Irrational Numbers and Density
sgm, 2018/NOV/01

The density of the real numbers is known to come from the density of the irrationals contained within them. Rational numbers are sparse within the reals and natural / counting numbers are sparser still. Let’s look at the binary representation of a restricted subset of integers and examine the parallels between that set and the reals:

		size
000 001 010 100	←→ $\mathscr{N}$	∞^1_c
[000, 111]	←→ $\mathscr{R}$	2^∞

To explain, 0 1 2 4 represent the natural numbers in the set just above left – and – the whole set represents the reals. As we increase the size of the set, *the naturals* ***only increment*** while *the reals* ***increase exponentially***.

Because the rationals are formed from two integers, they are just a little less sparse than the naturals with size ∞^2_c. Now we can calculate the actual numerical density of irrationals and rationals within the reals:

$$\frac{2^\infty - \infty^2_c}{2^\infty} \qquad \frac{\infty^2_c}{2^\infty}$$

So the rationals within the reals are sparse indeed. To imply they’re “dense” is a complete mis-characterization.

It’s natural to be curious about any classes of irrational numbers since they’re the bulk of the reals. Perhaps the simplest irrational number is √2. The next simplest is $\sqrt{\mathscr{P}}$ where $\mathscr{P}$ is prime. Next would be √a/b where a/b is rational. Then, linear combinations of those and so on. Before we try to classify groups of irrationals, let’s talk about functions specifically on $\mathscr{N}$:

There exists no deterministic function f
  such that f maps 0/1 onto the ***whole*** unit interval, [0,1]
    - this violates the definition of a function

which implies there is no f($\mathscr{N}$) that can map $\mathscr{N}$ onto $\mathscr{R}$
but Σf($\mathscr{N}$) ***can*** (see examples below) which implies
{Σf($\mathscr{N}$)} is closed under field operations and {Σf($\mathscr{N}$)} = $\mathscr{R}$

$\pi = 4\Sigma(-1)^n/(2n+1)$ $n=0-\infty$

$e = \Sigma\ 1/n!$ $n=0-\infty$

$1 = \Sigma\ 1/2^n$ $n=1-\infty$

Which demonstrates that ***any*** number, natural or transcendental, can be generated by an infinite series of a function on the naturals .. Let’s get back to categorizing any groups of irrationals. By observing two degrees of freedom defines the size of the rationals and that maximum density is defined by 2^∞, we can see that any function with higher degrees of freedom would have higher associated range-set density – placing that between sparse natural density and maximum irrational density.

The author is disappointed by not being able to go further in identification of larger classes of irrationals. However, it has been one of his life-goals to quantify density better than 1/0 (with meaningless incidental assignments on the boundaries). The rationals are sparse indeed; the irrationals are super-dense; now we can quantify those qualitative descriptions.

Note: a way to visualize the numbers 2^∞ and $\infty^2{}_c$ is to plot the associated functions of x side by side and their derivatives realizing the derivative of x^2 is simply 2x which is a ***linear*** function – and – that of 2^x is $\ln 2 * 2^x$. The latter derivative or rate-of-change is still an exponential function which indicates the density of the irrationals is ***HUGE*** compared to the density of their complement, the rationals.

This essay along with that at:
http://vixra.org/pdf/1805.0332v1.pdf
provide a purely axiomatic derivation of $\mathscr{R}$ based on $\mathscr{N}$ and further based on first principles in set theory.