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Two simples proofs of Fermat 's last theorem and Beal conjecture

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Abstract :If after 374 years the famous theorem of Fermat-Wiles was demonstrated in 110 pages by A. Wiles [4], the puspose of this article is to give a simple demonstration and deduce a proof of the Beal conjecture.

Résumé : Si après 374 ans le célèbre théorème de Fermat-Wiles a été démontré en 110 pages par A. Wiles [4], le but de cet article est de donner une simple démonstration et d'en déduire une preuve de la conjecture de Beal.

Keywords : Fermat, Fermat-Wiles theorem, Fermat's great theorem.

The Subject Classification Codes : 11D41 - 11G05 - 11G07 - 26B15 - 26B20 - 28A10 - 28A75 -

1-Introduction :

Set out by Pierre de Fermat [2], it was not until more than three centuries ago that Fermat's great theorem was published, validated and established by the British mathematician Andrew Wiles [4] in 1995.

In mathematics, and more precisely in number theory, the last theorem of Fermat [2], or Fermat's great theorem, or since his Fermat-Wiles theorem demonstration [4], is as follows : There are no non-zero integers a, b, and c such that : $a^n+b^n=c^n$, as soon as n is an integer strictly greater than 2 ".

The Beal conjecture is the following conjecture in number theory : If $a^x+b^y=c^z$ where a, b, c, x, y and z are positive integers with x, y, z > 2, then a, b, and c have a common prime factor. Equivalently, There are no solutions to the above equation in positive integers a, b, c, x, y, z with a, b and c being pairwise coprime and all of x, y, z being greater than 2.

If the famous Fermat-Wiles theorem has been demonstrated in **110 pages** by A. Wiles [4], the purpose of this article is to give a simple proof and deduce a proof of the Beal conjecture.

2. The proof of *Fermat 's last theorem*

Theorem :

There are no non-zero integers a, b, and c such that: $a^n+b^n=c^n$, with n an integer strictly greater than 2

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Lemma 1 :

If n, a, b and c are a non-zero integers with and $a^n+b^n=c^n$ then:

$$\int_0^b x^{n-1}-\left(\frac{c-a}{b}x+a\right)^{n-1}\frac{c-a}{b}dx=0$$

Proof :

$$a^n+b^n=c^n \Leftrightarrow \int_0^a n\,x^{n-1}dx+\int_0^b n\,x^{n-1}dx=\int_0^c n\,x^{n-1}dx$$

But as :

$$\int_0^c n\,x^{n-1}dx=\int_0^a n\,x^{n-1}dx+\int_a^c n\,x^{n-1}dx$$

So :

$$\int_0^b n\,x^{n-1}dx=\int_a^c n\,x^{n-1}dx$$

And as by changing variables we have :

$$\int_a^c n\,x^{n-1}dx=\int_0^b n\left(\frac{c-a}{b}y+a\right)^{n-1}\frac{c-a}{b}dy$$

Then :

$$\int_0^b x^{n-1}dx=\int_0^b\left(\frac{c-a}{b}y+a\right)^{n-1}\frac{c-a}{b}dy$$

It results:

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$$\int_0^b x^{n-1} - \left(\frac{c-a}{b}x + a\right)^{n-1} \frac{c-a}{b} dx = 0$$

Corollary 1 : If N, n, a, b and c are a non-zero integers with and $a^n + b^n = c^n$ then :
$$\int_0^{\frac{b}{N}} x^{n-1} - \left(\frac{c-a}{b}x + \frac{a}{N}\right)^{n-1} \frac{c-a}{b} dx = 0$$

Proof : It results from the proof of lemma 1 by replacing a, b and c respectively by $\frac{a}{N}$, $\frac{b}{N}$ and $\frac{c}{N}$.

Lemma 2 :

If $a^n + b^n = c^n$, where n, a, b and c are a non-zero integers with $n > 2$ and $a \leq b \leq c$. Then for an integer N big enough we have : $x^{n-1} - \left(\frac{c-a}{b}x + \frac{a}{N}\right)^{n-1} \frac{c-a}{b} \leq 0 \quad \forall x \in \left[0, \frac{b}{N}\right]$.

Proof :

Let $f(x,a,b,c,y) = x^{n-1} - \left(\frac{c-a}{b}x + y\right)^{n-1} \frac{c-a}{b}$. with $x, y \in \mathbb{R}^+$.

We have : $\frac{\partial f}{\partial x} = (n-1)x^{n-2} - (n-1)\left(\frac{c-a}{b}x + y\right)^{n-2} \left(\frac{c-a}{b}\right)^2$, $f(0,a,b,c,y) < 0$ and $\frac{\partial f}{\partial x}|_{x=0} < 0$. So, by continuity, $\exists \epsilon > 0$ such that $\forall u \in [0, \epsilon]$ we have $\frac{\partial f}{\partial x}|_{x=u} < 0$. So the function f is decreasing in $[0, \epsilon]$ and $\exists \epsilon' > 0, \epsilon \geq \epsilon' > 0$ such that we have :
$f(x,a,b,c,y) \leq 0 \quad \forall x \in [0, \epsilon'], \quad \forall y \in [0, \epsilon']$.

