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Abstract:

This paper shows the importance of two properties, which are at the base of the Riemann hypothesis. The key point of all the reasoning about the validity of the Riemann hypothesis is in the fact that only if the Riemann hypothesis is true, these two properties, which are satisfied by the non-trivial zeros, are both true.

$\zeta(s)=\zeta(1-s)$ and $\zeta(\bar{s})=\overline{\zeta(s)}$ at the base of the Riemann hypothesis

First of all, we introduce the functional equation of the Riemann zeta function:

$$\textbf{1.}\ \zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\ \zeta(1-s)\ \text{ for } s \in \mathbb{C} \setminus \{0,1\}$$ [1]

From this equation we follow this thought:

- Since the equation **(1.)** is valid for $s \in \mathbb{C} \setminus \{0,1\}$ [1], it will be satisfied also by the non-trivial zeros, which have 0<Re(s)<1.
- Since it is satisfied by non-trivial zeros, when *s* is a non-trivial zero one between 2^s, π^s, $\sin\left(\frac{\pi s}{2}\right)$, $\Gamma(1-s)$ and $\zeta(1-s)$ will cancel the equation **(1.)**.
- Since 2^s, π^s, and $\Gamma(1-s)$ don't cancel the equation **(1.)** and $\sin\left(\frac{\pi s}{2}\right)$ only for the trivial zeros, which are of the form $s=-2n$ for $n \in \mathbb{N} \setminus \{0\}$ [2], when *s* is a non-trivial zero only $\zeta(1-s)$ will cancel the equation **(1.)**.
- Since only $\zeta(1-s)$ cancels the equation **(1.)** for non-trivial zeros, when *s* is a non-trivial zeros $\zeta(1-s)=0$ and for the zero-product property $\zeta(s)=0$.
- Thus, we deduce that $\zeta(s)=\zeta(1-s)$, when *s* is a non-trivial zero.

From the previous reasoning we get the first property of non-trivial zeros, which is:

$$\textbf{2.}\ \zeta(s)=\zeta(1-s)$$

From the equation **(2.)** we observe that:

2.A) Non-trivial zeros are symmetric with respect to the point $P\left(\frac{1}{2};0\right)$

Now, we introduce the mirror symmetry formula of the Riemann zeta function:

$$\textbf{3.}\ \zeta(\bar{s})=\overline{\zeta(s)}$$ [1] [3]

Following a similar thought to that done before…

- Since the equation **(3.)** is valid for $s \in \mathbb{C} \setminus \{1\}$ [1] , it will be satisfied also by the non-trivial zeros.

- Hence, we deduce that $\zeta(\overline{s}) = \overline{\zeta(s)}$, when s is a non-trivial zeros.

… we obtain the second property of non-trivial zeros, which is:

$$\textbf{4.}\ \zeta(\overline{s}) = \overline{\zeta(s)}$$

From the equation **(4.)** we observe that:

4.A) The equation **(4.)** implies that non-trivial zeros are symmetric with respect to the real line. In fact:

- Since s is a non-trivial zero, $\zeta(s) = 0$.
- Since $\zeta(s) = 0$, $\overline{\zeta(s)} = 0$ and so $\zeta(s) = \overline{\zeta(s)}$
- Since $\overline{\zeta(s)} = \zeta(\overline{s})$, $\zeta(s) = \zeta(\overline{s})$
- Thus, since $\zeta(s) = \zeta(\overline{s})$, non-trivial zeros are symmetric with respect to the real line.

So for the observation **4.A,** we can rewrite the equation **(4.)** in this way for the non-trivial zeros:

$$\textbf{5.}\ \zeta(s) = \zeta(\overline{s})$$

At this point, we have shown that non-trivial zeros satisfy both equations **(2.)** and **(5.).** Hence the following system is valid for non-trivial zeros:

$$\textbf{6.}\begin{cases} \zeta(s) & = \zeta(1-s) \\ \zeta(s) & = \zeta(\overline{s}) \end{cases}$$

From now on, we will consider non-trivial zeros in couples so that:

$$\textbf{7.}\ Z_n = (s_{1(n)}; s_{2(n)}),$$

where Z_n is nth couple of non-trivial zeros, $s_{1(n)}$ is the first non-trivial zero of the nth couple and $s_{2(n)}$ is the second non-trivial zero of the nth couple.

Now, we rewrite the system **(6.)**, using the formula **(7.)**:

$$\textbf{8.}\begin{cases} Z_n = (s_n; 1 - s_n) \\ Z_n = (s_n; \overline{s_n}) \end{cases}$$

From the system **(8.)** we observe that:

8.A) A couple Z_n of non-trivial zeros is made by two non-trivial zeros so that let $i\,t_1$ and $i\,t_2$ their imaginary parts, $|i\,t_1| = |i\,t_2|$.

8.B) For the first equation of the system **(8.)** the two non-trivial zeros of a couple Z_n are symmetric with respect to the point $P\left(\frac{1}{2};0\right)$, according to the observation **2.A**.

8.C) For the second equation of the system **(8.)** the two non-trivial zeros of a couple Z_n are symmetric with respect to the real line, according to the observation **4.A**.

From the observations **8.B)** and **8.C)** we obtain the key statement, which is at the base of the Riemann hypothesis:

9. *" The non-trivial zeros of the Riemann zeta function are situated in the complex plane, arranged in couples, symmetric with respect to the point* $P\left(\frac{1}{2};0\right)$ *and to the real line ,so that let* $i\,t_1$ *and* $i\,t_2$ *the imaginary parts of the two non-trivial zeros of the couple,* $|\,i\,t_1\,| = |\,i\,t_2\,|$*".*

In the paper [4] we have shown that, after having studied all the possible cases, also the limit ones, only if the real part of the non-trivial zeros is equal to $\frac{1}{2}$, that is when the Riemann hypothesis is true, the statement **(9.)** is satisfied by all couples of non-trivial zeros Z_n and this is a proof of the Riemann hypothesis.

In the paper [4] we have used a system in which we have considered also the symmetry about the critical line $x=\frac{1}{2}$; however in this paper it has turned out to be unnecessary, since it is only a consequence of the combination between the two equations of the system **(8.).**

References:

[1]: *https://de.wikipedia.org/wiki/Riemannsche_%CE%B6-Funktion*
[2]: *https://www.academia.edu/37289964/Proposal_of_solution_of_the_Riemann_hypothesis_part_1*
[3]: *http://functions.wolfram.com/ZetaFunctionsandPolylogarithms/Zeta/04/02/01/*
[4]: *https://www.academia.edu/37498670/Proposal_of_solution_of_the_Riemann_hypothesis_part_2*