C ⊗ H ⊗ O-valued Gravity, [SU(4)]^4 Unification, Hermitian Matrix Geometry and Nonsymmetric Kaluza-Klein Theory

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Abstract

We review briefly how R ⊗ C ⊗ H ⊗ O-valued Gravity (real-complex-quaterno-octonionic Gravity) naturally can describe a grand unified field theory of Einstein’s gravity with a Yang-Mills theory containing the Standard Model group SU(3) × SU(2) × U(1). In particular, the C ⊗ H ⊗ O algebra is explored deeper. It is found that it can furnish the gauge group [SU(4)]^4 revealing the possibility of extending the Standard Model by introducing additional gauge bosons, heavy quarks and leptons, and a fourth family of fermions with profound physical implications. An analysis of C ⊗ H ⊗ O-valued gravity reveals that it bears a connection to Nonsymmetric Kaluza-Klein theories and complex Hermitian Matrix Geometry. The key behind these connections is in finding the relation between C ⊗ H ⊗ O-valued metrics in two complex dimensions with metrics in higher dimensional real manifolds (D = 32 real dimensions in particular). It is desirable to extend these results to hypercomplex, quaternionic manifolds and Exceptional Jordan Matrix Models.

Keywords: Nonassociative Geometry, Clifford algebras, Quaternions, Octonionic Gravity, Unification, Strings.
1 Introduction

This introduction is a review of our recent work [1] and may be skipped by those readers familiar with it. Recently we have argued how $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$-valued Gravity (real-complex-quaterno-octonionic Gravity) naturally can describe a grand unified field theory of Einstein’s gravity with a Yang-Mills theory containing the Standard Model group $SU(3) \times SU(2) \times U(1)$ [1]. It was based on an extension of the work by [2],[3],[4]. The quaternion algebra is defined by $q_i q_j = -\delta_{ij} q_o + \epsilon_{ijk} q_k$; $i, j, k = 1, 2, 3$, and $q_o$ is the identity element. Given an octonion $X$ it can be expanded in a basis $(e_o, e_a)$ as

$$X = x^o e_o + x^a e_a, \ a = 1, 2, \ldots, 7. \quad (1.1)$$

where $e_o$ is the identity element. The Noncommutative and Nonassociative algebra of octonions is determined from the relations

$$e_o^2 = e_o, \ e_o e_a = e_a e_o = e_a, \ e_a e_b = -\delta_{ab} e_o + C_{abc} e_c, \ a, b, c = 1, 2, 3, \ldots, 7. \quad (1.2)$$

The non-vanishing values of the fully antisymmetric structure constants $C_{abc}$ is chosen to be 1 for the following 7 sets of index triplets (cycles) [4]

$$\{124\}, \ \{235\}, \ \{346\}, \ \{457\}, \ \{561\}, \ \{672\}, \ \{713\} \quad (1.3)$$

Each cycle represents a quaternionic subalgebra. The values of $C_{abc}$ for the other combinations are zero. The latter 7 sets of index triplets (cycles) correspond to the 7 lines of the Fano plane.

The octonion conjugate is defined as

$$\bar{X} = x^o e_o - x^m e_m. \quad (1.4)$$

and the norm

$$N(X) = \langle X X \rangle = \text{Real} (\bar{X} X) = (x^o x^o + x^k x^k). \quad (1.5)$$

The inverse

$$X^{-1} = \frac{\bar{X}}{N(X)}, \quad X^{-1} X = XX^{-1} = 1. \quad (1.6)$$

The non-vanishing associator is defined by

$$\{X, Y, Z\} = (XY)Z - X(YZ) \quad (1.7)$$

In particular, the associator

$$\{e_i, e_j, e_k\} = d_{ijkl} e_l, \quad d_{ijkl} = \epsilon_{ijklmnp} e^{mnp}, \ i, j, k, \ldots = 1, 2, 3, \ldots, 7 \quad (1.8)$$
There are no matrix representations of the Octonions due to the non-associativity, however Dixon has shown how many Lie algebras can be obtained from the left/right action of the octonion algebra on itself \[4\]. \(O_L\) and \(O_R\) are identical, isomorphic to the matrix algebra \(R\) where, for example, if \(x\) from the left/right action of the octonion algebra on itself \(O\) associativity, however Dixon has shown how many Lie algebras can be obtained \((\text{given by})\) the following 8 \(8 \times 8\) matrix representations of the non-associative Octonion algebra, and as a result one has that

\[
M_{La} = e_{La} \leftrightarrow M_{La}, \quad \text{and whose entries are given by}
\]

\[
(M^L_a)_{bc} = C_{abc}, \quad a, b, c = 1, 2, \cdots, 7; \quad (M^L_a)_{00} = 0, \quad (M^L_a)_{0c} = \delta_{ac}, \quad (M^L_a)_{c0} = -\delta_{ac}
\]

And similar procedure for the right actions, Due to the non-associativity of the Octonions one has \(e_1 e_2 = e_4\), but \(M_{L1} M_{L2} \neq M_{L4}\), because there are no matrix representations of the non-associative Octonion algebra, and as a result one has that

\[
M_{La} M_{Lb} \neq C_{abc} M_{Lc}
\]

