A NEW PROOF OF THE STRONG GOLDBACH CONJECTURE

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Abstract

The Goldbach conjecture dates back to 1742 ; we refer the reader to [1]-[2] for a history of the conjecture. Christian Goldbach stated that every odd integer greater than seven can be written as the sum of at most three prime numbers. Leonhard Euler then made a stronger conjecture that every even integer greater than four can be written as the sum of two primes. Since then, no one has been able to prove the Strong Goldbach Conjecture.

The only best known result so far is that of Chen [3], proving that every sufficiently large even integer N can be written as the sum of a prime number and the product of at most two prime numbers. Additionally, the conjecture has been verified to be true for all even integers up to 4.10^{18} in 2014 , Jërg [4] and Tomás [5]. In this paper, we prove that the conjecture is true for all even integers greater than 8.

Key words: prime number, Goldbach Function, Goldbach Set , Strong Goldbach Conjecture , Sebastian Martin Ruiz Conjecture

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Introduction

The Goldbach conjecture asserts that every even integer greater than 4 is equal to the sum of two primes , for example $10 = 3+7 = 5+5$ and $16 = 3+13 = 5+11$. At present there is no proof of this conjecture in sight.

for every integer positive n we have $2n = (n-k)+(n+k)$,So let

$$\mathcal{S}_n = \Big\{ k \in \mathbb{N} : 0 \leq k \leq n \mid \; n-k \in \mathbb{P} \;\; and \;\; n+k \in \mathbb{P} \Big\}$$

In this paper we yield the more detailed proofs of the binary Goldbach's theorem using only this set S_n .

1 About new set S_n

Corollary 1 *Let n be positive integer greater than 4 , and let:*

$$\mathcal{S}_n^m = \Big\{ \; k \in \mathbb{N} : 0 \leqslant k \leqslant n \mid \; n-k \in \mathbb{P} \; \Big\} \tag{1}$$

Then for every positive integer n great than 4 we have : $\mathcal{S}_n^m \neq \varnothing$

PROOF. Let n be positive integer greater than 4 . So if we pose $k = n-2$ then $n-k = 2 \in \mathbb{P}$ $\square$

Corollary 2 *Let n be positive integer greater than 4 , and let:*

$$\mathcal{S}_n^M = \Big\{ \; k \in \mathbb{N} : 0 \leqslant k \leqslant n \mid \; n+k \in \mathbb{P} \; \Big\} \tag{2}$$

Then for every positive integer n great than 4 we have : $\mathcal{S}_n^M \neq \varnothing$

To prove this corollary we need to used Bertrand's postulate

Lemma 3 *Bertrand's postulate*
for every positive integer n great than 2 there is always at least one prime p such that $n \prec p \prec 2n$.

PROOF. There are many proofs of Bertrand's postulate ,First proved by Chebyshev (1850) and we refer to read Erdos's proof [6] and Ramanujan proof [7] $\square$

PROOF. Of corollary 2
For every positive integer k between 0 and n we have always $n \leqslant n+k \leqslant 2n$, and from Lemma 3 there is always at least one prime p such that $n \prec p \prec 2n$
So if we pose $k = p-n$ then $n+k = p \in \mathbb{P}$. $\square$

Example 4 *For $n \doteq 40$ and $n \doteq 29$ we have :*

$\mathcal{S}^m_{40} \doteq \{3\ \ 9\ \ 11\ \ 17\ \ 21\ \ 23\ \ 27\ \ 29\ \ 33\ \ 35\ \ 37\}$ *and* $\mathcal{S}^M_{40} \doteq \{1\ \ 3\ \ 7\ \ 13\ \ 19\ \ 21\ \ 27\ \ 31\ \ 33\ \ 39\}$
$\mathcal{S}^m_{29} \doteq \{6\ \ 10\ \ 12\ \ 16\ \ 18\ \ 22\ \ 24\ \ 26\}$ *and* $\mathcal{S}^M_{29} \doteq \{2\ \ 8\ \ 12\ \ 14\ \ 18\ \ 24\}$

Remark 5 *For every integer $n \succeq 2$ we have :*

$$\mathcal{S}^m_{2n+1} \subset 2\mathbb{N} \quad and \quad \mathcal{S}^m_{2n} \subset 2\mathbb{N}+1$$
$$\mathcal{S}^M_{2n+1} \subset 2\mathbb{N} \quad And \quad \mathcal{S}^M_{2n} \subset 2\mathbb{N}+1$$

Proposition 6 *Let n be positive integer greater than 4 , and let $\delta_m(n) = card\Big(\mathcal{S}^m_n\Big)$, then :*

$$\delta_m(n) = \pi(n) \tag{3}$$

with : $\pi(n) = \Big\{Number\ \ of\ \ numbers\ \ prime\ \ less\ \ than\ \ n\Big\}$

PROOF. Let n be positive integer greater than 4 ,then :

$$\delta_m(n) = \sum_{\substack{k=0\\ n-k\in\mathbb{P}}}^{n-1} 1 = \sum_{\substack{2\leqslant p\leqslant n\\ p\in\mathbb{P}}} 1 = \pi(n) \qquad \square$$

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Proposition 7 *Let n be positive integer greater than 4 , and let $\delta_M(n) = card\Big(\mathcal{S}^M_n\Big)$, then :*

$$\delta_m(n) = \pi(2n) - \pi(n) \tag{4}$$

with : $\pi(n) = \Big\{Number\ \ of\ \ numbers\ \ prime\ \ less\ \ than\ \ n\Big\}$

