ELEMENTARY SET THEORY CAN BE USED TO PROVE FERMAT’S LAST THEOREM (FLT) V.5

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Abstract. An open problem is proving FLT simply for each n ∈ N, n > 2. Our proof of FLT is based on our algebraic identity, denoted, for convenience, as \( r^n + s^n = t^n \) with \( r, s, t > 0 \) as functions of variables. For \( n \in \mathbb{N}, n > 0 \): We relate \( r, s, t \) for which \( r^n + s^n = t^n \) holds with \( x, y, z > 0 \) for which \( x^n + y^n = z^n \) holds. We infer as true by direct argument (not BWOC), for any given \( n > 2 \), that \( \{ (x, y, z) | x, y, z \in \mathbb{N}, x^n + y^n = z^n \} \implies \{ (r, s, t) | r, s, t \in \mathbb{N}, r^n + s^n = t^n \} \).

In addition, we show, for \( n > 2 \), that \( \{ (r, s, t) | r, s, t \in \mathbb{N}, r^n + s^n = t^n \} = \emptyset \).

Thus, for \( n \in \mathbb{N}, n > 2 \), it is true that \( \{ (x, y, z) | x, y, z \in \mathbb{N}, x^n + y^n = z^n \} = \emptyset \).

1. Introduction

FLT states, for \( n \in \mathbb{N}, n > 2, x, y, z \in \mathbb{N}, x, y, z > 0 \) that \( x^n + y^n = z^n \) does not hold. It is well known that a simple proof of FLT for every \( n \in \mathbb{N}, n > 2 \) is lacking.

For \( n \in \mathbb{N} \) : We use basics to devise a direct proof, not the expected BWOC.

Per Sect. 3, an identity with very restricted integral triples for \( n \in \mathbb{N}, n > 2 \) is :

\[
(4q^n)^\frac{1}{n} + (p - 2q^n)^\frac{1}{n} = (p + 2q^n)^\frac{1}{n}.
\]

Basic conditions : \( n \in \mathbb{N}, n > 0, p \in \mathbb{R}, p > 0, q \in \mathbb{Q}, q > 0 \) such that \( p > 2q^n \).

Denote \( r \) for \( (4q^n)^\frac{1}{n} \); \( s \) for \( (p - 2q^n)^\frac{1}{n} \); and \( t \) for \( (p + 2q^n)^\frac{1}{n} \) throughout the paper.

Therefore, \( r, s, t \in \mathbb{N}, r, s, t > 0, r \neq s \) for which \( r^n + s^n = t^n \) holds, is similar to, thus comparable to \( x, y, z \in \mathbb{N}, x, y, z > 0, x \neq y \) for which \( x^n + y^n = z^n \) holds.

We begin, in Sect 2, below, with \( r, s, t, x, y, z \in \mathbb{R} \) to subsequently infer a relation between included \( r, s, t \in \mathbb{N}, r, s, t > 0 \) and included \( x, y, z \in \mathbb{N}, x, y, z > 0 \).

We argue from an equality of two sets to an equality of the two respective subsets since an equality of two sets, with both sets nonempty or both sets empty, implies that the respective two subsets are equal, with both nonempty or both empty.

A consistent argument in Sect. 2 requires, for \( n = 1, 2 \), with \( r, s, t, x, y, z > 0 \), that \( \{ (r, s, t) | r, s, t \in \mathbb{N}, r^n + s^n = t^n \} = \{ (x, y, z) | x, y, z \in \mathbb{N}, x^n + y^n = z^n \} \) be true; it is clearly true for \( n = 1, 2 \), but solely with \( q \in \mathbb{Q}, q = \frac{5}{3}, \frac{2}{3} \), respectively; so, \( \{ (r, s, t) | r, s, t \in \mathbb{N}, r^n + s^n = t^n \} = \{ (x, y, z) | x, y, z \in \mathbb{N}, x^n + y^n = z^n \} \) would be false should, instead, \( q \in \mathbb{R} - \mathbb{Q} \). So, we must exclude \( q \in \mathbb{R} - \mathbb{Q} \) from our proof.

That \( \{ (r, s, t) | r, s, t \in \mathbb{N}, r^n + s^n = t^n \implies \{ (x, y, z) | x, y, z \in \mathbb{N}, x^n + y^n = z^n \} \) is true is shown, in section 2, below, for \( n \in \mathbb{N}, n > 2, x, y, z, r, s, t > 0, p \in \mathbb{R}, q \in \mathbb{Q} \); therefore, equation \( \{ (x, y, z) | x, y, z \in \mathbb{N}, x^n + y^n = z^n \} = \emptyset \) (which is FLT) is true since we show in section 3, below, that \( \{ (r, s, t) | r, s, t \in \mathbb{N}, r^n + s^n = t^n \} = \emptyset \).