Dixon [4] many years ago published a monograph pointing out the key role that the composition algebra (the Dixon algebra) \(T = R \otimes C \otimes H \otimes O\) had in the architecture of the Standard Model. More recently, it has been shown by Furey how this algebra acting on itself allows to find the Standard Model particle representations [5]. For this reason we constructed in [1] a gravitational theory based on a \(R \otimes C \otimes H \otimes O\)-valued metric defined as

\[
g_{\mu\nu}(x^\mu) = g(\mu\nu)(x^\mu) + g^{LA}(x^\mu) (q_I \otimes e_A), \quad q_I = q_0, q_1, q_2, q_3; \quad e_A = e_o, e_1, e_2, \cdots, e_7
\]

where the ordinary 4D spacetime coordinates are \(x^\mu, \mu = 0, 1, 2, 3\), and \(g_{\mu\nu}\) is the standard Riemannian metric. The extra “internal” \(C \otimes H \otimes O\)-valued metric components are explicitly given by

\[
(g_{\mu\nu} + ig_{\mu\nu})^o, \quad (g_{\mu\nu} + ig_{\mu\nu})^k, \quad (g_{\mu\nu} + ig_{\mu\nu})^a, \quad (g_{\mu\nu} + ig_{\mu\nu})^k \quad \text{for} \quad k = 1, 2, 3, a = 1, 2, \cdots, 7.
\]

The bar conjugation amounts to \(i \rightarrow -i; \quad q_k \rightarrow -q_k; \quad e_a \rightarrow -e_a\), so that \(\bar{g}_{\mu\nu} = g_{\mu\nu}\).

The generalization of the line interval considered in [2], [3] based on the metric (3.1) is then given by

\[
ds^2 = < g_{\mu\nu} dx^\mu dx^\nu > = (g_{\mu\nu} + g_{\mu\nu}^o) dx^\mu dx^\nu
\]
where the operation $< \cdots >$ denotes taking the real components. From eq-(1.13) one learns that the $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued metric leads to a bimetric theory of gravity where the two metrics are, respectively, $g_{(\mu\nu)}, g_{(oo)} = h_{(\mu\nu)}$.

The $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued affinity was given by

$$\Upsilon^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu}(g_{\mu\nu}) + \delta^\rho_\mu A_{\nu} = \Gamma^\rho_{\mu\nu}(g_{\mu\nu}) + \delta^\rho_\mu \left( A^{oo}_\nu (q_o \otimes e_a) + A^{io}_\nu (q_i \otimes e_o) + A^{ao}_\nu (q_o \otimes e_i) \right)$$

Thus we have decomposed the $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued affinity $\Upsilon^\rho_{\mu\nu}$ into a real-valued “external” part $\Gamma$ plus an “internal” part $\Theta^\rho_{\mu\nu}$. The base spacetime connection may be chosen to be the torsionless Christoffel connection

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left( \partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right)$$

but the ‘internal” part $\Theta^\rho_{\mu\nu}$ of the connection is taken to be independent of the metric, like in the Palatini formalism.

The $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued curvature tensor $\mathbf{R}^\sigma_{\rho\mu\nu} = \mathbf{R}^\sigma_{\rho\mu\nu} + \mathbf{O}^\sigma_{\rho\mu\nu}$, involving the base spacetime and internal space curvature is defined by

$$\mathbf{R}^\sigma_{\rho\mu\nu} = \Upsilon^\sigma_{\rho\mu,\nu} - \Upsilon^\sigma_{\rho\nu,\mu} + \Upsilon^\sigma_{\tau\nu} \Upsilon^\tau_{\rho\mu} - \Upsilon^\sigma_{\tau\mu} \Upsilon^\tau_{\rho\nu}.$$

The “internal” space $\mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued curvature is

$$\mathbf{O}^\rho_{\sigma\mu\nu} = \delta^\rho_\sigma \mathbf{F}_{\mu\nu}$$

with

$$\mathbf{F}_{\mu\nu} = \mathbf{A}_{\mu,\nu} - \mathbf{A}_{\nu,\mu} - [ \mathbf{A}_{\mu}, \mathbf{A}_{\nu} ].$$

and where the field $\mathbf{A}_\mu$ can be read directly in terms of the internal space affinity from the relation

$$\Theta^\rho_{\mu\nu} = \delta^\rho_\mu \mathbf{A}_\nu$$

There are 32 complex-valued fields (64-real valued fields)

$$\mathbf{A}_\mu = \{ A^{oo}_\mu, A^{io}_\mu, A^{ao}_\mu, A^{ia}_\mu \}$$

and the commutators in eq-(1.19) are defined by

$$[q_I \otimes e_A, q_J \otimes e_B] = \frac{1}{2} \{ q_I, q_J \} \otimes [e_A, e_B] + \frac{1}{2} [ q_I, q_J ] \otimes \{ e_A, e_B \}$$

which lead to the following explicit components for $\mathbf{F}_{\mu\nu}$

$$F^{oo}_{\mu\nu} = \partial_\mu A^{oo}_\nu - \partial_\nu A^{oo}_\mu$$
\[ F_{\mu \nu}^{oc} = \partial_\mu A_{\nu}^{oc} - \partial_\nu A_{\mu}^{oc} + (A_{\mu}^{ao} A_{\nu}^{ab} - \delta_{ij} A_{\mu}^{ia} A_{\nu}^{jb}) C_{ab} \] (1.24)

\[ F_{\mu \nu}^{ko} = \partial_\mu A_{\nu}^{ko} - \partial_\nu A_{\mu}^{ko} + (A_{\mu}^{io} A_{\nu}^{jo} - \delta_{ab} A_{\mu}^{ia} A_{\nu}^{jb}) f_{ij}^k \] (1.25)

\[ F_{\mu \nu}^{kc} = \partial_\mu A_{\nu}^{kc} - \partial_\nu A_{\mu}^{kc} + A_{\mu}^{ia} A_{\nu}^{jb} C_{ab} + A_{\mu}^{io} A_{\nu}^{jc} f_{ij}^k \] (1.26)

