(pk mk qk) or an Unexpected Inconsistency

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Abstract. This note proves the inconsistency of the Peano arithmetic (PA) by deriving both a strengthened form of the strong Goldbach conjecture and its negation.

Notations. Let $\mathbb{N}$ denote the natural numbers starting from 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3.

Theorem. The Peano arithmetic (PA) is inconsistent.

Proof. We define the set $S_g := \{ (pk, mk, qk) | k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \}$ and we consider the following two cases:

(G) The numbers $m$ in the components $mk$ take all integer values $x \geq 4$. 
$\neg$(G) The numbers $m$ in the components $mk$ do not take all integer values $x \geq 4$.

The case $\neg$(G) means that for each $k \geq 1$ there is an $n_k$, $n \geq 4$, different from all the $mk$, where all pairs $(p, q)$ of odd primes, which determine the numbers $m$, are used in $S_g$. For each $k \geq 1$, such an $n_k$ can be written as some $pk$ when $n$ is prime, as some $pk'$ when $n$ is composite and not a power of 2, or as $4k'$ when $n$ is a power of 2; $p \in \mathbb{P}_3$; $k, k' \in \mathbb{N}$.

The expression $pk'$ for $n_k$, $k' = k$ or $k' \neq k$, is a first component of $S_g$ triples and the expression $4k'$ for $n_k$ is component of the triple $(3k', 4k', 5k')$. Thus, for each $k \geq 1$, $n_k$ is a component of the same triples that exist when there is no such $n_k$, i.e. in the case (G).

Therefore, we obtain that the triples of $S_g$ are the same in both cases, (G) and $\neg$(G). So, the components $mk$ are the same in both cases, (G) and $\neg$(G), which is a contradiction.

Actually, the above argument uses a strengthened form of the strong Goldbach conjecture and its negation:

**Strengthened strong Goldbach conjecture (SSGB):** Every even integer greater than 6 can be expressed as the sum of two different primes.

$\neg$SSGB: There is an even integer greater than 6 that cannot be expressed as the sum of two different primes.

SSGB is equivalent to saying that all integers $x \geq 4$ appear as $m$ in a component $mk$ of $S_g$. Therefore, SSGB is equivalent to the case (G) and the negation $\neg$SSGB is equivalent to the case $\neg$(G). We have seen above that the $S_g$ triples are the same in these two cases. This means that both SSGB and $\neg$SSGB hold.