PROOF. Let n be positive integer greater than 4 ,then :

$$\delta_M(n) = \sum_{\substack{k=0\\ n+k\in\mathbb{P}}}^{n-1} 1 = \sum_{\substack{n\leqslant p\leqslant 2n-1\\ p\in\mathbb{P}}} = \sum_{\substack{2\leqslant p\leqslant 2n-1\\ p\in\mathbb{P}}} 1 - \sum_{\substack{2\leqslant p\leqslant n\\ p\in\mathbb{P}}} 1 = \pi(2n) - \pi(n) \qquad \square$$

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Theorem 8 (Principal) *Let n be positive integer greater than 4 , and let :*

$$\mathcal{S}_n = \Bigg\{k \in \mathbb{N} : 0 \leq k \leq n \mid\ \ n-k \in \mathbb{P}\ \ and\ \ n+k \in \mathbb{P}\Bigg\} \doteq \mathcal{S}^m_n \bigcap \mathcal{S}^M_n \tag{5}$$

Then for every positive integer $n \geq 4$ we have : $\mathcal{S}_n \neq \varnothing$.

PROOF. Let n be positive integer greater than 4 , and we put :
$\mathbb{C}_n = \Big\{\ k \in \mathbb{N} : 0 \leqslant k \leqslant n \mid k \notin \mathbb{P}\Big\}$ and $\mathbb{P}_n = \Big\{\ p \in \mathbb{N} : 0 \leqslant p \leqslant n \mid p \in \mathbb{P}\Big\}$.
Suppose we have that : $\mathcal{S}_n = \varnothing$, then : $\forall\ k \in \mathcal{S}_n^M \Rightarrow k \notin \mathcal{S}_n^m$
and as $\forall\ i \in \Big\{1, 2, ..., \pi(2n) - \pi(n)\Big\}$
We have : $k_i \in \mathcal{S}_n^M \Longrightarrow n + k_i \in \mathbb{P}$ i.e $n + k_i$ is an odd number .
since $\forall\ i \in \Big\{1, 2, ..., \pi(2n) - \pi(n)\Big\}$ we have : $0 \leqslant n - k_i \leqslant n$.
So according to the assumed hypothesis we have $n - k_i \in \mathbb{C}_n$
Now , if n is an even number , then every number n , $n-2$,$n-4$,...,$n-n$ is the element of $\in \mathbb{C}_n$
if n is an odd number , then every number $n-1$,$n-3$,...,$n-n+1$ is in $\mathbb{C}_n$
Hence : $Card\Big(\mathbb{C}_n\Big) = \frac{n}{2} + \pi(2n) - \pi(n)$,Since $[[0, n]] = \mathbb{C}_n \cup \mathbb{P}_n$
Then : $Card\Big([[1, n]]\Big) = Card\Big(\mathbb{C}_n \cup \mathbb{P}_n\Big) \Rightarrow Card\Big(\mathbb{C}_n\Big) + Card\Big(\mathbb{P}_n\Big) = n + 1$
Finally :

$$\frac{n}{2} + \pi(2n) - \pi(n) + \pi(n) = n + 1$$

then $2\pi(2n) - 2 = n$ which is absurd .
So $\exists\ i \in \Big\{1, 2, ..., \pi(2n) - \pi(n)\Big\}$ such that $n + k_i \in \mathbb{P}$ *and* $n - k_i \in \mathbb{P}$.
as a result for any positive integer $n \succeq 4$, we have $\mathcal{S}_n^M \neq \varnothing$. . $\square$

2 Proof of Goldbach Conejcture and Sebastian Martin Ruiz Conjecture

Conjecture 9 (GoldBach Conjecture) *Let $n \in \mathbb{N}$ such that $n \succeq 3$.*
Then : $\exists\ p, q \in \mathbb{P}$ such that $2n = p + q$.

PROOF. Let $n \in \mathbb{N}$ Such that $n \succeq 4$.
Then from theorem 8 : $\exists\ k \in \mathcal{S}_n$ *such that* $n - k \in \mathbb{P}$ *et* $n + k \in \mathbb{P}$. since $n - k + n + k = 2n$.
Then for every positive integer $n \succeq 4$ they exist two prime numbers p et q such that :

$$2n = p + q \ \ with\ \ p = n - k \ \ and\ \ p = n + k \ \ , k \in \mathcal{S}_n$$

$\square$

Conjecture 10 (Sebastian Martin Ruiz Conjecture) *Let $n \in \mathbb{N}$ Such that $n \succeq 3$.*
Then : $\exists\ k \in \{1, 2, ..., n-1\}$ such that $\phi(n^2 - k^2) = (n-1)^2 - k^2$.
with ϕ is Euler's totient : $\phi(n) = Card\{k \in \mathbb{N} : n \leqslant n/\ \ pgcd(n, k) = 1\}$

PROOF. Let $n \in \mathbb{N}$ Such that $n \succeq 4$.
Then : $\exists\, k \in \mathcal{S}_n$ 8 *such that* $n-k \in \mathbb{P}$ *and* $n+k \in \mathbb{P}$.
Then : $\phi((n-k)(n+k)) = \phi(n-k)\phi(n+k)$ (because $gcd(n-k, n+k) = 1$).
Since $n-k$ and $n+k$ are prime numbers
then $\phi(n-k) = n-k-1$ and $\phi(n+k) = n+k-1$.
So we have : $\phi((n-k)(n+k)) = (n-k-1)(n+k-1) = (n-1)^2 - k^2$ □

References

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