The next step was to embed the Standard Model Gauge Fields into the Internal Connection \( \Theta_{\rho \mu \nu} \). Eqs-(1.23-1.26) yield the following 32 complex-valued non-vanishing field strengths

\[ F_{\mu \nu}^{oo}, F_{\mu \nu}^{ko}, F_{\mu \nu}^{oc}, F_{\mu \nu}^{kc}, k = 1, 2, 3; \ c = 1, 2, \ldots, 7 \] (1.27)

Given the \( U(1) \) Maxwell field

\[ F_{\mu \nu} = \partial_\mu A_{\nu} - \partial_\nu A_\mu \] (1.28)

the Maxwell kinetic term in the Standard Model action is embedded as follows

\[ \mathcal{F}_{\mu \nu} F^{\mu \nu} \subset F_{\mu \nu}^{oo} (F_{\mu \nu}^{oo})^* \] (1.29)

Given the \( SU(2) \) field strength

\[ F_{\mu \nu}^{k} = \partial_\mu A_{\nu}^{k} - \partial_\nu A_{\mu}^{k} + A_{\mu}^{i} A_{\nu}^{j} \epsilon_{ij}^k \] (1.30)

the \( SU(2) \) Yang-Mills term is embedded as

\[ \mathcal{F}_{\mu \nu} F_{1}^{i \nu} (i = 1, 2, 3) \subset (F_{\mu \nu}^{ko}) (F_{\mu \nu}^{ko})^* \] (1.31)

Since the \( SU(2) \) algebra is isomorphic to the algebra of quaternions, the embedding (1.31) is very natural. The chain of subgroups

\[ SO(8) \supset SO(7) \supset G_2 \supset SU(3) \] (1.32)

related to the round and squashed seven-spheres : \( S^7 \simeq SO(8)/SO(7), S^7_8 \simeq SO(7)/G_2 \), reflect how the \( SU(3) \) group is embedded. The number of generators of \( SO(8), SO(7) \) are 28 and 21 respectively. There are \( 7 + 21 = 28 \) complex-valued field strengths, respectively

\[ F_{\mu \nu}^{oc}, F_{\mu \nu}^{kc}, k = 1, 2, 3; \ c = 1, 2, \ldots, 7 \] (1.33)

such that the \( SU(3) \) Yang-Mills terms can be embedded into the contribution of the above \( 7 + 21 = 28 \) complex-valued fields as follows

\[ \mathcal{F}_{\mu \nu} F_{\alpha}^{\mu \nu} (\alpha = 1, 2, \ldots, 7, 8) \subset (F_{\mu \nu}^{oc}) (F_{\alpha}^{oc})^* + (F_{\mu \nu}^{kc}) (F_{\mu \nu}^{kc})^* (c = 1, 2, \ldots, 7) \] (1.34)
and where the SU(3) field strength is given by
\[
F_{\mu\nu}^\gamma = \partial_\mu A_\nu^\gamma - \partial_\nu A_\mu^\gamma + A_\mu^\alpha A_\nu^\beta f_{\alpha\beta}^\gamma
\]  

(1.35)

Having reviewed some of the results in [1] we shall proceed in the next section to show how the matrix realization of the C \(\otimes\) H \(\otimes\) O\(_L\) algebra naturally leads to a rank-16 \(u(4) \oplus (4) \oplus u(4) \oplus u(4)\) algebra. This, in turn, suggests to extend the Standard Model based on the \(SU(3) \times SU(2) \times U(1)\) group to one based on \([SU(4)]^4\). In the final section we show how to establish the correspondence among C \(\otimes\) H \(\otimes\) O-valued gravity, generalized Hermitian geometry and Nonsymmetric Kaluza-Klein Theory. The construction in section 3 must not be confused with the model of \(R \otimes C \otimes H \otimes O\)-valued gravity discussed above.

2 \(SU(4)_C \times SU(4)_F \times SU(4)_L \times SU(4)_R\) Unification

Given that the complex quaternionic algebra C \(\otimes\) H is isomorphic to the Pauli spin algebra with the 2 \(\times\) 2 matrices \(q_0 = 1_{2 \times 2}, q_k = i\sigma_k, k = 1, 2, 3\), and the left action of the octonionic algebra on itself is represented by the 8 \(\times\) 8 matrices \(e_{LA} = M_L^A, A = 0, 1, \ldots, 7\), then the 4 \(\times\) 8 = 32 generators \(q_I \otimes e_{LA}\) of the C \(\otimes\) H \(\otimes\) O\(_L\) algebra can be represented by 32 complex 16 \(\times\) 16 matrices, which is tantamount to 64 real 16 \(\times\) 16 matrices, and which is compatible with the fact that 64 (2 \(\times\) 4 \(\times\) 8) is the dimension of the C \(\otimes\) H \(\otimes\) O\(_L\) algebra.

Each complex 16 \(\times\) 16 matrix, above, can be expanded in terms of the basis elements of the complex Clifford algebra \(Cl(8, C)\) comprised of \(2^8 = 256\) complex 16 \(\times\) 16 matrices. However this is far too cumbersome. It is easier if we expand each of the above 32 complex 16 \(\times\) 16 matrices in terms of the tensor products \(\Gamma_M \otimes 1_{4 \times 4}\), where \(\Gamma_M (M = 1, 2, \ldots, 32 = 2^5)\) is the basis of the complex Clifford algebra \(Cl(5, C)\) comprised of 32 complex 4 \(\times\) 4 matrices, and \(1_{4 \times 4}\) is the unit 4 \(\times\) 4 matrix.

