(pk mk qk) or an Unexpected Inconsistency

Ralf Wüsthofen

**Abstract.** This note proves the inconsistency of the Peano arithmetic (PA) by deriving both a strengthened form of the strong Goldbach conjecture and its negation.

**Notations.** Let \( \mathbb{N} \) denote the natural numbers starting from 1 and let \( \mathbb{P}_3 \) denote the prime numbers starting from 3.

**Theorem.** *The Peano arithmetic (PA) is inconsistent.*

**Proof.** We define the set \( S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \} \).

Then, \( S_g \) is the same under these two conditions:

(i) For each \( k \geq 1 \), there is an \( nk, n \geq 4 \), different from all the \( mk \).

(ii) For each \( k \geq 1 \), there is no \( nk, n \geq 4 \), different from all the \( mk \).

The reason is that for each \( k \geq 1 \) such an \( nk \) from (i) can be written as some \( pk \) when \( n \) is prime, as some \( pk' \) when \( n \) is composite and not a power of 2, or as \( 4k' \) when \( n \) is a power of 2; \( p \in \mathbb{P}_3; k, k' \in \mathbb{N} \). As each of these expressions \( pk \), \( pk' \) and \( 4k' \) for \( nk \) is a \( S_g \) triple component, \( S_g \) does not change regardless of whether (i) or (ii) applies.

We note that in the definition of \( S_g \) all pairs \( (p, q), p < q, \) of odd primes are used. This excludes the possibility of an \( nk \) from (i) where \( nk = (p_nk + q_nk) / 2 \) with a pair of primes \( (p_n, q_n) \) not used in \( S_g \).

So, if all the triples of \( S_g \) are the same in both cases, then all the \( mk \) are the same in both cases. This is a contradiction since the case "\( nk \) exists" means that the numbers \( m \) do not take all integer values \( x \geq 4 \) and the case "\( nk \) does not exist" means that the numbers \( m \) take all integer values \( x \geq 4 \). \( \square \)

Actually, the above argument uses a strengthened form of the strong Goldbach conjecture and its negation:

**Strengthened strong Goldbach conjecture (SSGB):** *Every even integer greater than 6 can be expressed as the sum of two different primes.*

\( \neg \text{SSGB}: \) *There is an even integer greater than 6 that cannot be expressed as the sum of two different primes.*

SSGB is equivalent to saying that all integers \( x \geq 4 \) appear as \( m \) in a component \( mk \) of \( S_g \). Therefore, SSGB is equivalent to the case (ii) and the negation \( \neg \text{SSGB} \) is equivalent to the case (i). We have seen above that the \( S_g \) triples are the same in these two cases. This means that both SSGB and \( \neg \text{SSGB} \) hold.