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For any given \( n \in \mathbb{N}, n > 0 \) : Letting \( r, s, t \) respectively denote \((4q^n)\frac{1}{2}, (p-2q^n)\frac{1}{2}\), and \((p+2q^n)\frac{1}{2}\), with \( q \in \mathbb{Q}, p \in \mathbb{R}, p,q > 0 \) for which \( r^n + s^n = t^n \) holds, and with \( x, y, z \) such that \( x^n + y^n = z^n \) holds, all the sets that we use in Sect. 2, below, are:

Let \( A \) be \((r,s,t)|r,s,t \in \mathbb{N}, r,s,t > 0\) : An infinite of elements for \( n > 0; \) for \( n = 1\), for example \( \{3\pi, 4\pi, 7\pi\} \), \( n = 2\), e.g., \( \{3\pi, 4\pi, 5\pi\}\), for \( n = 3\), e.g. \((1,2,9)\).

Let \( B \) be \((r,s,t)|r,s,t \in \mathbb{N}, r,s,t > 0\) : An infinite of elements for \( n > 0; \) for \( n = 1\), e.g. \( \{12,7\}\), and, for \( n = 2\), e.g. \( \{12,5\}\), for \( n = 3\) e.g. \((4-2\sqrt{2})\frac{1}{2} (4+2\sqrt{2})\frac{1}{2}, 2\).

Let \( C \) be \((r,s,t)|r,s,t \in \mathbb{N}, r,s,t > 0\) : An infinite of elements for \( n = 1, 2\) for \( n = 1\), e.g. \( \{3, 4, 7\}\), and, for \( n = 2\), e.g. \( \{3, 4, 5\}\); likely no elements for \( n \geq 3\).

Let \( D \) be \((x,y,z)|x,y,z \in \mathbb{R}, x,y,z > 0\) : An infinite of elements for \( n > 0; \) for \( n = 1\), e.g. \( \{6\pi, 8\pi, 14\pi\}\), for \( n = 2\), e.g. \( \{6\pi, 8\pi, 10\pi\}\), for \( n = 3\), e.g. \( \{2, 4, 264\}\).

Let \( E \) be \((x,y,z)|x,y,z \in \mathbb{N} : An infinite of elements for \( n > 0; \) for \( n = 1\), e.g. \( \{48, 14\}\); for \( n = 2\), e.g. \( \{48, 10\}\); for \( n = 3\), e.g. \( \{(32 - 16\sqrt{2})\frac{1}{2} (32 + 16\sqrt{2})\frac{1}{2}, 2\}\).

Let \( F \) be \((x,y,z)|x,y,z \in \mathbb{R}, x,y,z > 0\) : An infinite of elements for \( n = 1, 2\); for \( n = 1\), e.g. \( \{6, 8, 14\}\), and, for \( n = 2\), e.g. \( \{6, 8, 10\}\); likely no elements for \( n \geq 3\).

Let \( G \) be \((x,y,z, s\in \mathbb{N}) : An infinite of elements for \( n > 0; \) for \( n = 1\), for example \( \{(12\pi^2)/7\}, \) and, for \( n = 2\), for example \( \{(12\pi^2)/5\}, \) with \( n = 3\), e.g. \( \{(2-1)/9\}\).

Let \( H \) be \((x,y,z, s\in \mathbb{N}) : An infinite of elements for \( n > 0; \) for \( n = 1\), for example \( \{12\pi/7\}, \) and, for \( n = 2\), e.g. \( \{12\pi/5\}\), for \( n = 3\); e.g. \( \{(4 - 2\sqrt{2})\frac{1}{2} (4 + 2\sqrt{2})\frac{1}{2}, 2\}\).

Let \( J \) be \((r,s,t)\in \mathbb{N} : An infinite of elements for \( n = 1, 2\); for \( n = 1\), for example \( \{3-4\}/7\}, \) and, for \( n = 2\), e.g. \( \{3-4\}/5\}; \) likely no elements for \( n \geq 3\).

Let \( K \) be \((x,y,z)\in \mathbb{N} : An infinite of elements for \( n > 0; \) for \( n = 1\), for example \( \{(48\pi^2)/14\}, \) and, for \( n = 2\), e.g. \( \{(48\pi^2)/10\}\), for \( n = 3\), e.g. \( \{(4 \cdot 2)/72\}\).

Let \( L \) be \((x,y,z)\in \mathbb{N} : An infinite of elements for \( n > 0; \) for \( n = 1\), e.g. \( \{48/14\}\) and, for \( n = 2\), e.g. \( \{48/10\}\), for \( n = 3\), e.g. \( \{(32 - 16\sqrt{2})\frac{1}{2} (32 + 16\sqrt{2})\frac{1}{2}\}/2\).

Let \( M \) be \((x,y,z)\in \mathbb{N} : An infinite of elements for \( n = 1, 2\); for \( n = 1\), for example \( \{(6/8)/14\}\) and, for \( n = 2\), e.g. \( \{(6/8)/10\}\); likely no elements for \( n \geq 3\).