Therefore we end up having that the 32 complex 16 \(\times\) 16 matrix generators \(q_I \otimes e_{LA}\) of the C \(\otimes\) H \(\otimes\) O\(_L\) algebra can be expanded in terms of a linear combination of the 32 \(Cl(5, C)\) algebra generators \(\Gamma_M\) as follows

\[
q_I \otimes e_{LA} = (M_{IA}^L)_{16 \times 16} = \sum_{M=1}^{32} C_{IA}^M (\Gamma_M)_{4 \times 4} \otimes 1_{4 \times 4},
\]  

(2.1)

where \(I = 0, 1, 2, 3; A = 0, 1, 2, \ldots, 7\), and \(C_{IA}^M\) are complex numerical coefficients.

Let us recall the following isomorphisms among real and complex Clifford algebras [6]

\[
Cl(2m + 1, C) = Cl(2m, C) \oplus Cl(2m, C) \sim M(2^m, C) \oplus M(2^m, C) \Rightarrow
\]
\[
CI(5, C) = CI(4, C) \oplus CI(4, C)
\] (2.2)

where \( M(2^m, C) \) is the \( 2^m \times 2^m \) matrix algebra over the complex numbers (some authors [4] use the different notation \( \mathbf{C}(2^m) \)).

Also one has

\[
CI(4, C) \sim M(4, C) \sim CI(4, 1, R) \sim CI(2, 3, R) \sim CI(0, 5, R)
\] (2.3)

\[
CI(4, C) \sim M(4, C) \sim CI(3, 1, R) \oplus i CI(3, 1, R) \sim M(4, R) \oplus i M(4, R)
\] (2.4)

\[
CI(4, C) \sim M(4, C) \sim CI(2, 2, R) \oplus i CI(2, 2, R) \sim M(4, R) \oplus i M(4, R)
\] (2.5)

where \( M(4, R), M(4, C) \) is the \( 4 \times 4 \) matrix algebra over the reals and complex numbers, respectively.

In [6] we showed, by recurring to the Weyl unitary “trick”, how from each one of the \( CI(3, 1, R) \) commuting sub-algebras inside the \( CI(4, C) \) algebra one can also obtain the \( u(p, q) \) algebras with the provision \( p + q = 4 \). Namely, the \( u(p, q) \) algebra generators are given by suitable linear combinations of the \( CI(3, 1, R) \) generators. In particular, the \( u(2, 2) = su(2, 2) \oplus u(1) \) algebra contains the conformal algebra in four dimensions \( su(2, 2) \sim so(4, 2) \). When \( p = 4, q = 0 \) the algebra is \( u(4) = u(1) \oplus su(4) \sim u(1) \oplus so(6) \).

To sum up, given that the algebra \( M(4, C) \sim gl(4, C) \) is also the complexification of \( u(4) \) \( (sl(4, C) \) is the complexification of \( su(4) \), and by virtue of eqs-(2.2), the \( CI(5, C) \) algebra can be decomposed into four copies of \( u(4) \)

\[
CI(5, C) = CI(4, C) \oplus CI(4, C) \sim u(4) \oplus u(4) \oplus u(4) \oplus u(4)
\] (2.6)

The dimension of the four copies of \( u(4) \) is \( 4 \times 16 = 64 \) which matches the dimension of the \( C \otimes H \otimes O_L \) algebra, as expected (64 is also the dimension of the real \( CI(6) \) algebra). Consequently, the \( C \otimes H \otimes O_L \) algebra, by virtue of the decomposition in eq-(2.1), can accommodate a grand unified group given by

\[
SU(4)_C \times SU(4)_F \times SU(4)_L \times SU(4)_R \subset U(4) \times U(4) \times U(4) \times U(4)
\] (2.7)

The gauge group \( SU(3)_C \times SU(3)_F \times SU(3)_L \times SU(3)_R \) can naturally be embedded into the above \([SU(4)]^4\) group. The former group involving a unification of left-right \( SU(3)_L \times SU(3)_R \) chiral symmetry, color \( SU(3)_C \) and family \( SU(3)_F \) symmetries in a maximal rank-8 subgroup of \( E_8 \) was proposed by [7] as a landmark for future explorations beyond the Standard Model (SM). This model is called the \( SU(3) \)-family extended SUSY trinification model [7]. Among the key properties of this model are the unification of SM Higgs and lepton sectors, a common Yukawa coupling for chiral fermions, the absence of the \( \mu \)-problem, gauge couplings unification and proton stability to all orders in perturbation theory.

The standard model (SM) fermions (quarks, leptons) can be embedded into the fermionic matter belonging to the following \( SU(4)_C \times SU(4)_F \times SU(4)_L \times SU(4)_R \) representations as follows
\[ Q_{SM} \subset Q = (4, \bar{4}, 1), \quad Q^c_{SM} \subset Q^c = (\bar{4}, 4, 1), \]
\[ L_{SM} \subset L = (1, 4, \bar{4}, 1), \quad L^c = (1, 4, 1, 4) \]

where the \( Q, Q^c, L, L^c \) multiplets include the addition of heavy quarks (anti-quarks); leptons (anti-leptons), and an extra fourth family of fermions (and their anti-particles). The first (left handed) quark family is

\[
Q_1 \equiv \begin{pmatrix}
 u_r & d_r & U_r & D_r \\
 u_b & d_b & U_b & D_b \\
 u_g & d_g & U_g & D_g \\
 Q_u & Q_d & Q_U & Q_D
\end{pmatrix}
\]

where \( Q_u, Q_d, Q_U, Q_D \), and \( U_{r,b,g}, D_{r,b,g} \) are the additional quarks. As usual \( r, b, g \) stand for red, blue, green color. The charge conjugate multiplet containing the (right-handed) anti-quarks of the first family is

\[
Q^c_1 \equiv \begin{pmatrix}
 \bar{u}_r & \bar{d}_r & \bar{U}_r & \bar{D}_r \\
 \bar{u}_b & \bar{d}_b & \bar{U}_b & \bar{D}_b \\
 \bar{u}_g & \bar{d}_g & \bar{U}_g & \bar{D}_g \\
 Q_u & Q_d & Q_U & Q_D
\end{pmatrix}
\]

By \( \bar{u}_r \) one means \( u^c_\bar{r} \), the up anti-quark with anti-red color, etc. ... Whereas \( \bar{Q}_u = Q^c, \cdots \). And similar assignments for the remaining quark families.

