(pk mk qk) or an Unexpected Inconsistency

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Abstract. This note proves the inconsistency of the Peano arithmetic (PA) by deriving both a strengthened form of the strong Goldbach conjecture and its negation.

Notations. Let \( \mathbb{N} \) denote the natural numbers starting from 1 and let \( \mathbb{P}_3 \) denote the prime numbers starting from 3.

Theorem. The Peano arithmetic (PA) is inconsistent.

Proof. We define the set \( S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \} \) and we consider the following two cases:

\[(G) \text{ The numbers } m \text{ in the components } mk \text{ take all integer values } x \geq 4.
\]
\[\neg(G) \text{ The numbers } m \text{ in the components } mk \text{ do not take all integer values } x \geq 4.\]

The case \( \neg(G) \) means that for each \( k \geq 1 \) there is an \( n_k, n \geq 4, \) different from all the \( mk, \) where all pairs \( (p, q) \) of odd primes, which determine the numbers \( m, \) are used in \( S_g. \) For each \( k \geq 1, \) such an \( n_k \) can be written as some \( pk \) when \( n \) is prime, as some \( pk' \) when \( n \) is composite and not a power of 2, or as \( 4k' \) when \( n \) is a power of 2; \( p \in \mathbb{P}_3; k, k' \).

The expression \( pk' \) for \( n_k \) with \( k' = k \) or \( k' \neq k \) is a first component of \( S_g \) triples and the expression \( 4k' \) for \( n_k \) is component of the triple \( (3k', 4k', 5k'). \) Thus, the \( S_g \) triples are always the same, regardless of whether \( n_k \) as a component of them exists or not.

Therefore, we obtain that the triples of \( S_g \) are the same in both cases, \( (G) \) and \( \neg(G). \) So, the components \( mk \) are the same in both cases, \( (G) \) and \( \neg(G), \) which is a contradiction.

Actually, the above argument uses a strengthened form of the strong Goldbach conjecture and its negation:

**Strengthened strong Goldbach conjecture (SSGB):** Every even integer greater than 6 can be expressed as the sum of two different primes.

\[\neg(SSGB) \text{: There is an even integer greater than 6 that cannot be expressed as the sum of two different primes.}\]

SSGB is equivalent to saying that all integers \( x \geq 4 \) appear as \( m \) in a component \( mk \) of \( S_g. \) Therefore, SSGB is equivalent to the case \( (G) \) and the negation \( \neg(SSGB) \) is equivalent to the case \( \neg(G). \) We have seen above that the \( S_g \) triples are the same in these two cases. This means that both SSGB and \( \neg(SSGB) \) hold.