

# (pk mk qk) or an Unexpected Inconsistency

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**Abstract.** This note proves the inconsistency of the Peano arithmetic (PA) by deriving both a strengthened form of the strong Goldbach conjecture and its negation.

**Notations.** Let  $\mathbb{N}$  denote the natural numbers starting from 1 and let  $\mathbb{P}_3$  denote the prime numbers starting from 3.

**Theorem.** *The Peano arithmetic (PA) is inconsistent.*

*Proof.* We define  $S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \}$  and we consider the following two statements on  $S_g$  :

- (G) For each  $k \geq 1$ , there is no  $nk, n \geq 4$ , different from all the  $mk$ .
- $\neg(G)$  For each  $k \geq 1$ , there is an  $nk, n \geq 4$ , different from all the  $mk$ .

The case (G) means that the numbers  $m$  take all integer values  $x \geq 4$  and the case  $\neg(G)$  means that the numbers  $m$  do not take all integer values  $x \geq 4$ . So, we have

**(1)** There is a difference in  $S_g$  under the two conditions (G) and  $\neg(G)$ .

On the other hand, for each  $k \geq 1$  such an  $nk$  from  $\neg(G)$  can be written as some  $pk$  when  $n$  is prime, as some  $pk'$  when  $n$  is composite and not a power of 2, or as  $4k'$  when  $n$  is a power of 2;  $p \in \mathbb{P}_3; k, k' \in \mathbb{N}$ . The expressions  $pk, pk'$  for  $nk$  are first components of  $S_g$  triples and  $4k'$  is component of the triple  $(3k', 4k', 5k')$ . Since these same triple components also exist in case (G), we obtain that all the triples of  $S_g$  are the same in both cases:

**(2)** There is no difference in  $S_g$  under the two conditions (G) and  $\neg(G)$ .

As (2) is the negation of (1), this is a contradiction in PA.

Note: If  $nk$  from  $\neg(G)$  could be an arithmetic mean of a pair of primes not used in  $S_g$  or if  $nk$  could be a number that cannot be expressed by a  $S_g$  triple component, then  $S_g$  would be different under the conditions (G) and  $\neg(G)$ , without causing any contradiction. However, these two options are excluded by the characteristics of  $S_g$ .

□

Actually, the above argument uses a strengthened form of the strong Goldbach conjecture and its negation:

**Strengthened strong Goldbach conjecture (SSGB):** *Every even integer greater than 6 can be expressed as the sum of two different primes.*

$\neg$ **SSGB:** *There is an even integer greater than 6 that cannot be expressed as the sum of two different primes.*

SSGB is equivalent to saying that all integers  $x \geq 4$  appear as  $m$  in a component  $mk$  of  $S_g$ . Therefore, SSGB is equivalent to the case (G) and the negation  $\neg$ SSGB is equivalent to the case  $\neg(G)$ . We have seen above that the  $S_g$  triples are the same in these two cases. This means that both SSGB and  $\neg$ SSGB hold.