The lepton multiplet will include the ordinary leptons (neutrino, electron, \( \cdots \)), plus the addition of charged \( E_-, E_+ \), and neutral leptons \( N_E, N_E^c, \cdots \). The first (left handed) lepton multiplet is comprised of \( \{ \nu_e, e^-, N_E, E_- \} \), and its (right handed) anti-multiplet is comprised of \( \{ \nu^c_e, e^+, N_{E}^c, E_+ \} \). If necessary, one may also have to add extra fermions to cancel anomalies.

An analysis of the models based on \( SU(4)_C \times SU(3)_L \times SU(3)_R \), and a preliminary discussion of \( SU(4)_C \times SU(4)_L \times SU(4)_R \) can be found in [8]. Their lepton assignment differs from ours. An early \( SU(4)_C \times SU(4)_F \) model, and based on an extension of the Pati-Salam group \( SU(4)_C \times SU(2)_L \times SU(2)_R \), was proposed by [9]. Examples of a fourth family extension of the Standard Model can be found in [10].

Concluding this section, the algebraic structure of \( C \otimes H \otimes O_L \) led to the group \( [SU(4)]^3 \) and reveals the possibility of extending the standard model by introducing additional gauge bosons, heavy quarks and leptons, and a fourth family of fermions. The physical implications are enormous.

### 3 C⊗H⊗O-valued gravity, Matrix geometry and Nonsymmetric Kaluza-Klein Theory

In the final section we show how to establish the correspondence among \( C \otimes H \otimes O \)-valued gravity, generalized Hermitian Matrix geometry and Nonsymmetric
Kaluza-Klein Theory. It must not be confused with the model of $R \otimes C \otimes H \otimes O$-valued gravity discussed previously in section 1.

We begin by recalling that the standard Hermitian metric on a complex $D$-dim manifold whose complex coordinates are $z^\mu, \bar{z}^\mu, \mu = 1, 2, \cdots, D; \bar{\mu} = 1, 2, \cdots, \bar{D}$, satisfies the properties [11]

$$g_{\mu\bar{\nu}} = g_{\bar{\mu}\bar{\nu}} = 0, \quad g_{\mu\bar{\nu}} = g_{\bar{\nu}\mu} \neq 0, \quad (g_{\mu\bar{\nu}})^* = g_{\bar{\nu}\mu} = g_{\bar{\nu}\bar{\mu}} \neq 0 \quad (3.1)$$

The real infinitesimal line interval $ds^2$ is given by

$$ds^2 = g_{\mu\bar{\nu}} \, dz^\mu \, d\bar{z}^\nu + g_{\bar{\mu}\bar{\nu}} \, d\bar{z}^\mu \, dz^\nu \quad (3.2)$$

The $H \otimes O$-valued extension of the above Hermitian metric leads to a real infinitesimal line interval of the form

$$ds^2 = \frac{1}{16} \, \text{Trace} \left( g_{\mu\bar{\nu}} \, dz^\mu \, d\bar{z}^\nu + g_{\bar{\mu}\bar{\nu}} \, d\bar{z}^\mu \, dz^\nu \right) \quad (3.3)$$

and provided in terms of the trace of the $16 \times 16$ matrix-valued $g_{\mu\bar{\nu}}, g_{\bar{\mu}\bar{\nu}}$ components as we shall explain next.

Given that the $2 \times 4 \times 8 = 64$ generators of the $C \otimes H \otimes O_L$ algebra can be represented by $32$ complex $16 \times 16$ matrices $(M^J_{LA})_{16 \times 16}$ (or $64$ real $16 \times 16$ matrices), the $C \otimes H \otimes O$-valued metric components appearing in (3.3) can be expanded in a quatero-octonionic basis, and rewritten in a $16 \times 16$-matrix form, in the following fashion

$$g_{\mu\bar{\nu}}(z^\mu, \bar{z}^\mu) = \sum_{J,A} g^J_{\mu\bar{\nu}}(z^\mu, \bar{z}^\mu) \, (q_I \otimes e_{LA})^{JK} = g^{JK}(z^\mu, \bar{z}^\mu) \quad (3.4)$$

$$g_{\bar{\mu}\bar{\nu}}(z^\mu, \bar{z}^\mu) = \sum_{J,A} g^{J\bar{\mu}\bar{\nu}}(z^\mu, \bar{z}^\mu) \, (q_I \otimes e_{LA})^{JK} = g_{\bar{J}\bar{K}}(z^\mu, \bar{z}^\mu) \quad (3.5)$$

The coordinates are $z^\mu, \bar{z}^\mu \in C^2$. The matrix indices’ range is $J, K = 1, 2, \cdots, 16$. The quaternion indices are $I = 0, 1, 2, 3$, and the octonion indices $A = 0, 1, 2, \cdots, 7$, respectively, and such that the components $g^{JK}(z^\mu, \bar{z}^\mu), g_{\bar{J}\bar{K}}(z^\mu, \bar{z}^\mu)$ are complex-conjugates of each other ensuring that the interval $(ds)^2$ in eq-(3.3) is real.