As $\frac{b}{N} \in [0, \epsilon']$ for an integer N big enough , It follows that $\forall x \in \left[0, \frac{b}{N}\right]$ we have :
$f(x,a,b,c,\frac{a}{N}) \leq 0 \quad \forall x \in \left[0, \frac{b}{N}\right]$.

Proof of Theorem:
If $a^n + b^n = c^n$, where n, a, b and c are a non-zero integers with $n > 2$ and $a \leq b \leq c$. Then for an integer N big enough, it results from the **lemma 2** that we have :
$$f(x,a,b,c,\frac{a}{N}) = x^{n-1} - \left(\frac{c-a}{b}x + \frac{a}{N}\right)^{n-1} \frac{c-a}{b} \leq 0 \quad \forall x \in \left[0, \frac{b}{N}\right]$$

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And by using the **corollary 1**, we have $\int_0^{\frac{b}{N}} x^{n-1}-\left(\frac{c-a}{b}x+\frac{a}{N}\right)^{n-1}\frac{c-a}{b}dx=0$ **.**

So : $x^{n-1}-\left(\frac{c-a}{b}x+\frac{a}{N}\right)^{n-1}\frac{c-a}{b}=0 \;\; \forall x\in\left[0,\frac{b}{N}\right]$

And therefore $\frac{c-a}{b}=1$ because $f(x,a,b,c,\frac{a}{N})$ is a null polynomial as it have more than n zeros. So $c=a+b$ and $a^n+b^n\neq c^n$ which is absurde .

3- The proof of Beal conjecture :

Corollary : **[Beal conjecture]**

If $a^x+b^y=c^z$ where a, b, c, x, y and z are positive integers with x, y, z > 2, then a, b, and c have a common prime factor.

Equivalently, there are no solutions to the above equation in positive integers a, b, c, x, y, z with a, b and c being pairwise coprime and all of x, y, z being greater than 2.

Proof :

Let $a^x+b^y=c^z$

If a, b and c are not pairwise coprime, then by posing $a=ka'$, $b=kb'$, and $c=kc'$.

Let $a'=u'^{yz}$, $b'=v'^{xz}$, $c'=w'^{xy}$ and $k=u^{yz}$, $k=v^{xz}$, $k=w^{xy}$

As $a^x+b^y=c^z$, we deduce that $(uu')^{xyz}+(vv')^{xyz}=(ww')^{xyz}$.

So : $k^x u'^{xyz}+k^y v'^{xyz}=k^z w'^{xyz}$

This equation does not look like the one studied in the first theorem. But if a, b and c are pairwise coprime, we have $k=1$ and $u=v=w=1$ and we will have to solve the equation : $u'^{xyz}+v'^{xyz}=w'^{xyz}$

The equation $u'^{xyz}+v'^{xyz}=w'^{xyz}$ have a solution if at least one of the equations : $(u'^{xy})^z+(v'^{xy})^z=(w'^{xy})^z$, $(u'^{xz})^y+(v'^{xz})^y=(w'^{xz})^y$, $(u'^{yz})^x+(v'^{yz})^x=(w'^{yz})^x$, have a solution .

So by the proof given in the proof of the first Theorem we must have : $x\leq 2$ or $y\leq 2$, or $z\leq 2$.

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We therefore conclude that if $a^x + b^y = c^z$ where a, b, c, x, y, and z are positive integers with $x, y, z > 2$, then a, b, and c have a common prime factor.

4- Important notes :

1- If a, b, and c are not pairwise coprime, someone, by applying the proof given in the corollary like this : $a = u^{yz}$, $b = v^{xz}$, $c = w^{xy}$ will have $u^{xyz} + v^{xyz} = w^{xyz}$, and could say that all the x,y and z are always smaller than 2. What is false: $7^3 + 7^4 = 14^3$.
The reason is sipmle: it is the common factor k which could increase the power, for example if $k = c'^r$ in the proof, then $c^z = (kc')^z = c'^{(r+1)z}$. You can take the example : $2^r + 2^r = 2^{r+1}$ where $k = 2^r$.

2- These techniques do not say that the equation $a^n + b^n = c^n$ where $a, b, c \in]0, +\infty[$, has no solution since in the proof the equation $X^2 + Y^2 = Z^2$ can have a sloution. We take $a = X^{\frac{2}{n}}$, $b = Y^{\frac{2}{n}}$ and $C = Z^{\frac{2}{n}}$.

3 – In [3] I proved the abc conjecture which implies only that the equation $a^x + b^y = c^z$ has only a finite number of solution with a, b, c, x, y, z a positive integers, a, b and c being pairwise coprime and all of x, y, z being greater than 2.

5- Conclusion :

The techniques used in this article have allowed to prove both the Fermat' last theorem and the Beal' conjecture and have shown that the Beal conjecture is only a corollary of the Fermat' last theorem.

Bibliography :

[1] https://en.wikipedia.org/wiki/Beal conjecture

[2] https://en.wikipedia.org/wiki/Fermat last theorem

[3] M. Mghiar , la preuve de la conjecture abc, iosr journal of mathematics (iosr-jm), *e-issn: 2278-5728, p-issn: 2319-765x. volume 14, issue 4 ver. i (jul - aug 2018), pp 22-26.*

[4] Andrew Wiles, Modular elliptic curves and Fermat's last Theorem, Annal of mathematics, volume 10,142, pages 443-551, september-december, 1995.