2. Our Direct Proof Using Sets and Respective Subsets

**Proposition 2.1.** For any given \( n \in \mathbb{N}, n > 0 \) : \( H = L \), with \( H, L \neq \emptyset \).

**Proof.** For any given \( n \in \mathbb{N}, n > 0 \) : \( x = \frac{(4q^n)\frac{1}{2}(p-2q^n)\frac{1}{2}}{(p+2q^n)\frac{1}{2}} \in G \), so \( x = y \in G \), and \( y = z \in K \) are equally restricted, as follows : With any given \( q \in \mathbb{Q}, q > 0 \), unrestricted \( p \in \mathbb{R}, p > 0 \) varies such that \( x = \frac{y}{t} \in G \) takes any given \( \frac{y}{t} \in K \); also, \( rs/t < r, xy/z < x \). Clearly, \( K \) includes \( G \). Thus, for any given \( n > 0 \) it is true that \( \{\frac{x}{y}\in G\} = \{\frac{y}{z}\in K\} \). Since we focus on \( x, y, z, r, s, t \in \mathbb{N} \) : Let \( z, t \in \mathbb{N} \) which exist for each \( n > 0 \), implying \( \{\frac{z}{x} \in H : C \subseteq G \} = \{\frac{z}{x} \in L : C \subseteq H \} \) with \( H, L \neq \emptyset \). □

**Proposition 2.2.** Existing \( x, y, z \in \mathbb{F} \) are rational multiples of existing \( r, s, t \in \mathbb{F} \).

**Proof.** For any given \( n \in \mathbb{N}, n > 0 \) : Define constant \( 2\alpha/\beta \) with \( \alpha \in \mathbb{Q}, \alpha > 0 \). Equation \( x = \frac{y}{y} \in H \) yields \( \frac{\alpha}{\beta} \in L \) for which \( \{r \cdot s \cdot t \cdot \alpha \cdot \beta \in H \} = \{x \cdot y \in L \}; \{t \cdot \alpha \cdot \beta \in H \} = \{z \cdot y \in L \}; \{s \cdot r \cdot \alpha \cdot \beta \in H \} = \{y \cdot x \in L \}; \{r \cdot s \cdot \alpha \cdot \beta \in H \} = \{r \cdot s \cdot \alpha \cdot \beta \in H \} \).

So, \( \{r \cdot s \cdot t \cdot \alpha \cdot \beta \in J \} = \{x, y \in M \subseteq \mathbb{C} \} \) with \( J, M \neq \emptyset \) or \( J, M = \emptyset \).

Consequently, \( \{r \cdot s \cdot t \cdot \alpha \cdot \beta \in \mathbb{F}, C \subseteq \mathbb{F} \} \) : The Fermat triple \( (x, y, z) \in F \) is a rational multiple of \( (r, s, t) \in \mathbb{C} \) with \( F, C \neq \emptyset \), or \( F, C = \emptyset \). □
Thus, for $n \in \mathbb{N}, n > 0$ we prove Props. 2.1-2.2 with $p \in \mathbb{R}$, and $q \in \mathbb{Q}$.

### 3. Results and Conclusion

With $(4q^n)^{\frac{1}{n}}, (p - 2q^n)^{\frac{1}{n}}, (p + 2q^n)^{\frac{1}{n}} \in \mathbb{Q}$, or $r, s, t \in \mathbb{Q}$, respectively, of Sect. 1:

Term $(4q^n)^{\frac{1}{n}} \in \mathbb{Q}$ reduces to $2^{\frac{1}{n}}q \in \mathbb{Q}$. So, such $2^{\frac{1}{n}}q \in \mathbb{Q}$ and $r \in \mathbb{Q}$ are identical.

Thus, for $n \in \mathbb{N}, n > 0$ we prove Props. 2.1-2.2 with $p \in \mathbb{R}$, and $q \in \mathbb{Q}$.

Hence, for $n \in \mathbb{N}, n > 0$ : There are no values, with $q \in \mathbb{Q}$, for $2^{\frac{1}{n}}q \in \mathbb{Q}$.

For $n \in \mathbb{N}, n > 2$, the fact that such $r \in \mathbb{N}$ is impossible shows that $C = \emptyset$.

|For $n \in \mathbb{N}, n > 2$, the fact that such $r \in \mathbb{N}$ is impossible shows also that $\{r \cdot \alpha | r, s, t \in A\} \neq x | x, y, z \in \mathbb{D} \}$ : Term $x$ can be integral, e.g., $2^3 + 3^3 = (35)^{\frac{1}{2}}$.

However, we show $\{r \cdot \alpha | r, s, t \in C\} = \{x | x, y, z \in \mathbb{F}\}$ to be true, in Sect 2, above.]

Per our proof of proposition 2.2, above, it is true that $C \implies F$.

Consequently, $F = \emptyset$. In other words, for $n \in \mathbb{N}, n > 2$, the following is true:

Equation $x^n + y^n = z^n$ does not hold for $(x, y, z)$ with $x, y, z \in \mathbb{N}, x, y, z > 0$.

QED.