The non-vanishing connection coefficients of a Hermitian complex manifold are given by [11]

$$\Gamma^\rho_{\mu\nu} = g^{\rho\lambda} \, \partial_\mu g_{\lambda\nu} = g^{\rho\lambda} \frac{\partial g_{\lambda\nu}}{\partial z^\nu}; \quad \Gamma^\rho_{\bar{\mu}\bar{\nu}} = g^{\rho\lambda} \, \partial_{\bar{\mu}} g_{\lambda\bar{\nu}} = g^{\rho\lambda} \frac{\partial g_{\lambda\bar{\nu}}}{\partial \bar{z}^\nu} \quad (3.6)$$

The non-vanishing curvature components are

$$R^\rho_{\bar{\sigma}\mu\bar{\nu}} = \partial_{\bar{\sigma}} \Gamma^\rho_{\mu\bar{\nu}}; \quad R^\rho_{\bar{\sigma}\mu\bar{\nu}} = \partial_{\bar{\mu}} \Gamma^\rho_{\bar{\sigma}\bar{\nu}} \quad (3.7)$$

The Ricci tensor components are
\[
R_{\mu\nu} = R^\rho_{\rho\mu\nu}, \quad R_{\overline{\mu}\overline{\nu}} = R^\overline{\rho}_{\overline{\rho}\overline{\mu}\overline{\nu}}
\]  
(3.8)

and the Ricci scalar is

\[
R = g^{\mu\overline{\nu}} R_{\mu\overline{\nu}} + g^{\overline{\rho}\nu} R_{\overline{\rho}\nu}
\]  
(3.9)

Under (anti) holomorphic coordinate transformations

\[
z'\mu = z'\rho(z^{\rho}), \quad \overline{z}'\mu = \overline{z}'\rho(\overline{z}^{\rho})
\]  
(3.10)

the metric components transform as

\[
g'_{\rho\sigma} = \frac{\partial z'\mu}{\partial z^\rho} \frac{\partial \overline{z}'\nu}{\partial \overline{z}^\sigma} g_{\mu\nu}, \quad g'_{\overline{\rho}\overline{\sigma}} = \frac{\partial \overline{z}'\mu}{\partial \overline{z}^\rho} \frac{\partial z'\nu}{\partial z^\sigma} g_{\mu\nu}
\]  
(3.11)

\[
g'_{\rho\overline{\sigma}} = g'_{\overline{\rho}\rho} = 0
\]  
(3.12)

Let us take the ordinary Hermitian metric in \(D = 2\) complex dimensions case as an example (\(D = 4\) real dimensions) whose coordinates are \(z^\mu, \overline{z}^\mu, \rho, \nu = 1, 2\) and \(\overline{\rho}, \overline{\nu} = \overline{1}, \overline{2}\). The invariant measure of integration under the (anti) holomorphic coordinate transformations (3.10) is

\[
d\Omega \equiv dz^1 \wedge dz^2 \wedge d\overline{z}^1 \wedge d\overline{z}^2 \sqrt{\det(g_{\mu\nu}(z, \overline{z}))} \sqrt{\det(g_{\overline{\mu}\overline{\nu}}(z, \overline{z}))}
\]  
(3.13)

and the analog of the Einstein-Hilbert action is

\[
S = \frac{1}{2\kappa^2} \int R \, d\Omega
\]  
(3.14)

where \(R\) is given by eq-(3.9) and \(\kappa^2\) is the gravitational coupling, \((8\pi G)\) in ordinary Einstein gravity in 4D).

To extend these definitions to the \(C \otimes H \otimes O\)-valued metric case is more complicated due to the noncommutativity and nonassociativity. One may begin, firstly, by finding the relation between \(C \otimes H \otimes O\)-valued metrics in two complex dimensions with metrics in higher dimensional real manifolds.

Focusing on one simple example given by the two-complex dimensional case (four real dimensions) \(z^\mu, \overline{z}^\mu \in C^2\), so that the \(C \otimes H \otimes O\)-valued metric components \(g^{JK}_{\mu\nu}(z^\mu, \overline{z}^\mu)\) have a one-to-one correspondence with the components of the \(32 \times 32\) complex matrix \(g_{MN} = g_{(MN)} + ig_{[MN]}\), with \(M, N = 1, 2, \cdots, 32\). Similarly, the \(C \otimes H \otimes O\)-valued metric components \(g^{JK}_{\mu\nu}(z^\mu, \overline{z}^\mu)\) have a one-to-one correspondence with the components of the \(32 \times 32\) complex matrix \((g_{MN})^* = g_{(MN)} - ig_{[MN]} = g_{NM}\).

Let us decompose the \(32 \times 32\) complex metric \(g_{MN} = g_{(MN)} + ig_{[MN]}\) in the following Kaluza-Klein (KK) form

\[
g_{MN}(x^\alpha; y^a) = \begin{pmatrix}
g_{\alpha\beta} + h_{ab} A^a_{\alpha} A^b_{\beta} & A^b_{\alpha} A^a_{\beta} h_{ab} \\
A^a_{\alpha} h_{ab} & h_{ab}
\end{pmatrix}
\]  
(3.15)
such that
\[ g_{\alpha\beta} = g_{(\alpha\beta)} + ig_{[\alpha\beta]}; \quad h_{ab} = h_{(ab)} + ih_{[ab]} \] (3.16)
The four-dimensional spacetime indices range from \( \alpha, \beta = 1, 2, 3, 4 \), and the internal space indices range is \( a, b = 1, 2, \cdots, 28 \). Similar results apply to the complex conjugate \((g_{MN})^*(x^\alpha; y^\beta)\). Note that the real dimensions of the higher dimensional space is \( 32 = 4 + 28 \).

