefore we have $x(2(a + b) - x) = |CD|^2 = a^2x^2/(4(a + b)b)$. Solving the equation for $x$, we get $x = 8b(a + b)^2/d^2$. Therefore $|AD| = 2(a + b) - x = 2a^2(a + b)/d^2$. Therefore $r_4 = b|AD|/|AB| = a^2b/d^2 = r_1$.
by Proposition 2.1. Meanwhile $\varepsilon_3$ and the incircle of the curvilinear triangle made by $\alpha$, $\gamma$ and $t$ have radius $a^2b/d^2$ [5, Theorem 9]. Therefore the last circle coincides with $\varepsilon_1$, i.e., $\varepsilon_1$ touches $t$. While we have also shown that $\varepsilon_1$ and $\varepsilon_2$ are congruent in [4]. This proves (i) and the first half part of (ii).

Let $\zeta$ be the circle with center $C$ passing through $G$. We invert the figure in $\zeta$. Then the circles $\alpha$ and $\delta_\alpha$ are orthogonal to $\zeta$, i.e., they are fixed by the inversion. The line $u$, which intersects $\zeta$, is inverted to a circle intersecting $\zeta$ touching $\alpha$ and $\delta_\alpha$ passing through $C$. Therefore $\gamma$ is the inverse of $u$. This implies that the points of intersection of $\gamma$ and $u$ lie on $\zeta$. Hence $C$ is the midpoint of the arc of $\gamma$ cut by $u$. Therefore $\varepsilon_2$ touches $\gamma$ at $C$. This proves the second half part of (ii).

\[ \square \]

Circles of radius $r_A$ are called Archimedean circles [2]. Therefore we now have six non-Archimedean congruent circles $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_6$. Exchanging the roles of $\alpha$ and $\beta$, we get another six non-Archimedean congruent circles of radius $ab^2/(2a + b)^2$, which are denoted in Figure 5.

### 3. The Circle Associated with a Point on $\gamma$

For a circle $\delta$ touching $\alpha$ externally and $\gamma$ internally, if $P$ is the point of intersection of $\gamma$ and the internal common tangent of $\delta$ and $\alpha$ closer to $B$, we say that $\delta$ is associated with $P$. As mentioned in the proof of Theorem 2.2, the circle $\delta_\alpha$ is associated with the point $B$ (see Figure 6). We can also consider that the point circle $A$ is associated with the point $A$ itself, because the perpendicular to $AB$ at $A$ can be considered as the internal common tangent of the point circle $A$ and $\alpha$. Let $I$ be the point of intersection of $\gamma$ and the axis. The next theorem gives the circle associated with the point $I$.

**Theorem 3.1.** The internal common tangent of $\alpha$ and $\varepsilon_1$ passes through $I$.

**Proof.** Let $\rho$ be the circle with center $I$ passing through $O$. We invert the figure in $\rho$ (see Figure 7). Then $\alpha$ and $\beta$ are fixed. While $t$, which intersects $\rho$, is inverted into the circle with center $I$ touching $\alpha$ and $\beta$ intersecting $\rho$. Therefore $\gamma$ is the inverse of $t$. Hence the figure consisting of $\alpha$, $\gamma$ and $t$ is fixed by the inversion. This implies that $\varepsilon_1$ is also fixed. Since $\alpha$ and $\varepsilon_1$ are orthogonal to $\rho$, their point of tangency lies on $\rho$, and their common internal tangent passes through $I$. \[ \square \]
The proof also shows that the points of intersection of \( \gamma \) and \( t \) lies on \( \rho \). Therefore \( I \) is the midpoint of the arc of \( \gamma \) cut by \( t \).

**References**