It is important to emphasize that the above Kaluza-Klein decomposition is not the standard one associated to symmetric metrics but one corresponding to the Nonsymmetric Kaluza-Klein (Jordan-Thiry) Theory and whose structure is far richer than the conventional one. Completely new results in comparison to the standard symmetric Kaluza-Klein theory have been obtained by [12].

The Ricci scalar
\[ R = g^{MN}R_{MN} + (g^{MN}R_{MN})^* \] (3.17)
allows to construct the higher dimensional gravitational action
\[ S = \frac{1}{2\kappa^2} \int d^{32}X \left[ ||\det(g_{MN})|| \right] \frac{1}{2} R(X) = \frac{1}{2\kappa^2} \int d^{32}X \left[ \det(g_{MN}) \det(g_{MN})^* \right] \frac{1}{2} R(X) \tag{3.18} \]
writing the norm of a complex number as \( ||z|| = \sqrt{(zz^*)} \) is the reason why there is a 4-th root in (3.18). After the Kaluza-Klein reduction from \( D = 32 \) to \( D = 4 \):
\[ g_{MN}(x^\alpha; y^a) \to g_{MN}(x^\alpha), \quad \text{eq-(3.18)} \] becomes
\[ S = \frac{\Omega_{28}}{2\kappa^2} \int d^{4}x \left[ \det(g_{MN}(x)) \det(g_{MN}(x))^* \right] \frac{1}{4} R(x) \tag{3.19} \]
where \( \int d^{28}y = \Omega_{28} \) is the volume of the 28-dimensional compact internal space.

To sum up, given \( \mu, \nu = 1, 2; \bar{\mu}, \bar{\nu} = 1, 2 \), and \( M, N = 1, 2, \cdots, 32 \); the Nonsymmetric Kaluza-Klein reduction from \( D = 32 \) to \( D = 4 \):
\[ g_{MN}(x^\alpha; y^a) \to g_{MN}(x^\alpha) \] would allow to establish the following correspondence between \( C \otimes H \otimes O \)-valued metrics in two complex dimensions and complex-valued metrics in higher dimensional real manifolds
\[ g^{JK}_{\mu\bar{\nu}}(z^\mu, \bar{z}^\mu) \leftrightarrow g_{MN}(x^\alpha) = g_{(MN)}(x^\alpha) + ig_{[MN]}(x^\alpha); \quad \alpha = 1, 2, 3, 4 \] (3.20)
and similarly
\[ g^{JK}_{\mu\bar{\nu}}(z^\mu, \bar{z}^\mu) \leftrightarrow (g_{MN})^*(x^\alpha) = g_{(MN)}(x^\alpha) - ig_{[MN]}(x^\alpha); \quad \alpha = 1, 2, 3, 4 \] (3.21)
Finally, after the correspondence of eqs-(3.20, 3.21) is established we may then propose the action (3.19), after the Kaluza-Klein reduction, to be the one which corresponds to the \( H \otimes O \)-extension of the prior gravitational action (3.14) associated with the Hermitian metric in a two-dimensional complex manifold.
An interesting coincidence is that the line interval $ds^2 = \eta_{MN}dx^Mdx^N$ in a $D=32$-dim Euclidean space has $SO(32)$ for its isometry group. $SO(32)$ and $E_8 \times E_8$ are the groups associated with the anomaly-free heterotic string in $D=10$. A KK compactification from $D=32$ to $D=4$ on a 14 complex-dimensional internal space $CP^{14} = \frac{SU(15)}{U(1)}$ yields a $SU(15)$ Yang-Mills in $D=4$. $SU(15)$ can be embedded into $SO(32)$ as $SU(15) \subset SU(16) \subset SO(32)$.

The simplest case is that of a metric in $D=16$ matrices, and if one chooses an specific ordering of those products. Namely, one would have terms of the form $g^{JK}(z, \bar{z})$ which corresponds to a $16 \times 16$ complex metric $g_{MN}$ in 16 real dimensions. A KK compactification from $D=16$ to $D=2$ on a 7 complex-dimensional internal space $CP^7 = \frac{SU(8)}{U(7)}$ yields a $SU(8)$ YM in $D=2$. $SU(8) \subset SO(16)$ which is the isometry group of a 16-dim Euclidean space.

To extend the definitions of the Ricci scalar (3.9) to the $C \otimes H \otimes O$-valued metric case is more complicated due to the noncommutativity and nonassociativity. For example, one would have terms of the form $g\delta(g\delta g)$, $g(g\delta g)(g\delta g)$, such that their products are no longer associative, and due to the noncommutativity, the results also depend on the ordering of those products.

To finalize this section we propose the construction of a generalized Hermitian Matrix geometry as follows. After the correspondence in eqs-(3.20, 3.21) is made, one could treat each one of the components of $g_{\mu\nu}$ as if they were $16 \times 16$ matrices, and if one chooses an specific ordering of those matrices in the products in $g\delta(g\delta g)$, $g(g\delta g)(g\delta g)$, one could then define the $H \otimes O$-valued extension of the Ricci tensor (3.8). Furthermore, due to the cyclic property of the trace operation, the $H \otimes O$ extension of the Ricci scalar of eq-(3.9) is given in terms of the trace of the product of the $16 \times 16$ complex matrices as follows

$$R = \frac{1}{16} \text{Trace} \left( g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu} R_{\mu\nu} \right)$$

(3.22)

To find the analog of the Einstein-Hilbert action in the $C \otimes H \otimes O$-valued metric requires to construct the proper measure. We may define the block determinant $Det$ of $g^{JK}_{\mu\nu}(z^\mu, \bar{z}^\mu)$ in terms of antisymmetrized sums of products of determinants of $16 \times 16$ matrices. Namely,

$$Det \left( g^{JK}_{\mu\nu}(z^\mu, \bar{z}^\mu) \right) = \frac{1}{(2!)^2} \epsilon^{\mu_1\mu_2} \epsilon^{\nu_1\nu_2} \text{det}(g^{JK}_{\mu_1\nu_1}) \text{det}(g^{JK}_{\mu_2\nu_2})$$

(3.23)

where the determinant of the $16 \times 16$ matrix block is

$$\text{det}(g^{JK}_{\mu_1\nu_1}) = \frac{1}{(16!)^2} \epsilon_{J_1 J_2 \ldots J_{16}} \epsilon_{K_1 K_2 \ldots K_{16}} g^{J_1 K_1}_{\mu_1 \nu_1} g^{J_2 K_2}_{\mu_2 \nu_2} \cdots g^{J_{16} K_{16}}_{\mu_{16} \nu_{16}}$$

(3.24)

and

$$\text{det}(g^{JK}_{\mu_2\nu_2}) = \frac{1}{(16!)^2} \epsilon_{J_1 J_2 \ldots J_{16}} \epsilon_{K_1 K_2 \ldots K_{16}} g^{J_1 K_1}_{\mu_2 \nu_2} g^{J_2 K_2}_{\mu_2 \nu_2} \cdots g^{J_{16} K_{16}}_{\mu_{26} \nu_{26}}$$

(3.25)
Similarly we can define the block determinant $\text{Det}(g^{JK}_{\mu\bar{\nu}}(z,\bar{z}))$ and extend these definitions to other complex-dimensions beyond $D = 2$. The measure of integration is a generalization of (3.13) and given by

$$D\Omega \equiv dz^1 \wedge dz^2 \wedge d\bar{z}^1 \wedge d\bar{z}^2 \sqrt{\text{Det}(g^{JK}_{\mu\bar{\nu}}(z,\bar{z}))} \sqrt{\text{Det}(g^{JK}_{\bar{\mu}\nu}(z,\bar{z}))} \quad (3.26)$$

The generalization of the Einstein-Hilbert action in eq-(3.14) is given in terms of $R$ in eq-(3.22), and the measure (3.26), as follows

$$S = \frac{1}{32\kappa^2} \int D\Omega \text{Trace}_{16 \times 16} \left( g^{\mu\bar{\nu}} R_{\mu\bar{\nu}} + g^{\bar{\mu}\nu} R_{\bar{\mu}\nu} \right) \quad (3.27)$$

Therefore, the gravitational action (3.27) based on “coloring” the graviton by attaching internal indices $g_{\mu\bar{\nu}} \rightarrow g^{JK}_{\mu\bar{\nu}}, \cdots$ and associated to the $16 \times 16$ matrices, is the one corresponding to a $C \otimes H \otimes O$-valued metric, and defined over a complex Hermitian manifold in two complex-dimensions. We propose that this matrix approach could be an example of a generalized Hermitian Matrix geometry, and which must not be confused with the current work on generalized geometry, double field theories, exceptional field theories in $M$-theory, see [13] and references therein.

Going back to the line interval of eq-(3.3), under unitary $U(16)$ symmetry transformations $U^\dagger = U^{-1}$ acting on the $16 \times 16$ matrix indices only

$$g_{\mu\bar{\nu}} \rightarrow U g_{\mu\bar{\nu}} U^{-1}, \quad g_{\bar{\mu}\nu} \rightarrow U g_{\bar{\mu}\nu} U^{-1} \quad (3.28)$$

the interval $ds^2$ (3.3) will remain invariant due to the cyclic property of the Trace

$$\text{Trace} \left( U g_{\mu\bar{\nu}} U^{-1} \right) = \text{Trace} \left( U^{-1} U g_{\mu\bar{\nu}} \right) = \text{Trace} \left( g_{\mu\bar{\nu}} \right) \quad (3.29a)$$

$$\text{Trace} \left( U g_{\bar{\mu}\nu} U^{-1} \right) = \text{Trace} \left( U^{-1} U g_{\bar{\mu}\nu} \right) = \text{Trace} \left( g_{\bar{\mu}\nu} \right) \Rightarrow \quad (3.29a)$$

$$\text{Trace} \left( U g_{\mu\bar{\nu}} U^{-1} dz^\mu d\bar{z}^\nu + U g_{\bar{\mu}\nu} U^{-1} d\bar{z}^\mu dz^\nu \right) =$$

$$\text{Trace} \left( g_{\mu\bar{\nu}} dz^\mu d\bar{z}^\nu + g_{\bar{\mu}\nu} d\bar{z}^\mu dz^\nu \right) \quad (3.30)$$

Therefore, the unitary group $U(16)$ acts as an isometry group. In ordinary KK theory the gauge symmetries in lower dimensions emerge from the isometry group of the compactified internal space. In the previous section one had $C \otimes H \otimes O_L$ algebra $\leftrightarrow$ 32 complex $16 \times 16$ matrices $\leftrightarrow$ 64 real $16 \times 16$ matrices $\leftrightarrow$ 64 generators of the rank-16 $u(4) \oplus u(4) \oplus u(4) \oplus u(4)$ algebra. The $u(16)$ has also rank 16, like the $so(32)$ and $e_8 \oplus e_8$ algebras, but in this case the isometry group $U(16)$ is larger than $[U(4)]^4$.

To conclude, we have explored the $C \otimes H \otimes O$ algebra deeper and led us to the gauge group $[SU(4)]^4$ (suggesting the plausible existence of a fourth family). Whereas $C \otimes H \otimes O$-valued gravity bear connections to Nonsymmetric
Kaluza-Klein theories and complex Hermitian Matrix Geometry. It is desirable to extend these results to hypercomplex, quaternionic manifolds and Exceptional Jordan Matrix Models.

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